Abstracts

Stability of Loday constructions

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(joint work with Ayelet Lindenstrauss)

For a commutative ring spectrum R and a commutative R-algebra spectrum A the Loday construction $\mathcal{L}^R_X(A)$ for a finite simplicial set X generalizes the concept of topological Hochschild homology of A which corresponds to the case where X is the circle, S^1, and R is the sphere spectrum S. Of particular interest is the case $X = T^n = (S^1)^n$, an n-torus, as $\mathcal{L}^S_{T^n}(A)$ is the target of an iterated trace map from the n-fold iterated algebraic K-theory of A. The homotopy groups of $\mathcal{L}^S_{T^n}(A)$ are in general difficult to calculate.

If we assume that R and A are cofibrant, then the homotopy type of $\mathcal{L}^R_X(A)$ only depends on the homotopy type of X. In several classes of examples it actually only depends on the homotopy type of ΣX. In this case one says that $R \to A$ is stable. For such $R \to A$ one can for instance determine the homotopy type of $\mathcal{L}^R_{T^n}(A)$ in terms of $\mathcal{L}^R_{S^k}(A)$ for $1 \leq k \leq n$ and the homotopy groups of the latter are known in many examples, such as when R is the sphere spectrum and A is the Eilenberg-MacLane spectrum $H\mathbb{F}_p$ for any prime p [2].

In the talk we present several different notions of stability together with their structural properties and we discuss examples an non-examples of stability.

A strong notion of stability is the following: Let $R \to A$ be a cofibration of commutative S-algebras with R cofibrant. We call $R \to A$ *multiplicatively stable* if for every pair of pointed simplicial sets X and Y an equivalence $\Sigma X \simeq \Sigma Y$ implies that $\mathcal{L}^R_X(A) \simeq \mathcal{L}^R_Y(A)$ as augmented commutative A-algebras. There are also linear variants of stability.

An easy stability result says that for any augmented commutative R-algebra spectrum A, $A \to R$ and $R \to \mathcal{L}^R_{\Sigma X}(A; R) \to R$ are multiplicatively stable.

Dundas and Tenti show [3] that the 2-torus is a witness for the fact that $H\mathbb{Q}[t]/t^2$ is *not* stable and in [4] we show that $H\mathbb{Q} \to \mathbb{Q}[t]/t^m$ is not multiplicatively stable for all $m \geq 2$ by using the m-torus as a witness.

In [4] we also show that for any commutative Hopf algebra spectrum $\mathcal{H}$ and every equivalence $\Sigma(X_+) \simeq \Sigma(Y_+)$ in the infinity category of pointed spaces $\mathcal{S}_*$, there is an equivalence $\mathcal{L}_X(\mathcal{H}) \simeq \mathcal{L}_Y(\mathcal{H})$. This generalizes a result by Berest, Ramadoss, Yeung for commutative Hopf algebras over a field [1].

Other concrete examples are that $HR \to HR/(a_1, \ldots, a_n)$ is multiplicatively stable if R is a commutative ring and $(a_1, \ldots, a_n)$ is a regular sequence and if $R \to A$ is a cofibration of commutative S-algebras with R cofibrant, then $A \to \mathcal{L}^R_{\Sigma X}(A)$ is multiplicatively stable for all $X \in sSets_*$ [5].

We show [5] that stability satisfies certain inheritance properties: If $f \colon A \to B$ is multiplicatively stable, then so is $C \wedge_R f \colon C \wedge_R A \to C \wedge_R B$. Multiplicative stability is closed under pushouts: If $R \to B$ and $R \to C$ are multiplicatively stable, then so is $R \to B \wedge_R C$.

Multiplicative stability is also closed under forming Loday constructions: If $R \to A$ is multiplicatively stable, then so is $R \to \mathcal{L}_Z^R(A)$ for any Z. If $S \to A$ and $S \to B$ are cofibrations of commutative S-algebras and if A and B are multiplicatively stable, then for connected X and Y with $\Sigma X \simeq \Sigma Y$, there is an equivalence

$$\mathcal{L}_X^S(A \times B) \simeq \mathcal{L}_Y^S(A \times B)$$

of commutative S-algebras.

Beware, however, that stability is not transitive: If $R \to A$ and $A \to B$ satisfy stability then this does *not* imply that $R \to B$ is stable. A concrete example is $\mathbb{Q} \to \mathbb{Q}[t]$ and $\mathbb{Q}[t] \to \mathbb{Q}[t]/t^m$.

Dundas and Tenti [3] show that for $k \to A$ smooth, the map $Hk \to HA$ is stable. We develop an adequate generalization of this phenomenon for ring spectra [5]. We show that for every simplicial set X there is a weak equivalence of commutative R-algebras

$$\mathcal{L}_X^R(\mathbb{P}_R(M)) \simeq \mathbb{P}_R(X_+ \wedge M),$$

in particular, if $\Sigma X \simeq \Sigma Y$, then $\mathcal{L}_X^R(\mathbb{P}_R(M)) \simeq \mathcal{L}_Y^R(\mathbb{P}_R(M))$ as commutative R-algebra spectra. Here, $\mathbb{P}_R(M)$ is the free commutative R-algebra spectrum generated by an R-module spectrum M.

For ring spectra there are several non-equivalent notions of étale maps. Let $R \to A \to B$ be a sequence of cofibrations of commutative S-algebras with R cofibrant. Then this sequence *satisfies étale descent* if for all connected X the canonical map

$$\mathcal{L}_X^R(A) \wedge_A B \to \mathcal{L}_X^R(B)$$

is an equivalence.

We call a map of cofibrant S-algebras $\varphi\colon R \to A$ *really smooth* if it can be factored as $R \xrightarrow{i_R} \mathbb{P}_R(M) \xrightarrow{f} A$ where i_R is the canonical inclusion, M is an R-module, and $R \xrightarrow{i_R} \mathbb{P}_R(M) \xrightarrow{f} A$ satisfies étale descent.

We establish, for instance, that periodic complex topological K-theory, KU, is stable and we deduce with the Galois descent property of $KO \to KU$ that periodic real topological K-theory, KO, is also stable.

References

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