Structural Reflection and the HOD Conjecture

3. Lecture: Consequences of ultraexactness

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Introduction

In this lecture, we want derive strong consequences of the ultraexactness of a cardinal.

These results will rely on arguments showing that, given an elementary embedding $j : X \longrightarrow \mathrm{V}_\zeta$ witnessing the ultraexactness of a cardinal $\lambda < \zeta$, for certain ordinals $\lambda < \alpha \in X \cap \zeta$, we can find functions

$$f : \mathrm{L}_\alpha(\mathrm{V}_{\lambda+1}) \longrightarrow \mathrm{L}_{j(\alpha)}(\mathrm{V}_{\lambda+1})$$

in X with the property that

$$f \restriction (X \cap \mathrm{L}_\alpha(\mathrm{V}_{\lambda+1})) \;=\; j \restriction (X \cap \mathrm{L}_\alpha(\mathrm{V}_{\lambda+1})).$$

This will allow us to show that e.g., the successor of an ultraexacting cardinal is measurable in HOD.

In the following, fix

- a cardinal λ,
- an ordinal $\zeta > \lambda$ such that V_ζ is sufficiently elementary in V,
- an elementary submodel X of V_ζ with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$, and
- an elementary embedding $j : X \longrightarrow \mathrm{V}_\zeta$ with $j(\lambda) = \lambda$, $j \restriction \lambda \neq \mathrm{id}_\lambda$ and $j \restriction \mathrm{V}_\lambda \in X$.

Extensions to $V_{\lambda+1}$

Remember that, if ξ is an ordinal of countable cofinality and $k : \mathrm{V}_\xi \longrightarrow \mathrm{V}_\xi$ is an elementary embedding, then the map

$$k_+ : \mathrm{V}_{\xi+1} \longrightarrow \mathrm{V}_{\xi+1};\ A \longmapsto \bigcup\{k(A \cap V_\alpha) \mid \alpha < \xi\}$$

is the unique Σ_0-elementary function from $\mathrm{V}_{\xi+1}$ to $\mathrm{V}_{\xi+1}$ extending k.

If $i : \mathrm{V} \longrightarrow M$ is an I2-embedding with least non-trivial fixed point λ, then

$$(i \upharpoonright \mathrm{V}_\lambda)_+ = i \upharpoonright \mathrm{V}_{\lambda+1} : \mathrm{V}_{\lambda+1} \longrightarrow \mathrm{V}_{\lambda+1}$$

is Σ_1-elementary. Conversely, every non-trivial Σ_1-elementary function from $\mathrm{V}_{\lambda+1}$ to itself can be extended to an I2-embedding.

Lemma

The map

$$(j \upharpoonright \mathrm{V}_\lambda)_+ : \mathrm{V}_{\lambda+1} \longrightarrow \mathrm{V}_{\lambda+1}$$

is an elementary embedding that is contained in X and satisfies

$$(j \upharpoonright \mathrm{V}_\lambda)_+ \upharpoonright (X \cap \mathrm{V}_{\lambda+1}) \; = \; j \upharpoonright (X \cap \mathrm{V}_{\lambda+1}).$$

Proof.

Since $j \upharpoonright \mathrm{V}_\lambda \in X$, elementarity implies that $(j \upharpoonright \mathrm{V}_\lambda)_+ \in X$.

Moreover, the equality

$$(j \upharpoonright \mathrm{V}_\lambda)_+ \upharpoonright (X \cap \mathrm{V}_{\lambda+1}) \; = \; j \upharpoonright (X \cap \mathrm{V}_{\lambda+1}).$$

holds by the definition of $(j \upharpoonright \mathrm{V}_\lambda)_+$.

The elementarity of j and the above equality then imply that $(j \upharpoonright \mathrm{V}_\lambda)_+$ is an elementary embedding of $\mathrm{V}_{\lambda+1}$ into itself in X. Finally, the correctness properties of X ensure that $(j \upharpoonright \mathrm{V}_\lambda)_+$ also has this property in V. $\square$

Extensions below $\Theta^{\mathrm{L}(\mathrm{V}_{\lambda+1},E)}$

Lemma

Let γ be an ordinal in X such that there is a surjection $s : V_{\lambda+1} \longrightarrow \gamma$ in X with $j(s) \in X$. Then there exists a unique function $j_\gamma : \gamma \longrightarrow j(\gamma)$ that is an element of X and satisfies

$$j \restriction (X \cap \gamma) \;=\; j_\gamma \restriction (X \cap \gamma).$$

Proof.

Define $j_\gamma : \gamma \longrightarrow j(\gamma)$ to be the unique function satisfying

$$j_\gamma(s(x)) \;=\; j(s)((j \restriction \mathrm{V}_\lambda)_+(x))$$

for all $x \in \mathrm{V}_{\lambda+1}$. Then j_γ possesses all of the listed properties. □

Remember that, given $E \subseteq \mathrm{V}_{\lambda+1}$, we let $\Theta^{\mathrm{L}(\mathrm{V}_{\lambda+1},E)}$ denote the least ordinal γ such that $\mathrm{L}(\mathrm{V}_{\lambda+1},E)$ does not contain a surjection from $\mathrm{V}_{\lambda+1}$ onto γ.

The basic structure theory of $\mathrm{L}(\mathrm{V}_{\lambda+1},E)$ now yields the following statement:

Lemma

If $E \in X \cap V_{\lambda+2}$ with $j(E) = E$ and $\gamma \in X \cap \Theta^{\mathrm{L}(\mathrm{V}_{\lambda+1},E)}$ with $j(\gamma) \in X$, then there is a surjection $s : V_{\lambda+1} \longrightarrow \gamma$ in X with $j(s) \in X$.

Corollary

If $E \in X \cap V_{\lambda+2}$ with $j(E) = E$ and $\gamma \in X \cap \Theta^{\mathrm{L}(\mathrm{V}_{\lambda+1},E)}$ with $j(\gamma) = \gamma$, then there is a function $j_\gamma : \gamma \longrightarrow \gamma$ in X with

$$j \restriction (X \cap \gamma) \ = \ j_\gamma \restriction (X \cap \gamma).$$

An application: the consistency strength of ultraexactingness

Corollary

If $E \in X \cap V_{\lambda+2}$ with $j(E) = E$ and $\gamma \in X \cap \Theta^{\mathrm{L}(\mathrm{V}_{\lambda+1}, E)}$ is a limit ordinal with $j(\gamma) = \gamma$, then there is a function

$$j^{\gamma} : \mathrm{L}_{\gamma}(\mathrm{V}_{\lambda+1}, E) \longrightarrow \mathrm{L}_{\gamma}(\mathrm{V}_{\lambda+1}, E)$$

in X with

$$j \restriction (X \cap \mathrm{L}_{\gamma}(\mathrm{V}_{\lambda+1}, E)) \;=\; j^{\gamma} \restriction (X \cap \mathrm{L}_{\gamma}(\mathrm{V}_{\lambda+1}, E)).$$

Definition (Woodin)

The *Internal Axiom I0* holds at a cardinal λ if for all $\lambda < \gamma < \Theta^{\mathrm{L}(\mathrm{V}_{\lambda+1})}$, there exists a non-trivial elementary embedding

$$j : \mathrm{L}_\gamma(\mathrm{V}_{\lambda+1}) \longrightarrow \mathrm{L}_\gamma(\mathrm{V}_{\lambda+1})$$

with $\mathrm{crit}(j) < \lambda$.

Corollary (Aguilera–Bagaria–Goldberg–L.)

Internal Axiom I0 holds at every ultraexacting cardinal.

Theorem (Woodin)

The following statements are equiconsistent over ZFC:

- There exists an I0-embedding.
- Internal Axiom I0 holds at some cardinal.

An application: fragments of HOD-Berkeleyness

Definition

Given an inner model N and ordinals $\rho < \vartheta$, the ordinal ρ is *N-ϑ-Berkeley* if for every $\alpha < \rho$ and every transitive set M in N that contains the ordinal ρ as an element and has cardinality less than ϑ in N, there exists a non-trivial elementary embedding $j : M \longrightarrow M$ with $\alpha < \mathrm{crit}(j) < \rho$.

Theorem (Aguilera–Bagaria–L.)

If $E \in V_{\lambda+2} \cap X$ with $j(E) = E$, then λ is HOD-$\Theta^{L(V_{\lambda+1},E)}$-Berkeley.

An application: Icarus sets

Definition

An element E of $V_{\lambda+2}$ is a *strong Icarus set* if there exists a non-trivial elementary embedding $i : L(V_{\lambda+1}, E) \longrightarrow L(V_{\lambda+1}, E)$ with the property that $\mathrm{crit}(i) < \lambda$ and $i(E) = E$.

Theorem (Aguilera–Bagaria–L.)

Let $E \in V_{\lambda+2} \cap X$ be such that $j(E) = E$. If $E^{\#}$ exists, then there is an elementary embedding $i : L(V_{\lambda+1}, E) \longrightarrow L(V_{\lambda+1}, E)$ with $i \restriction V_\lambda = j \restriction V_\lambda$ and $i(E) = E$.

Corollary

If λ is an ultraexacting cardinal with the property that $\mathrm{V}^{\#}_{\lambda+1}$ exists, then there is an I0-embedding $j : \mathrm{L}(\mathrm{V}_{\lambda+1}) \longrightarrow \mathrm{L}(\mathrm{V}_{\lambda+1})$.

Pick $E \in X \cap \mathrm{V}_{\lambda+2}$ such that $j(E) = E$ and $E^{\#}$ exists.

Set $\Theta = \Theta^{\mathrm{L}(\mathrm{V}_{\lambda+1}, E)}$ and $\Theta' = \Theta^{\mathrm{L}(\mathrm{V}_{\lambda+1}, E^{\#})}$. Then $\Theta < \Theta'$ holds.

Moreover, we have $E^{\#}, \Theta, \Theta' \in X$ with $j(E^{\#}) = E^{\#}$, $j(\Theta) = \Theta$ and $j(\Theta') = \Theta'$.

Hence, our results yield a function $j^{\Theta} : \mathrm{L}_{\Theta}(\mathrm{V}_{\lambda+1}, E) \longrightarrow \mathrm{L}_{\Theta}(\mathrm{V}_{\lambda+1}, E)$ in X with

$$j^{\Theta} \restriction (\mathrm{L}_{\Theta}(\mathrm{V}_{\lambda+1}, E) \cap X) \ = \ j \restriction (\mathrm{L}_{\Theta}(\mathrm{V}_{\lambda+1}, E) \cap X).$$

If we define

$$U \ = \ \{A \in \mathrm{V}_{\lambda+2}^{\mathrm{L}(\mathrm{V}_{\lambda+1}, E)} \mid j \restriction \mathrm{V}_{\lambda} \in j^{\Theta}(A)\} \ \in \ X,$$

then it can be shown that U is a normal $\mathrm{L}(\mathrm{V}_{\lambda+1}, E)$-ultrafilter and taking an ultrapower of $\mathrm{L}(\mathrm{V}_{\lambda+1}, E)$ by U yields an elementary embedding

$$i : L(V_{\lambda+1}, E) \longrightarrow L(V_{\lambda+1}, E)$$

with $i \restriction V_{\lambda} = j \restriction V_{\lambda}$ and $i(E) = E$.

Thank you for listening!