Higher topological Hochschild homology of rings of integers in number fields

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For any strictly commutative ring spectrum A and for any simplicial set X one can define the simplicial commutative ring spectrum $A \otimes X$ as $(A \otimes X)_n = \bigwedge_{x \in X_n} A$. Similarly, if M is an A-module spectrum and X is a pointed simplicial set, we define $(M, A) \otimes X$ by placing M at the basepoint of X and A at all other simplices of X. Important examples of this construction are

- *topological Hochschild homology of A with coefficients in M*, $THH(A, M)$, given by $(M, A) \otimes \mathbb{S}^1$,
- *higher topological Hochschild homology of order n of A with coefficients in M*,
$$THH^{[n]}(A, M) = (M, A) \otimes \mathbb{S}^n, n \geq 1, \text{ and}$$
- *torus homology* where we tensor with $(\mathbb{S}^1)^n = \mathbb{T}^n$.

Ordinary topological Hochschild homology is the target of a trace map from algebraic K-theory. This trace map factors via topological cyclic homology, TC, and the latter is often a very good approximation of algebraic K-theory.

If we consider iterated algebraic K-theory, then we can use an iteration of the trace map and obtain torus homology as the natural target of such a trace map. Using the standard cell structure of an n-dimensional torus gives us a method of calculating torus homology from higher topological Hochschild homology.

An important class of examples are rings of integers in number fields. As a starting point we consider higher THH of the integers with coefficients in the residue field $\mathbb{F}_p$, $THH^{[n]}(\mathbb{Z}, \mathbb{F}_p)$. Bökstedt [2] calculated

$$THH_*(\mathbb{Z}, \mathbb{F}_p) \cong \mathbb{F}_p[x_{2p}] \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(x_{2p-1}).$$

There is a well-known description of iterated Tor-algebras due to Cartan [4]: If we start with a polynomial algebra over $\mathbb{F}_p$ generated by an element w of even degree, then we call this algebra $B^1_{\mathbb{F}_p}(w)$. Iteratively, we define

$$B^{n+1}_{\mathbb{F}_p}(w) := \mathrm{Tor}^{B^n_{\mathbb{F}_p}(w)}(\mathbb{F}_p, \mathbb{F}_p)$$

for all n. The case $n = 2$ immediately gives

$$B^2_{\mathbb{F}_p}(w) = \mathrm{Tor}^{B^1_{\mathbb{F}_p}(w)}(\mathbb{F}_p, \mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}(\varepsilon w)$$

where the degree of εw is one higher than the degree of w.

$$B^3_{\mathbb{F}_p}(w) = \mathrm{Tor}^{B^2_{\mathbb{F}_p}(w)}(\mathbb{F}_p, \mathbb{F}_p) \cong \Gamma_{\mathbb{F}_p}(\varrho^0 \varepsilon w)$$

where the latter denotes a divided power algebra. As the base field is of characteristic p this algebra splits into a tensor product of truncated polynomial algebras

$$\Gamma_{\mathbb{F}_p}(\varrho^0 \varepsilon w) \cong \bigotimes_k \mathbb{F}_p[\varrho^k \varepsilon w]/(\varrho^k \varepsilon w)^p;$$

here $\varrho^k \varepsilon w$ corresponds to the p^kth divided power of $\varrho^0 \varepsilon w$. For each of the tensor factors we obtain again a periodic resolution and we get

$$B^4_{\mathbb{F}_p}(w) = \mathrm{Tor}^{B^3_{\mathbb{F}_p}(w)}(\mathbb{F}_p, \mathbb{F}_p) \cong \bigotimes_k \Gamma_{\mathbb{F}_p}(\varphi^0 \varrho^k \varepsilon w) \otimes \Lambda_{\mathbb{F}_p}(\varepsilon \varrho^k \varepsilon w).$$

From here on the iteration process yields terms of a form that already occured before.

Theorem 1 [Dundas-Lindenstrauss-R]
For all $n \geq 1$ and for all primes p:

$$THH^{[n]}_*(\mathbb{Z}, \mathbb{F}_p) \cong B^n_{\mathbb{F}_p}(x_{2p}) \otimes_{\mathbb{F}_p} B^{n+1}_{\mathbb{F}_p}(y_{2p-2}).$$

A crucial ingredient in the proof is the following

Lemma
Let C be a commutative augmented $H\mathbb{F}_p$-algebra and assume that there is an isomorphism of graded commutative $\mathbb{F}_p$-algebras $\pi_* C \cong \Lambda_{\mathbb{F}_p}(x)$ where $|x| = m > 0$. Then there is a zigzag of equivalences of commutative augmented $H\mathbb{F}_p$-algebras between C and $H\mathbb{F}_p \vee \Sigma^m H\mathbb{F}_p$.

This Lemma was suggested by Mike Mandell. Our proof uses a Postnikov argument in the world of commutative $H\mathbb{F}_p$-algebras. With the help of this result we can split off the bottom Postnikov piece of $THH(\mathbb{Z}, \mathbb{F}_p)$ and obtain an iterated homotopy pushout diagram for $THH^{[2]}(\mathbb{Z}, \mathbb{F}_p)$:

$$\begin{array}{ccccc}
THH(\mathbb{Z}, \mathbb{F}_p) & \longrightarrow & H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p & \longrightarrow & H\mathbb{F}_p \\
\downarrow & & \downarrow f & & \downarrow \\
H\mathbb{F}_p & \longrightarrow & E & \longrightarrow & THH^{[2]}(\mathbb{Z}, \mathbb{F}_p)
\end{array}$$

A Tor-spectral sequence calculations yields that E has the exterior algebra $\Lambda_{\mathbb{F}_p}(z_{2p+1})$ as $\pi_*(E)$ and hence we know that $E \sim H\mathbb{F}_p \vee \Sigma^{2p+1} H\mathbb{F}_p$. The map f factors via the augementation and unit and this yields with another Tor-spectral sequence calculation that

$$\pi_* THH^{[2]}(\mathbb{Z}, \mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}(z_{2p+1}) \otimes_{\mathbb{F}_p} \Gamma_{\mathbb{F}_p}(a_{2p}).$$

For the iteration of this argument we use that we can express higher THH via an iterated bar construction. For instance $THH^{[3]}(\mathbb{Z}, \mathbb{F}_p)$ is equivalent to the diagonal of the bisimplicial commutative augmented $H\mathbb{F}_p$-algebra $B(H\mathbb{F}_p, B(H\mathbb{F}_p, H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p, E), H\mathbb{F}_p)$ and as the module structure of E over $H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p$ reduces to the $H\mathbb{F}_p$-module structure this bar construction splits as a bisimplicial commutative augmented $H\mathbb{F}_p$-algebra into

$$B(H\mathbb{F}_p, B(H\mathbb{F}_p, H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p, H\mathbb{F}_p), H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} B(H\mathbb{F}_p, \underline{E}, H\mathbb{F}_p)$$

where $\underline{E}$ denotes the constant simplicial commutative augmented $H\mathbb{F}_p$-algebra on E. For higher n there is a similar splitting and we get that $THH^{[n+1]}(\mathbb{Z}, \mathbb{F}_p)$ is equivalent to the diagonal of an n-fold reduced iterated bar construction on $H\mathbb{F}_p \vee \Sigma^{2p-1}H\mathbb{F}_p$ smashed with an $(n-1)$-fold iterated bar construction on $\underline{E}$. The square zero extensions E and $H\mathbb{F}_p \vee \Sigma^{2p-1}H\mathbb{F}_p$ can be modelled as the Eilenberg Mac Lane spectra on a simplicial commutative algebra and this allows for a comparison of the above iterated bar constructions with iterated algebraic bar constructions on exterior algebras. The homology groups of such bar constructions were determined in [1] and this gives the proof of Theorem 1.

Let $\mathcal{O}$ denote the ring of integers in a number field and let P be a non-trivial prime ideal in $\mathcal{O}$ with residue field $\mathcal{O}/P = \mathbb{F}_q$ where $q = p^\ell$ for some prime p. Higher *THH* detects ramification:

Theorem 2 [Dundas-Lindenstrauss-R]
For all $n \geq 1$:

$$THH_*^{[n]}(\mathcal{O}_P^\wedge, \mathcal{O}/P) \cong B^n_{\mathbb{F}_q}(x) \otimes_{\mathbb{F}_q} B^{n+1}_{\mathbb{F}_q}(y)$$

where

(i) $|x| = 2$ and $|y| = 0$ if A is ramified over $\mathbb{Z}$ at P, and
(ii) $|x| = 2p$ and $|y| = 2p - 2$, if A is unramified over $\mathbb{Z}$ at P.

Lindenstrauss and Madsen determined the topological Hochschild homology groups of rings of integers in [6].

In the unramified case we show that we have an isomorphism

$$THH_*^{[n]}(\mathcal{O}_P^\wedge, \mathcal{O}/P) \cong THH_*^{[n]}(\mathbb{Z}_p^\wedge, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{F}_q.$$

This uses the Lindenstrauss-Madsen result and an iterative spectral sequence argument. In the ramified case the important input is that the first Hochschild homology group (and therefore also the first *THH*-group) is isomorphic to $\mathbb{F}_q$. This fact ensures that the differentials in the Brun spectral sequence [3, p. 30]

$$THH_*(\mathcal{O}_P^\wedge/P, \mathrm{Tor}_{*,*}^{\mathcal{O}_P^\wedge}(\mathcal{O}_P^\wedge/P, \mathcal{O}_P^\wedge/P)) \Rightarrow THH_*(\mathcal{O}_P^\wedge, \mathcal{O}_P^\wedge/P)$$

have to vanish and we obtain that

$$THH_*(\mathcal{O}_P^\wedge, \mathcal{O}/P) \cong \mathbb{F}_q[u] \otimes_{\mathbb{F}_q} \Lambda_{\mathbb{F}_q}(\tau)$$

with $|u| = 2$ and $|\tau| = 1$. From this point on the argument is the same as in the case of the rational integers.

References

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