

# Real Hochschild homology as an equivariant Loday construction

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joint work with Ayelet Lindenstrauss and Foling Zou

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$THR(A)$  can be identified with  $N_{D_2}^{O(2)}(A)$  and  $N_{D_2}^{O(2)}(A)$  can be modelled by a dihedral bar construction (Angelini-Knoll, Gerhardt, Hill).

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under some harsh assumptions on  $E$  and  $A$ .

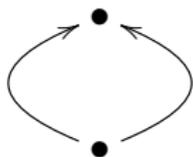
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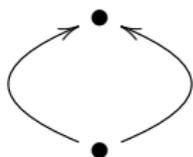
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Goal for today: Find geometric descriptions of Real Hochschild homology (and of  $i_{D_{2^m}}^{O(2)}(\mathrm{THR}(A))$ ) for all  $m$ .

Consider the flip-circle:  $S^\sigma$ :

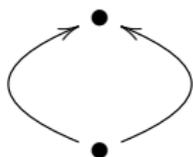


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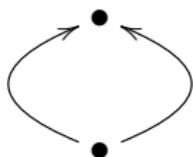


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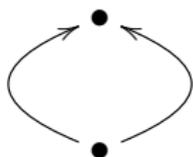
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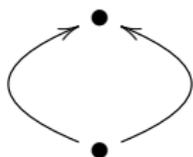
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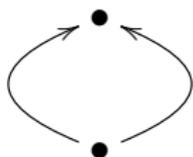
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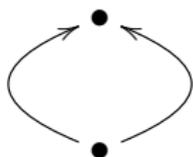
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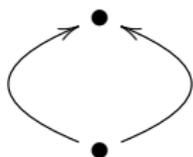
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Kristen Mazur 2013 ( $G = C_{p^n}$ ), Rolf Hoyer 2014 (general  $G$ ) showed:  $G$ -commutative monoids are  $G$ -Tambara functors.

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**Example** The Burnside  $G$ -Tambara functor,  $\underline{A} = \underline{A}^G$ , sends a finite  $G$ -set  $X$  to the group completion of the abelian monoid of iso classes of finite  $G$ -sets over  $X$ .

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The Weyl group  $W_G(H)$  acts on  $\underline{R}^{fix}(G/H) = R^H$  by  $[\gamma]r = \gamma r$ . These structure maps satisfy several compatibility relations...

**Example** The Burnside  $G$ -Tambara functor,  $\underline{A} = \underline{A}^G$ , sends a finite  $G$ -set  $X$  to the group completion of the abelian monoid of iso classes of finite  $G$ -sets over  $X$ .

$\underline{A}$  is initial in  $\text{Tamb}_G$  and a unit for the so-called box product of  $G$ -Mackey functors,  $\square$ .

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The pair  $(N_H^G, i_H^G)$  is an adjoint functor pair.

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The [Real  \$D\_{2m}\$ -Hochschild homology of  \$\underline{R}\$](#)  is the graded  $D_{2m}$ -Mackey functor

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For our Loday construction we would a priori need a  $D_{2m}$ -Tambara functor.

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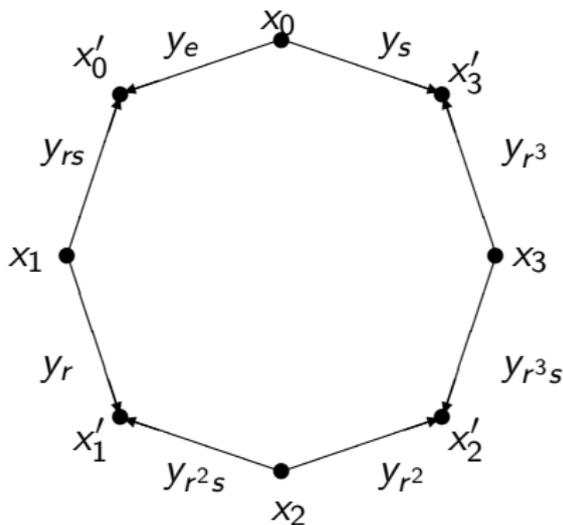
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Idea of proof: Set  $(\varphi^* \underline{R})(H/K) := \underline{R}(\varphi H / \varphi K)$  and diligently show that this is alright. □

So it remains to show that  $\mathcal{L}_{P_{2m}}^{D_{2m}}(\underline{R})$  is well-defined even if  $\underline{R}$  is just a  $D_2$ -Tambara functor (even just a discrete  $E_\sigma$ -ring).

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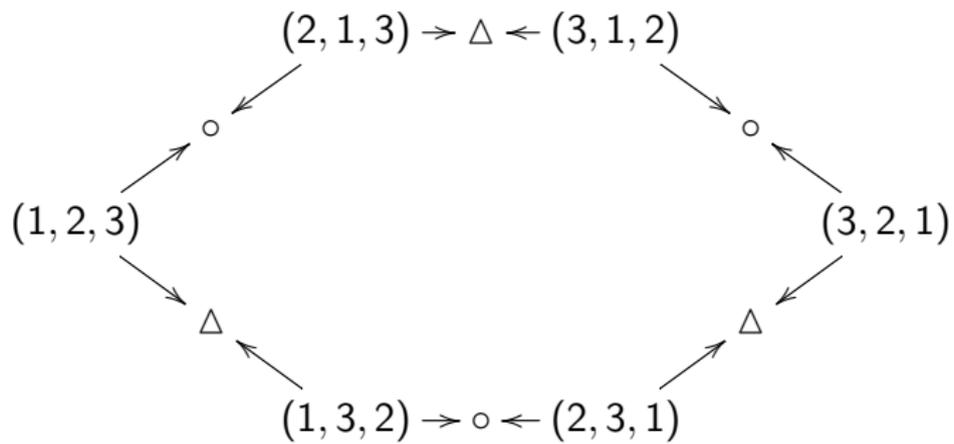
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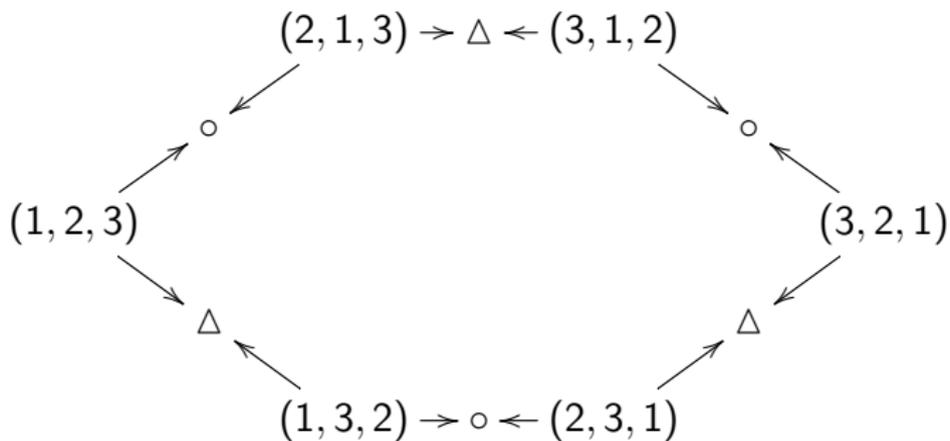
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We consider simplicial models of the 1-skeleta of the  $n$ th permutohedra and call them  $P_{\Sigma_n}$ .

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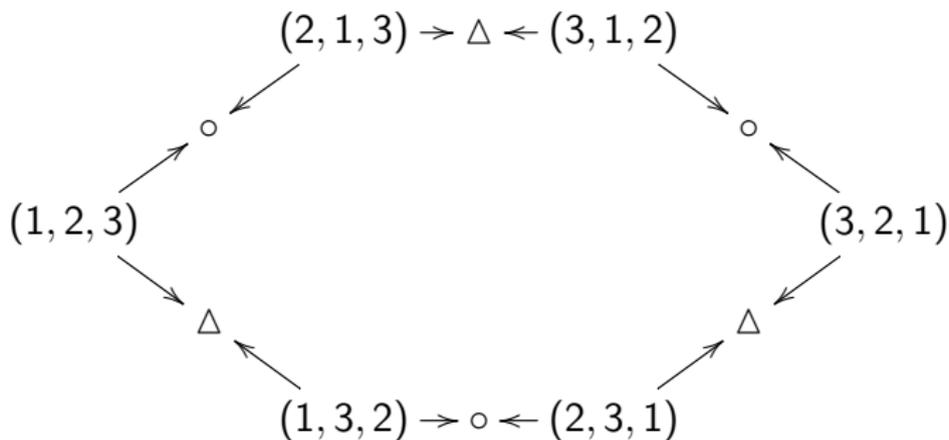


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If we choose as representatives for the orbits the points in the segment  $(1, 2.5, 2.5) = \Delta \leftarrow (1, 2, 3) \rightarrow \circ = (1.5, 1.5, 3)$ , then the vertices labelled with  $\circ$  give rise to a  $\Sigma_3/\langle(1, 2)\rangle$ -orbit and the  $\Delta$ -vertices assemble into a  $\Sigma_3/\langle(2, 3)\rangle$ -orbit.

We can define  $\mathcal{L}_{P_{\Sigma_3}}^{\Sigma_3}(\underline{R})$  for every  $\langle(1, 2)\rangle$ -Tambara functor  $\underline{R}$ .

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With  $C_2 := \langle(1, 2)\rangle$  this gives  $\mathcal{L}_{P_{\Sigma_3}^{\Sigma_3}}(\underline{R})$  as a pushout of

$$\begin{array}{ccc}
 N_e^{\Sigma_3} i_e^{C_2} \underline{R} & \longrightarrow & B(N_e^{\Sigma_3} i_e^{C_2} \underline{R}, N_e^{\Sigma_3} i_e^{C_2} \underline{R}, N_{C_2}^{\Sigma_3} \underline{R}) \\
 \downarrow & & \\
 B(N_{\langle(2,3)\rangle}^{\Sigma_3} (c_{(1,3)})^* \underline{R}, N_e^{\Sigma_3} i_e^{C_2} \underline{R}, N_e^{\Sigma_3} i_e^{C_2} \underline{R}) & & 
 \end{array}$$