Gluing algebras to points

Birgit Richter

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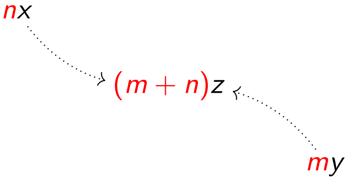

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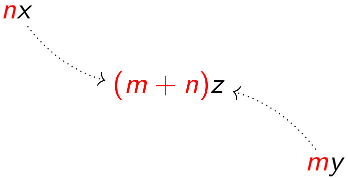

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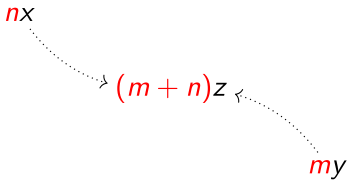

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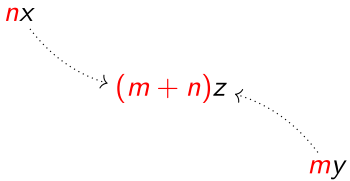

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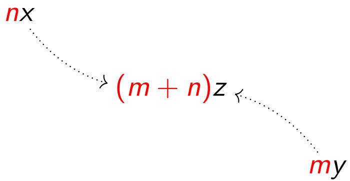

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Dold-Thom: $\pi_i(SP(X), [x_0]) \cong H_i(X; \mathbb{Z})$ for $i > 0$, if X is a connected CW complex.

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so m codifies a commutative multiplication. Note that m is also associative.

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Here, $b = \sum_{i=0}^{n}(-1)^i d_i$ where
$d_i(a_0 \otimes \ldots \otimes a_n) = a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots a_n$ for $i < n$ and
$d_n(a_0 \otimes \ldots \otimes a_n) = a_n a_0 \otimes \ldots \otimes a_{n-1}$.

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$$d_i(j) = \begin{cases} j, & j < i \\ i, & j = i < n, \quad (0, \quad j = i = n), \\ j-1, & j > i. \end{cases}$$

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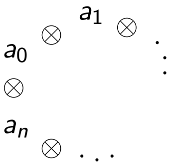
The circle had a cyclic ordering of the points, so A could be taken to be associative:



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If $f\colon [m] \to [n] \in \Delta$, then the induced map $f^*\colon \mathcal{L}_X(R)_n \to \mathcal{L}_X(R)_m$ is given by $f^*(\bigotimes_{x \in X_n} r_x) = \bigotimes_{y \in X_m} b_y$ with $b_y = \prod_{f(x)=y} r_x$ where the product over the empty set is defined to be $1 \in R$.

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The definition goes back to Pirashvili, 2000.

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  So one can view torus homology as iterated (topological) Hochschild homology.

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So iterating K-theory produces homotopically interesting objects.

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Theorem [Dundas-Lindenstrauss-R 2018; Mandell]
For all $n \geq 2$:

$$\pi_* \mathcal{L}_{S^n}(\mathbb{F}_p) \cong \mathrm{Tor}_{*,*}^{\pi_* \mathcal{L}_{S^{n-1}}(\mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p)$$

as a graded commutative algebra (with total grading).

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Calculating the homotopy groups of $\mathcal{L}_{S^1 \times S^1}(R)$ is difficult... But $\pi_* \mathcal{L}_{S^n}(R)$ is known for all n in many important cases.
Example: $R = H\mathbb{F}_p$. Bökstedt:

$$\pi_*(\mathsf{THH}(H\mathbb{F}_p)) \cong \mathbb{F}_p[\mu], \quad |\mu| = 2.$$

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Lindenstrauss-R, 2022: Thom spectra associated to Ω^∞-maps are stable, (real and complex) topological K-theory is stable and $HR \to HR/(a_1, \dots, a_n)$ is stable if R is a commutative ring and the sequence $(a_1, \dots, a_n)$ is regular, ...

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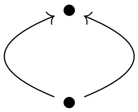

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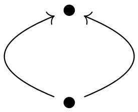

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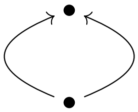

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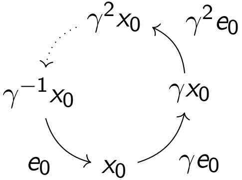
Some important known equivariant homology theories can be identified as equivariant Loday constructions.

[LRZ]: For $G = C_n =< \gamma >$ and $X = S^1_{rot}$



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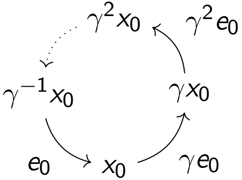

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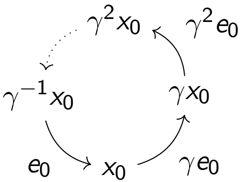

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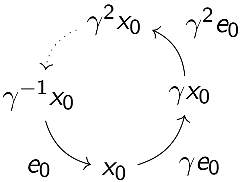

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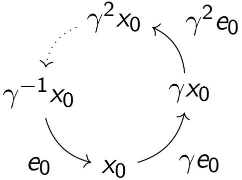

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Both these objects receive trace maps from equivariant versions of algebraic K-theory.