Detecting and describing ramification for structured ring spectra

Birgit Richter, DMV-ÖMG Tagung, September 30 2021

Joint work with Eva Höning

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Then $\mathbb{Z}[i] \supset (2) = (1+i)^2$ and 2 is the characteristic of the residue field $\mathbb{F}_2$, so (2) is wildy ramified.

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Here, using the cyclotomic polynomial one sees that the ideal (p) splits as $(1 - \zeta_p)^{p-1}$ in $\mathbb{Z}[\zeta_p]$.

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The fixed point condition is always satisfied in this situation, so the condition for being unramified is

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If X is a compact Hausdorff space and G is a finite group of homeomorphisms of X, then $C^0(X/G; \mathbb{R}) \to C^0(X; \mathbb{R})$ is a G-Galois extension iff G acts fixed-point free on X.

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We want to understand ramification of maps $A \to B$ in order to understand descent questions in algebraic K-theory: How close is $K(B)^{hG}$ to $K(A)$?
[Ausoni, Rognes, Clausen-Mathew-Naumann-Noel,...]

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$K(S) \simeq S \vee Wh^{\mathrm{Diff}}(*)$ where $Wh^{\mathrm{Diff}}(*)$ is the Whitehead spectrum and this in turn is related to the stable smooth h-cobordism space. [Waldhausen, Jahren, Rognes,...]

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Here, h is right adjoint to the composite map

$$B \wedge_A B \wedge G_+ \longrightarrow B \wedge_A B \longrightarrow B\,,$$

induced by the G-action and the multiplication on B.

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Rognes [2008]: This turns $KO \to KU$ into a C_2-Galois extension.

Note that on homotopy groups we get

$$\pi_*(KO) = \mathbb{Z}[\eta, y, \omega^{\pm 1}]/(2\eta, \eta^3, \eta y, y^2 - 4\omega) \xrightarrow{\pi_*(c)} \mathbb{Z}[u^{\pm 1}] = \pi_*(KU)$$

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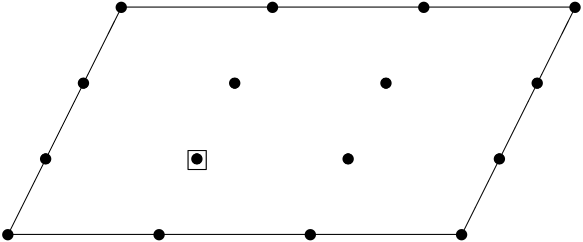
- $TMF[1/n] \to TMF(n)$ is $GL_2(\mathbb{Z}/n\mathbb{Z})$-Galois [MM-2015].



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We *do* have ramification, but we don't see yet, whether it's tame or wild.

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If $B = H\mathcal{O}_L$, then the spectral sequence collapses and $\hat{H}^*(G; \mathcal{O}_L) \cong \pi_{-*}(H\mathcal{O}_L)^{tG}$.

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Beware! If $A \to B$ is a map between connective commutative ring spectra, then often $B^{hG} \not\simeq A$, but $A \to \tau_{\geq 0} B^{hG}$ might be an equivalence (e.g. $ko \simeq \tau_{\geq 0} ku^{hC_2}$).

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We have $tmf(n)^{tGL_2(\mathbb{Z}/n\mathbb{Z})} \simeq *$ if and only if the order of $GL_2(\mathbb{Z}/n\mathbb{Z})$, is a unit in $\mathbb{Z}[\frac{1}{n}]$.

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- It also holds for instance if $n = 2 \cdot 3 \cdot \ldots \cdot p_m$ is the product of the first m prime numbers for any $m \geq 2$
- or for $n = 2 \cdot 3 \cdot 7 = 42$ but not for $n = 2 \cdot 3 \cdot 11$.