Abstracts

(Higher) Topological Hochschild homology – an overview

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When topological Hochschild homology, THH, of rings and ring spectra was first defined by Bökstedt in the mid 80's [5], there was no symmetric monoidal category of spectra developed, yet. Bökstedt used the diagram category of finite sets and injections in order to give a model for THH. Since the mid 90's there are other models, for instance one that mimics the definition of the Hochschild complex, one using a Tor-like definition and one using a suitable bar construction (see [13, chapter IX]). It was shown that THH of a ring is isomorphic to MacLane homology [19] and to stable K-theory [12]. The Dennis trace map $tr\colon K_*(R) \to HH_*(R)$ factors over $\mathsf{THH}_*(R)$ and the latter is a better approximation to algebraic K-theory than $HH_*(R)$; it also serves as the input for the construction for topological cyclic homology, $TC(R)$, and this approximates $K_*(R)$ very well in many cases.

Bökstedt calculated THH of the integers and of $\mathbb{F}_p$ [6]. His famous spectral sequence was used for instance by McClure and Staffeldt to determine the mod p homotopy groups of THH of the connective Adams summand [17]. We know THH in many more examples, for instance for local fields [14], number rings [15], $\mathbb{Z}/p^n$ [7] and connective complex topological K-theory [3].

For a discrete R-algebra A (R commutative), the center of A over R can be identified with the endomorphisms of A in the category of A-bimodules over R. Topological Hochschild cohomology of an R-algebra spectrum A can be defined as the derived spectrum of self-maps of A over the enveloping algebra $A \wedge_R^L A^o$ and can hence be viewed as a derived center of A over R. Angeltveit showed that this derived center depends on the chosen A_∞-structure, for instance different A_∞-structure of Morava K-theory, K_n, over Morava E-theory, E_n, give different $THH_{E_n}(K_n)$ [2].

Let A be a commutative R-algebra spectrum. Rognes defined in [20] when A is unramified over R and showed that in this case the canonical map $A \to THH^R(A)$ is a weak equivalence. We use this to show that the complexification map $ko \to ku$ is wildly ramified [11, Theorem 5.2]: $THH_*^{ko}(ku)$ is not equivalent to ku_* and it behaves like Hochschild homology of the Gaussian integers.

In the discrete case Weibel and Geller showed [22] that for an étale extension of commutative rings $R \to A$ Hochschild homology satisfies étale descent, $\mathsf{HH}_*(A) \cong A \otimes_R \mathsf{HH}_*(R)$, and if $R \to A$ is G-Galois for a finite group G this implies $\mathsf{HH}_*(A)^G \cong \mathsf{HH}_*(R)$. Both properties do not carry over to ring spectra: Akhil Mathew shows [16] that there is a C_p-Galois extension of commutative ring spectra for which étale descent fails for THH. In joint work with Ausoni we show that for the $H\mathbb{Q}$-dual of the Hopf map $\eta^*\colon F(S^2_+, H\mathbb{Q}) \to F(S^3_+, H\mathbb{Q})$ the S^1 homotopy fixed points of $THH(F(S^3_+, H\mathbb{Q}))$ are *not* homotopy equivalent to $THH(F(S^2_+, H\mathbb{Q}))$ although η^* is an S^1-Galois extension.

The category of commutative ring spectra is tensored over (pointed) simplicial sets. For a commutative ring spectrum A the standard simplicial model of $\mathsf{THH}(A)$ can be directly identified with $A \otimes S^1$ where $S^1 = \Delta^1/\partial\Delta^1$ is the standard simplicial model of the 1-sphere.

For any pointed simplicial set X we call $\pi_*(A \otimes X)$ the *X-homology of A*. In the discrete case this was defined by Pirashvili [18], but mentioned earlier for spheres for instance by Anderson [1] in the context of iterated Eilenberg-Moore spectral sequences. Basterra-McCarthy showed that topological André-Quillen homology can be viewed as the stabilization of the $A \otimes S^n$'s [4].

Higher topological Hochschild homology of order n of A is S^n-homology of A and denoted by $\mathsf{THH}^{[n]}(A)$. another important special case is torus homology [8]: if one considers n-fold iterated algebraic K-theory of A, $K^n(A)$, then the iteration of the trace map has $A \otimes (S^1)^n$ as the target.

We know $THH^{[n]}$ in some cases for all $n \geq 1$. For instance we show in [10, 3.6] that

$$\mathsf{THH}^{[n]}_*(H\mathbb{F}_p) \cong \mathsf{Tor}_{*,*}^{\mathsf{THH}^{[n-1]}_*(H\mathbb{F}_p))}(\mathbb{F}_p, \mathbb{F}_p), \quad n \geq 2.$$

These Tor-algebras were determined by Cartan [9] and can be explicitly written down as graded commutative $\mathbb{F}_p$-algebras. This result was also known to Basterra and Mandell. We also show in [11] that for all primes

$$\mathsf{THH}^{[2]}_*(H\mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}[x_1, x_2, \ldots]/p^n x_n = 0, x_n^p = px_{n+1}, \quad |x_1| = 2p.$$

Schlichtkrull gives a general identification for X-homology of commutative Thom spectra [21].

Ongoing work by Ausoni and Dundas makes progress on Rognes' red-shift conjecture using torus homology. They show that the generator v_{n-1} of connective Morava K-theory is not in the kernel of the unit map

$$k(n-1)_* \to k(n-1)_* K^n(H\mathbb{F}_p).$$

They prove this by showing that v_{n-1} is detected in $k(n-1)_*(H\mathbb{F}_p \otimes (S^1)^n)^{h(S^1)^n}$. It turns out that $\pi_*(H\mathbb{F}_p \otimes (S^1)^n)$ can be described by higher THH of $H\mathbb{F}_p$ because in this case torus homology does not see the attaching maps in the CW structure of the torus. In order to prove the red-shift conjecture for $\mathbb{F}_p$ they have to show that all powers of v_{n-1} also survive.

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