Exercises in Algebra (master): Homological Algebra

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Exercise sheet no 11 for the exercise class on the 23rd of June 2021

1 (Even more structure on $\mathsf{HH}_k^*(A)$) Consider two elements $f \in C_k^m(A)$ and $g \in C_k^n(A)$ where A is a k-algebra and $C_k^*(A)$ is the Hochschild cochain complex of A over k with coefficients in A. Define the operation of inserting g into the ith spot of f:

$$(f \circ_i g)(a_1 \otimes \ldots \otimes a_{m+n-1}) := f(a_1 \otimes \ldots \otimes a_{i-1} \otimes g(a_i \otimes \ldots \otimes a_{i+n-1}) \otimes a_{n+i} \otimes \ldots \otimes a_{m+n-1}).$$

We consider

$$f \circ g = \sum_{i=1}^{m} (-1)^{(i-1)(n-1)} f \circ_i g$$

and the bracket given by the antisymmetrization of the $\circ$-product:

$$[f, g] := f \circ g - (-1)^{(m-1)(n-1)} g \circ f.$$

(1) Show that the coboundary in the Hochschild cochain complex $\delta(f)$ agrees with $-[f, \mu]$ where $\mu\colon A \otimes_k A \to A$ is the multiplication map of A.

(2) You may use the fact, that the bracket induces a well-defined map on Hochschild cohomology (see Murray Gerstenhaber, The cohomology structure of an associative ring, Ann. of Math. (2) 78, 1963, 267–288 `https://doi.org/10.2307/1970343` for a proof):

$$[-,-]\colon \mathsf{HH}_k^m(A) \otimes \mathsf{HH}_k^n(A) \to \mathsf{HH}_k^{m+n-1}(A).$$

Make the bracket explicit on $\mathsf{HH}_k^1(A)$.

(3) The $\circ$-product interacts nicely with the cup-product. Prove that

$$(f \cup g) \circ h = (f \circ h) \cup g + (-1)^{m(p-1)} f \cup (g \circ h)$$

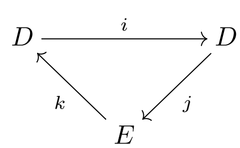
for f, g as above and $h \in C^p(A)$.

On $\mathsf{HH}_k^*(A)$, the bracket $[-,-]$ defines a graded Lie-algebra structure that satisfies

$$[f \cup g, h] = [f, h] \cup g + (-1)^{p(m-1)} f \cup [g, h].$$

Such a structure is called a *Gerstenhaber algebra*.

2 (Exact couples) Let D, E be two R-modules for some ring $0 \neq R$. Assume that we have R-linear maps $i\colon D \to D$, $j\colon D \to E$ and $k\colon E \to D$ with $\mathrm{im}(i) = \ker(j)$, $\mathrm{im}(j) = \ker(k)$ and $\mathrm{im}(k) = \ker(i)$. This is usually depicted as



Then (D, E, i, j, k) is an *exact couple*. Define $d = j \circ k$.

(1) Show that $d^2 = 0$.

(2) Define $D' = \mathrm{im}(i) = iD \subset D$ and let E' be the homology of E with respect to d.
Set $i'(i(x)) = i(i(x))$ for $x \in D$, $j'(i(x)) = [j(x)]$, where $[j(x)]$ denotes the homology class of $j(x)$. Finally, let $k'[y]$ be $k(y)$. Prove that the maps are well-defined and that (D', E', i', j', k') is again an exact couple.