An overview on K-theoretic red-shift

Birgit Richter

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$\vdots$

$v_4 : 30$

$v_3 : 14$

$v_2 : \ 6$

$v_1 : \ 2$

$v_0 : \ 0$

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Today, we'd say that $K_0(R)$ of a ring R is the Grothendieck group completion of the abeliand monoid of isomorphism classes of finitely generated projective R-modules, $\mathrm{Proj}(R)$.

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$K_1(R) = GL(R)/[GL(R), GL(R)]$ ('generalized determinant') and
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K-groups are notoriously hard to calculate, for instance we don't know all K-groups of $\mathbb{Z}$.

On the other hand:

$$K_i(\mathbb{F}_q) = \begin{cases} \mathbb{Z}, & i = 0, \\ 0, & i = 2j > 0, \\ \mathbb{Z}/(q^j - 1), & i = 2j - 1 \quad \text{[Quillen]}. \end{cases}$$

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$V(1)_*$ roughly cuts away p and v_1.

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The set of objects of $\mathcal{V}$ is just $\mathbb{N}_0$ (dimension), and

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The K-theory is

$$\mathcal{K}(\mathcal{V}) = \mathbb{Z} \times |BGL(\mathcal{V})|^{+}$$

where $GL(\mathcal{V})$ are weakly invertible matrices over $\mathcal{V}$.

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Then $GL_n(\mathcal{V})$ is the full subcategory of all *nxn*-matrices over $\mathcal{V}$, whose object-matrix is in $GL_n(\mathbb{N}_0)$.

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So, a matrix of objects $A \in GL_n(\mathbb{N}_0)$ is invertible, if it is invertible as an integral matrix.
Then $GL_n(\mathcal{V})$ is the full subcategory of all *nxn*-matrices over $\mathcal{V}$, whose object-matrix is in $GL_n(\mathbb{N}_0)$.
That $\mathcal{K}(\mathcal{V})$ classifies 2-vector bundles was shown by Baas-Dundas-Rognes (2004).

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Hahn-Wilson (2022): $BP\langle n\rangle = BP/v_{n+1}, v_{n+2}, \ldots$ satisfies red-shift.
Beware: $BP\langle n\rangle$ is *not* E_∞ by Lawson (2018) and Senger.

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I'll give a few more examples of cases where such Bott elements were determined. This is *not* a comprehensive list.

- Ausoni-Rognes (2011): $K(k(1))$ has Bott element v_2.

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- Bayındır (to appear): $K(ku/p)$ has Bott class b (as for $K(ku)$).
- Angelini-Knoll, Ausoni, Culver, Höning, Rognes (to appear): $K(BP\langle 2\rangle)$ has v_3 as a Bott class.

Note, that neither of $k(1)$, ku/p, $BP\langle 2\rangle$ are commutative, so these cases are *not* covered by Burklund-Schlank-Yuan, but $BP\langle 2\rangle$ is covered by Hahn-Wilson.

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Of course, ∞-categories are all over the place.