Exercises in Algebra (master): Homological Algebra

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Exercise sheet no 7 for the exercise class on the 26th of May 2021

1 (Equivalence of extensions)
Let p be an odd prime and consider the extensions

$$0 \longrightarrow \mathbb{Z}/p \xrightarrow{p\cdot} \mathbb{Z}/p^2 \xrightarrow{\pi} \mathbb{Z}/p \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{Z}/p \xrightarrow{2p\cdot} \mathbb{Z}/p^2 \xrightarrow{\pi} \mathbb{Z}/p \longrightarrow 0.$$

Prove that despite the fact that the middle group agrees, these two extensions are *not* equivalent.

2 (Some right derived things)

(1) Let R be an arbitrary ring $\neq 0$. Show that $\mathsf{Ext}^1_R(P, M) = 0$ for all R-modules M is equivalent to P being a projective R-module. Dually, $\mathsf{Ext}^1_R(N, I) = 0$ for all R-modules N is equivalent to I being an injective R-module.

(2) What are the Ext-groups $\mathsf{Ext}^*_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}/m)$ for natural numbers n and m? Relate your answer to the first exercise above.

(3) What is $\mathsf{Ext}^*_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}/p)$ for p a prime?

(4) Let A be a torsion abelian group. Show that $\mathsf{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \cong \mathsf{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$.

(5) Is $\mathsf{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$?

(6) Define the functor $T\colon \mathsf{Ab} \to \mathsf{Ab}$ by $T(A) = \ker(A \mapsto A \otimes \mathbb{Q})$, so $T(A)$ is the torsion subgroup of A. Show that T is a left exact additive functor and calculate its right derived functors.

3 (Free groups) Let F_n be a free group on n generators with $n \geqslant 2$. Show that there is a free resolution of the trivial $\mathbb{Z}[F_n]$-module $\mathbb{Z}$ of length one, *i.e.*, a short exact sequence

$$0 \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

such that P_1 and P_0 are free $\mathbb{Z}[F_n]$-modules.