Structural Reflection and the HOD Conjecture

1. Lecture: Large cardinals beyond HOD

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Introduction

In his work on the *Inner Model Problem*, Hugh Woodin proved the following surprising result:

Theorem (Simplified HOD Dichotomy, Woodin)

If δ is an extendible cardinal, then exactly one of the following statements holds:

- For every singular cardinal $\lambda > \delta$, the cardinal λ is singular in HOD and $(\lambda^+)^{\mathrm{HOD}} = \lambda^+$ holds.
- Every regular cardinal $\kappa \geq \delta$ is measurable in HOD.

Questions

Are there canonical extensions of ZFC that prove the second alternative holds? Are there such axioms that imply $\mathrm{V} \neq \mathrm{HOD}$?

All standard large cardinal axioms are compatible with the assumption that $\mathrm{V} = \mathrm{HOD}$ and therefore do not provide affirmative answers to these questions.

If we instead ask for extensions of ZF, then large cardinals beyond choice (e.g., Reinhardt cardinals) provide trivial affirmative answers to the second question.

In the following, we will observe that are more interesting things can be said about the relationship between V and HOD in this setting.

Definition (Goldberg & Schlutzenberg, ZF)

A cardinal λ is *rank-Berkeley* if for all $\alpha < \lambda < \beta$, there is a non-trivial elementary embedding $j : \mathrm{V}_\beta \longrightarrow \mathrm{V}_\beta$ with the property that $\alpha < \mathrm{crit}(j) < \lambda$ and λ is the first non-trivial fixed point of j.

Proposition (GB)

If $j : \mathrm{V} \longrightarrow \mathrm{V}$ is an elementary embedding, then the first non-trivial fixed point of j is a rank-Berkeley cardinal.

Proposition (ZF)

Rank-Berkeley cardinals are cardinals of countable cofinality that are regular in HOD.

Proof.

Assume, towards a contradiction, that a rank-Berkeley cardinal λ is singular in HOD.

Pick $\beta > \lambda$ such that V_β is sufficiently elementary in V.

Then there is an elementary embedding $j : \mathrm{V}_\beta \longrightarrow \mathrm{V}_\beta$ such that $\mathrm{cof}(\lambda)^{\mathrm{HOD}} < \mathrm{crit}(j)$ and λ is the first non-trivial fixed point of j.

Let $c : \mathrm{cof}(\lambda)^{\mathrm{HOD}} \longrightarrow \lambda$ be the least cofinal function in the canonical well-ordering of HOD.

Then c is definable from the parameter λ and hence $j(c) = c$.

Pick $\alpha < \mathrm{cof}(\lambda)^{\mathrm{HOD}}$ with $c(\alpha) > \mathrm{crit}(j)$. Then

$$c(\alpha) \; < \; j(c(\alpha)) \; = \; j(c)(j(\alpha)) \; = \; c(\alpha),$$

a contradiction. □

Exacting cardinals

We now want to isolate canonical fragments of rank-Berkeleyness that are compatible with the Axiom of Choice and still allow us to carry out the above argument.

The starting point for finding these fragments is the following classical result of Magidor:

Lemma (Magidor)

The following statements are equivalent for every cardinal κ:

- κ is a supercompact cardinal.
- For all ordinals $\zeta > \kappa$, there exists
  - an ordinal $\eta < \kappa$,
  - a cardinal $\bar{\kappa} < \eta$, and
  - a non-trivial elementary embedding $j : \mathrm{V}_\eta \longrightarrow \mathrm{V}_\zeta$ with $\mathrm{crit}(j) = \bar{\kappa}$ and $j(\bar{\kappa}) = \kappa$.

Theorem (L.)

The following statements are equivalent for every cardinal κ:

- For all cardinals $\zeta > \kappa$, there exists
  - an ordinal $\eta < \kappa$,
  - a cardinal $\bar{\kappa} < \eta$,
  - an elementary submodel X of V_η with $\mathrm{V}_{\bar{\kappa}} \cup \{\bar{\kappa}\} \subseteq X$, and
  - an elementary embedding $j : X \longrightarrow \mathrm{V}_\zeta$ with $\mathrm{crit}(j) = \bar{\kappa}$ and $j(\bar{\kappa}) = \kappa$.
- The cardinal κ is a strongly unfoldable cardinal.
- The cardinal κ is shrewd.

Definition (Aguilera–Bagaria–L.)

A cardinal λ is *exacting* if for all $\alpha < \lambda < \beta$, there exists

- an elementary submodel X of V_β with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$, and
- an elementary embedding $j : X \longrightarrow \mathrm{V}_\beta$ with $\alpha < \mathrm{crit}(j) < \lambda$ and $j(\lambda) = \lambda$.

Theorem (Aguilera–Bagaria–L.)

If λ is exacting, then λ is a singular cardinal that is regular in $\mathrm{HOD}_{\mathrm{V}_\lambda}$.

Corollary

If there is an exacting cardinal above an extendible cardinal, then eventually all regular cardinals are measurable in HOD.

Let λ be an exacting cardinal. Then there is a non-trivial elementary embedding $j : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda$ and results of Kunen imply $\mathrm{cof}(\lambda) = \omega$.

Assume, towards a contradiction, that λ is singular in $\mathrm{HOD}_{\mathrm{V}_\lambda}$. Then there is $z \in \mathrm{V}_\lambda$ such that λ is singular in $\mathrm{HOD}_{\{z\}}$.

Fix $\beta > \lambda$ such that V_β is sufficiently elementary in V. Pick $X \prec \mathrm{V}_\beta$ with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$ and an elementary embedding $j : X \longrightarrow V_\beta$ with $\mathrm{cof}(\lambda)^{\mathrm{HOD}_{\{z\}}} < \mathrm{crit}(j) < \lambda$, $j(\lambda) = \lambda$ and $j(z) = z$.

Results of Kunen imply that λ is the first non-trivial fixed point of j.

Let $c : \mathrm{cof}(\lambda)^{\mathrm{HOD}_{\{z\}}} \longrightarrow \lambda$ be the least cofinal function with respect to the canonical well-ordering of $\mathrm{HOD}_{\{z\}}$. Then $c \in X$ with $j(c) = c$.

If we pick $\alpha < \mathrm{cof}(\lambda)^{\mathrm{HOD}_{\{z\}}}$ with $c(\alpha) > \mathrm{crit}(j)$, then we have

$$c(\alpha) \ < \ j(c(\alpha)) \ = \ j(c)(j(\alpha)) \ = \ c(\alpha),$$

a contradiction.

The following result gives an alternative definition of exactingness that is often easier to check:

Lemma

The following statements are equivalent for every cardinal λ:

- λ is an exacting cardinal.
- There exists
  - an ordinal $\eta > \lambda$ with $\mathrm{V}_\eta \prec_{\Sigma_1} \mathrm{V}$,
  - an ordinal $\zeta > \lambda$ with $\mathrm{V}_\zeta \prec_{\Sigma_2} \mathrm{V}$,
  - an elementary submodel X of V_η with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$, and
  - an elementary embedding $j : X \longrightarrow \mathrm{V}_\zeta$ with $j(\lambda) = \lambda$ and $j \restriction \lambda \neq \mathrm{id}_\lambda$.

We now discuss the naturalness of the notion of exactingness.

First, note that as a fragment of Reinhardtness, this property is phrased in the standard format of large cardinal axioms.

Next, we show that exactingness is equivalent to a natural model-theoretic reflection principle.

For this purpose, remember that a cardinal λ is Jónsson if every structure in a countable first-order language whose domain has cardinality λ has a proper elementary substructure of cardinality λ.

The next result shows that exactingness is equivalent to a strengthening of this property that incorporates external features of the given structure.

Theorem (Aguilera–Bagaria–L.)

The following are equivalent for each cardinal λ with $\mathrm{V}_\lambda \prec_{\Sigma_1} \mathrm{V}$:

- λ is an exacting cardinal.
- For every class $\mathcal{C}$ of structures in a countable first-order language that is definable by a formula with parameters in $\mathrm{V}_\lambda \cup \{\lambda\}$, every structure of cardinality λ in $\mathcal{C}$ contains a proper elementary substructure of cardinality λ isomorphic to a structure in $\mathcal{C}$.
- For every class $\mathcal{C}$ of structures in a countable first-order language that is definable by a formula with parameters in $\mathrm{V}_\lambda \cup \{\lambda\}$, every structure of cardinality λ in $\mathcal{C}$ is isomorphic to a proper elementary substructure of a structure of cardinality λ in $\mathcal{C}$.

Ultraexacting cardinals

We now consider the possibility of further strengthening the notion of exacting cardinals.

Our motivation for the formulation of stronger notions comes from the observation that certain elements of $\mathrm{H}(\lambda^+)$ have to be missing from the domains of embeddings witnessing the exactingness of a cardinal λ.

The proof of the following result uses ideas from Woodin's proof of the Kunen Inconsistency:

Proposition

If λ is a cardinal, $\zeta > \lambda$ is an ordinal with $\mathrm{V}_\zeta \prec_{\Sigma_2} \mathrm{V}$, X is an elementary submodel of V_ζ with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$ and $j : X \longrightarrow \mathrm{V}_\zeta$ is an elementary embedding with $j(\lambda) = \lambda$ and $j \restriction \lambda \neq \mathrm{id}_\lambda$, then $\lambda^+ \nsubseteq X$ and $[\lambda]^\omega \nsubseteq X$.

The above proposition shows that we can strengthen the notion of exacting cardinals by demanding that certain sets are contained in the domains of the elementary embeddings witnessing the given property.

Arguments presented later in this lecture series will show that initial segments of the given elementary embeddings are canonical examples of sets that are, in general, not contained in their domains.

This motivates the following definition:

Definition (Aguilera–Bagaria–L.)

A cardinal λ is *ultraexacting* if for all $\alpha < \lambda < \beta$, there exist

- an elementary submodel X of V_β with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$, and
- an elementary embedding $j : X \longrightarrow \mathrm{V}_\beta$ with $\alpha < \mathrm{crit}(j) < \lambda$, $j(\lambda) = \lambda$ and $j \restriction \mathrm{V}_\lambda \in X$.

As before, the above definition is equivalent to a property that is often easier to check:

Lemma

The following statements are equivalent for every cardinal λ:

- λ is an ultraexacting cardinal.
- There exists
  - an ordinal $\eta > \lambda$ with $\mathrm{V}_\eta \prec_{\Sigma_1} \mathrm{V}$,
  - an ordinal $\zeta > \lambda$ with $\mathrm{V}_\zeta \prec_{\Sigma_2} \mathrm{V}$,
  - an elementary submodel $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X \prec \mathrm{V}_\eta$, and
  - an elementary embedding $j : X \longrightarrow \mathrm{V}_\zeta$ with $j(\lambda) = \lambda$, $j \restriction \lambda \neq \mathrm{id}_\lambda$ and $j \restriction \mathrm{V}_\lambda \in X$.

Thank you for listening!