Reflexive homology and involutive Hochschild homology as equivariant Loday constructions

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(joint work with Ayelet Lindenstrauss)

A non-equivariant Loday construction $\mathcal{L}_X(R)$ combines a finite simplicial set X and a commutative ring R into a simplicial commutative ring. For the circle, its homotopy groups are the Hochschild homology groups of R. Other important cases are higher dimensional spheres and tori.

Equivariantly for a finite group G, the input is a finite simplicial G-set and a G-commutative monoid: For G-spectra these are genuine commutative G ring spectra and for G-Mackey functor they are given by G-Tambara functors. We defined equivariant Loday constructions $\mathcal{L}_X^G(-)$ in these settings in [5]. In the following we will specialize to the group of order 2, C_2, to fixed point Tambara functors $\underline{R}^{\text{fix}}$ of a commutative ring R with C_2-action, and to the one-point compactification of the real sign-representation, S^σ. For well-behaved genuine commutative C_2-ring spectra A we identified $\mathcal{L}_{S^\sigma}^{C_2}(A)$ with the Real topological Hochschild homology of A, $THR(A)$, in [5]. In the talk, I explained a corresponding result for fixed point Tambara functors [4].

Involutive Hochschild cohomology was defined by Braun [1] and the corresponding homology theory, $\mathsf{iHH}_*^k(A;M)$, for associative k-algebras with anti-involution A and involutive A-bimodules M was developed by Fernàndez-València and Giansiracusa. We identify the latter with the homotopy groups of the C_2/C_2-level of our Loday construction [4]:

Theorem If 2 is invertible in R and if R is flat as an abelian group, then

$$\pi_*\mathcal{L}_{S^\sigma}^{C_2}(\underline{R}^{\text{fix}})(C_2/C_2) \cong \mathsf{iHH}_*^{\mathbb{Z}}(R;R).$$

Daniel Graves explored reflexive homology in [3]. This is the homology theory for the crossed simplicial group ΔR, where $R_n = C_2$ acts on the simplicial category Δ by reversing the simplicial structure. He showed that for a field of characteristic zero, k, involutive Hochschild homology and reflexive homology, $HR_*^{+,k}(A;M)$ of an associative k-algebra with anti-involution and an involutive A-bimodule M agree. We prove the following comparison result [4]:

Theorem If 2 is invertible in R and if R is flat as an abelian group, then

$$\pi_*\mathcal{L}_{S^\sigma}^{C_2}(\underline{R}^{\text{fix}})(C_2/C_2) \cong HR_*^{+,\mathbb{Z}}(R;R).$$

In particular, this identifies iHH_* and HR_*^+ in this generality. We also obtain identifications relative to an arbitrary commutative ground ring k under similar flatness conditions if 2 is invertible.

For an arbitrary finite group G there is no meaningful way for G to act on Δ. We propose

$$\pi_*\mathcal{L}_{SG}^{G}(\underline{R}^{\text{fix}})(G/G)$$

as a suitable homology theory for commutative rings R with G action if the order of G is invertible in R and if R is flat. Here, SG is the unreduced suspension of G and G acts on SG by permuting the arcs.

References

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