

Exercises in Algebra (master): Homological Algebra

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Summer term 2021

Exercise sheet no 6

for the exercise class on the 19th of May 2021

1 (Tor-calculations)

Let A and B be finitely generated abelian groups. Describe $\text{Tor}_n^{\mathbb{Z}}(A, B)$ for all $n \geq 0$.

2 (5-Lemma) Consider the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
 A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & A_5 \\
 \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\
 B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 & \xrightarrow{\beta_4} & B_5.
 \end{array}$$

Under which assumptions on f_1, f_2, f_4, f_5 can we deduce that the map f_3 is a monomorphism or an epimorphism? In particular, what assumptions are needed in order to ensure that f_3 is an isomorphism?

3 (Snake Lemma) <https://www.youtube.com/watch?v=etbcKWEKnvg>

(1) Use the long exact sequence in homology to prove the weak form of the Snake Lemma: If

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \longrightarrow & 0 \\
 & & \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \\
 0 & \longrightarrow & N' & \xrightarrow{h} & N & \xrightarrow{k} & N'' & \longrightarrow & 0
 \end{array}$$

is a commutative diagram with exact rows, then there is an exact sequence

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \ker(\alpha) & \longrightarrow & \ker(\beta) & \longrightarrow & \ker(\gamma) \\
 & & & & & & \searrow \\
 & & & & & & \longrightarrow \text{coker}(\alpha) & \longrightarrow & \text{coker}(\beta) & \longrightarrow & \text{coker}(\gamma) & \longrightarrow & 0.
 \end{array}$$

Specify the maps.

(2) Prove the stronger form: If

$$\begin{array}{ccccccccc}
 & & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \longrightarrow & 0 \\
 & & \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \\
 0 & \longrightarrow & N' & \xrightarrow{h} & N & \xrightarrow{k} & N'' & &
 \end{array}$$

is a commutative diagram with exact rows, then there is an exact sequence

$$\begin{array}{ccccccc}
 \ker(\alpha) & \longrightarrow & \ker(\beta) & \longrightarrow & \ker(\gamma) \\
 & & & & \searrow \\
 & & & & \longrightarrow \text{coker}(\alpha) & \longrightarrow & \text{coker}(\beta) & \longrightarrow & \text{coker}(\gamma).
 \end{array}$$