Loday constructions for Tambara functors

Birgit Richter
joint work with Ayelet Lindenstrauss and Foling Zou

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What are they?

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- for every pair of finite G-sets X and Y, applying M_* to $X \to X \sqcup Y \leftarrow Y$ gives the component maps of an isomorphism $\underline{M}(X) \oplus \underline{M}(Y) \cong \underline{M}(X \sqcup Y)$.

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The transfer T_π for $\pi\colon G/H \to G/K$ sends an $f \in G\text{-maps}(G/H, A)$ to $T_\pi(f)(gK) = \sum_{x \in \pi^{-1}(gK)} f(x)$.

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$\underline{A}$ is initial in Tamb_G and a unit for the so-called box product of G-Mackey functors, $\Box$.

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2. There is a natural isomorphism $X \otimes (Y \otimes \underline{R}) \cong (X \times Y) \otimes \underline{R}$.
3. On the category with objects finite sets with trivial G-action and morphisms consisting only of isomorphisms, the functor restricts to exponentiation $X \otimes \underline{R} = \Box_{x \in X} \underline{R}$.

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The pair (N_H^G, i_H^*) is an adjoint functor pair.

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The proof is by direct inspection, where we use the fact that $\underline{R}^c \Box \underline{R}^c \cong \underline{(R \otimes R)}^c$.

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$$\mathcal{L}_X^{C_p}(\underline{\mathbb{Z}}^c) \cong \begin{cases} \underline{\mathbb{Z}}^c, & \text{if } X^{C_p} \neq \varnothing, \\ \underline{A}, & \text{if } X^{C_p} = \varnothing. \end{cases}$$

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If $X^{C_p} = \varnothing$, then all orbits are free, so we just get $\underline{A}$ everywhere and $\underline{A} \Box \underline{A} \cong \underline{A}$.
If there is a fixed point somewhere, then we have one in every simplicial level. A fixed point corresponds to the orbit C_p/C_p, hence there we get $\underline{\mathbb{Z}}^c$.

The next result is a fun fact about fixed points: Let $G = C_p$ for p a prime and let X be any finite C_p-simplicial set.

Proposition[The hungry fixed points]

$$\mathcal{L}_X^{C_p}(\underline{\mathbb{Z}}^c) \cong \begin{cases} \underline{\mathbb{Z}}^c, & \text{if } X^{C_p} \neq \varnothing, \\ \underline{A}, & \text{if } X^{C_p} = \varnothing. \end{cases}$$

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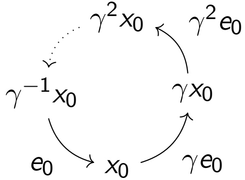

We have $(S^1_{\mathrm{rot}})_k = \{C_n \cdot x_k^0, C_n \cdot x_k^1, \cdots, C_n \cdot x_k^k\}$, where

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The simplicial identities imply that

$$\begin{aligned} d_j(x_k^0) =& x_{k-1}^0, \\ d_j(x_k^i) =& \begin{cases} x_{k-1}^{i-1} & 0 \leq j \leq i-1 \\ x_{k-1}^i & i \leq j \leq k \text{ and } i \neq k \end{cases} \\ d_k(x_k^k) =& \gamma^{-1} x_{k-1}^0. \end{aligned}$$

So for a C_n-Tambara functor $\underline{R}$ with $R := i_e^* \underline{R}$, there is

$$\mathcal{L}^{C_n}_{S^1_{\mathrm{rot}}}(\underline{R})_k = \square_{0 \le i \le k}(C_n \otimes \underline{R}) = (N_e^{C_n} R)^{\square(k+1)},$$

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and $d_i \colon (N_e^{C_n} R)^{\square(k+1)} \to (N_e^{C_n} R)^{\square k}$ is

$$d_i = \mathrm{id}^i \square \mu \square \mathrm{id}^{k-i} \qquad \text{for } 0 \le i < k$$

$$d_k = (\mu \square \mathrm{id}^{k-1}) \circ (\gamma^{-1} \square \mathrm{id}^k) \circ \tau$$

where $\mu \colon (N_e^{C_n} R)^{\square 2} \to N_e^{C_n} R$ is the multiplication and $\tau \colon (N_e^{C_n} R)^{\square(k+1)} \to (N_e^{C_n} R)^{\square(k+1)}$ moves the last coordinate to the front.

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where $\mu \colon (N_e^{C_n} R)^{\square 2} \to N_e^{C_n} R$ is the multiplication and $\tau \colon (N_e^{C_n} R)^{\square(k+1)} \to (N_e^{C_n} R)^{\square(k+1)}$ moves the last coordinate to the front. As $i_e^* \underline{R}$ is an e-Tambara functor, it can be identified with its value on e/e and that is $\underline{R}(C_n/e)$.

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Theorem The C_n-equivariant Loday construction for S^1_{rot} is

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For every subgroup $K < C_n$ we can identify the twisted cyclic nerve relative to K as

$$\underline{\mathrm{HC}}^{C_n}_K(i_K^* \underline{R}) =: \underline{\mathrm{HC}}^{C_n}(N_K^{C_n} i_K^* \underline{R}) \cong \mathcal{L}^{C_n}_{S^1_{\mathrm{rot}}/K}(\underline{R}).$$

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In particular, for $K = C_n$:

$$\mathcal{L}^{C_n}_{S^1_{\mathrm{rot}}/C_n}(\underline{R}) \cong \underline{\mathrm{HC}}^{C_n}_{C_n}(\underline{R}) = \underline{\mathrm{HC}}^{C_n}(\underline{R}).$$

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Theorem For A flat and well-pointed:

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We can identify μ_{2k+2} with the k-simplices of a reflection circle S^σ: