Separating rank-into-rank axioms through their descriptive consequences

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Joint work in progress with Vincenzo Dimonte (Udine).

Our starting point is the classical *Kunen Inconsistency*:

Theorem (Kunen)

Given an ordinal λ, there is no non-trivial elementary embedding $j : \mathrm{V}_{\lambda+2} \longrightarrow \mathrm{V}_{\lambda+2}$.

Shortly after Kunen's proof, people started studying large cardinal notions on the verge of this inconsistency result.

Definition (Gaifman, Kanamori–Reinhardt–Solovay)

- An *I3-embedding* is a non-trivial elementary embedding $j : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda$ for some limit ordinal λ.
- An *I2-embedding* is a non-trivial Σ_1-elementary embedding $j : \mathrm{V}_{\lambda+1} \longrightarrow \mathrm{V}_{\lambda+1}$.
- An *I1-embedding* is a non-trivial elementary embedding $j : \mathrm{V}_{\lambda+1} \longrightarrow \mathrm{V}_{\lambda+1}$.

Definition (Woodin)

An *I0-embedding* is a non-trivial elementary embedding $j : \mathrm{L}(\mathrm{V}_{\lambda+1}) \longrightarrow \mathrm{L}(\mathrm{V}_{\lambda+1})$ with $\mathrm{crit}(j) < \lambda$.

Results of Woodin show that, if $j : \mathrm{L}(\mathrm{V}_{\lambda+1}) \longrightarrow \mathrm{L}(\mathrm{V}_{\lambda+1})$ is an I0-embedding, then the model $\mathrm{L}(\mathrm{V}_{\lambda+1})$ possesses various structural features that generalize properties of determinacy models.

For example:

Theorem (Woodin)

If $j : \mathrm{L}(\mathrm{V}_{\lambda+1}) \longrightarrow \mathrm{L}(\mathrm{V}_{\lambda+1})$ is an I0-embedding, then λ^+ is a measurable cardinal in $\mathrm{L}(\mathrm{V}_{\lambda+1})$.

Given a cardinal $\nu > 0$ and an infinite cardinal μ, we equip the set ${}^{\mu}\nu$ of all functions from μ to ν with the topology whose basic open sets consists of all functions that extend a given function $s : \xi \longrightarrow \nu$ with $\xi < \mu$.

Next, we say that a map $\iota : X \longrightarrow Y$ between topological spaces is a *perfect embedding* if it induces a homeomorphism between X and the subspace $\mathrm{ran}(\iota)$ of Y.

Finally, given an infinite cardinal κ, we say that a subset of ${}^{\kappa}2$ has the *perfect set property* if it either has cardinality at most κ or it contains the range of a perfect embedding of ${}^{\mathrm{cof}(\kappa)}\kappa$ into ${}^{\kappa}2$.

Theorem (Cramer, Shi & Woodin)

If $j : \mathrm{L}(\mathrm{V}_{\lambda+1}) \longrightarrow \mathrm{L}(\mathrm{V}_{\lambda+1})$ is an I0-embedding, then every subset of ${}^{\lambda}2$ in $\mathrm{L}(\mathrm{V}_{\lambda+1})$ has the perfect set property.

Question

Do weaker large cardinal assumptions suffice to derive the above conclusion for smaller classes of definable subsets of ${}^{\lambda}2$?

The starting point of our project is the following result:

Theorem (L.–Müller)

If λ is a limit of measurable cardinals, then every subset of ${}^{\lambda}2$ that is definable by a Σ_1-formulas with parameters in $\mathrm{V}_\lambda \cup \{\lambda\}$ has the perfect set property.

Theorem (L.–Müller)

Let λ be a singular strong limit cardinal with the property that for every subset of ${}^{\lambda}2$ that is definable by a Σ_1-formula with parameters in $\mathrm{V}_\lambda \cup \{\lambda\}$ has the perfect set property. Then there is an inner model with a sequence of measurable cardinals of length $\mathrm{cof}(\lambda)$.

Question

Can we derive a stronger Perfect Set Theorem at limits of measurable cardinals?

What happens if we allow other *simple* parameters, like V_λ or a cofinal ω-sequence in λ, in our Σ_1-definitions?

Theorem (Dimonte–Iannella–L.)

If $\vec{\lambda}$ is a strictly increasing sequence of measurable cardinals with supremum λ, then the following statements hold in an inner model containing the sequence $\vec{\lambda}$:

- The sequence $\vec{\lambda}$ consists of measurable cardinals.
- There is a subset of ${}^{\lambda}2$ without the perfect set property that is definable by a Σ_1-formula with parameter V_{λ}.
- There is a subset of ${}^{\lambda}2$ without the perfect set property that is definable by a Σ_1-formula with parameter $\vec{\lambda}$.
- If $\vec{\mu}$ is an ω-sequence of regular cardinals with limit λ, then there is a subset of ${}^{\lambda}2$ without the perfect set property that is definable by a Σ_1-formula with parameters in $\mathbb{R} \cup \{\vec{\mu}\}$.

Descriptive properties of l2-embeddings

Remember that an I2-embedding is a non-trivial Σ_1-elementary embedding $j : \mathrm{V}_{\lambda+1} \longrightarrow \mathrm{V}_{\lambda+1}$.

Theorem (Dimonte–Iannella–L.)

If $j : \mathrm{V}_{\lambda+1} \longrightarrow \mathrm{V}_{\lambda+1}$ is an I2-embedding with critical sequence $\vec{\lambda}$, then every subset of ${}^{\lambda}2$ that is definable by a Σ_1-formula with parameters in $\mathrm{V}_\lambda \cup \{\mathrm{V}_\lambda, \vec{\lambda}\}$ has the perfect set property.

We will in fact show that the above conclusion holds for a larger collection of parameters that we will now define.

Lemma

The following statements are equivalent for every strictly increasing sequence $\vec{\lambda}$ with supremum λ:

- There is an I2-embedding with critical sequence $\vec{\lambda}$.
- There is a transitive class M with $\mathrm{V}_\lambda \subseteq M$ and an elementary embedding $j : \mathrm{V} \longrightarrow M$ with critical sequence $\vec{\lambda}$.

In the following, we will use the term "I2-embedding" for both types of embeddings.

Classical results of Martin show that I2-embeddings $j : V \longrightarrow M$ are $(\omega+1)$-*iterable*, i.e. there exists a commuting system

$$\langle\langle M^j_\alpha \mid \alpha \leq \omega\rangle, \langle j : M^j_\alpha \longrightarrow M^j_\beta \mid \alpha \leq \beta \leq \omega\rangle\rangle$$

of inner models and elementary embeddings with:

- $M^j_0 = V$ and $j_{0,1} = j$.
- If $n < \omega$, then $j_{n+1,n+2} = \bigcup\{j_{n,n+1}(j_{n,n+1} \restriction V_\alpha) \mid \alpha \in \mathrm{Ord}\}$.
- $\langle M^j_\omega, \langle j_{n,\omega} \mid n < \omega\rangle\rangle$ is a direct limit of

$$\langle\langle M^j_n \mid n < \omega\rangle, \langle j_{m,n} : M^j_m \longrightarrow M^j_n \mid m \leq n < \omega\rangle\rangle.$$

If $\vec{\lambda} = \langle \lambda_n \mid n < \omega\rangle$ is the critical sequence of j and $\lambda = \sup_{n<\omega} \lambda_n$, then:

- Given $m \leq n < \omega$, we then have $V_\lambda \subseteq M^j_\omega \subseteq M^j_n \subseteq M^j_m$, $\mathrm{crit}(j_{n,n+1}) = \lambda_n = j_{m,n}(\lambda_m)$, $j_{m,n}(\lambda) = \lambda$ and $j_{n,\omega}(\lambda_n) = \lambda$.
- $j_{0,\omega}(\lambda^+) = \lambda^+$ and $(2^\lambda)^{M^j_\omega} < \lambda^+$.
- $\vec{\lambda}$ is Prikry-generic over M^j_ω and hence $(2^\lambda)^{M^j_\omega[\vec{\lambda}]} < \lambda^+$.

Theorem (Laver)

Let $j : V \longrightarrow M$ be an I2-embedding with critical sequence $\langle \lambda_n \mid n < \omega \rangle$ and set $\lambda = \sup_{n<\omega} \lambda$.

If $d \in V_\lambda$ and $r : d \longrightarrow \mathrm{Ord}$ is a function, then the function

$$j_{0,\omega} \circ r : d \longrightarrow \mathrm{Ord}$$

is an element of M^j_ω.

Using Laver's result, we will be able to prove a strengthening of the above Perfect Set Theorem.

Theorem (Dimonte–Iannella–L.)

Let $j : V \longrightarrow M$ be an I2-embedding with critical sequence $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$, set $\lambda = \sup_{n<\omega} \lambda_n$ and let N be an inner model of $\mathbf{ZFC}$ with $M^j_\omega \cup \{\vec{\lambda}\} \subseteq N$ and $(2^\lambda)^N < \lambda^+$.

Then every subset of ${}^\lambda 2$ that is definable by a Σ_1-formula with parameters in $V^N_{\lambda+1}$ has the perfect set property.

- A subset of ${}^{\omega}\lambda$ is definable by a Σ_1-formula with parameters in $V^N_{\lambda+1}$ iff it is definable over V_λ by a Σ^1_2-formula with parameters in $V^N_{\lambda+1}$.

- A subset of ${}^{\omega}\lambda \times {}^{\omega}\lambda$ that is definable over V_λ by a Σ^1_1-formula with parameters in $V^N_{\lambda+1}$ can be represented as the projection $p[T]$ of the set $[T]$ of all cofinal branches through a subtree $T \in N$ of $({}^{<\omega}V_\lambda)^3$.

- We can build a *Shoenfield tree* for the Σ^1_2-subset of ${}^{\omega}\lambda$ defined by T.

  Let S^V_T denote the Shoenfield tree of T in V and let S^N_T denote the Shoenfield tree of T in N.

- Then $S^N_T \subseteq S^V_T$ and we can use Laver's theorem to find an embedding of S^V_T into S^N_T that is the identity on the first coordinate.

- We then know that $p[S^N_T]^V = p[S^V_T]^V$.

Lemma

Let $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$ be a strictly increasing sequence of infinite cardinals with limit λ and let $T \subseteq {}^{<\omega}a \times {}^{<\omega}b$ be a tree such that $p[T]$ does not contain the range of a perfect embedding of ${}^{\omega}\lambda$ into ${}^{\omega}a$.

If N is an inner model with $V_\lambda \cup \{T, \vec{\lambda}\} \subseteq N$, then $p[T]^V \subseteq N$.

- Assume that $p[S_T^V]^V$ has cardinality greater than $(2^\lambda)^N$.
- Then $p[S_T^N]^V = p[S_T^V]^V \nsubseteq N$.
- The lemma shows that $p[S_T^V]^V$ contains the range of a perfect embedding of ${}^{\omega}\lambda$ into itself.

Proposition (Dimonte–Iannella–L.)

If $j : \mathrm{V}_{\lambda+1} \longrightarrow \mathrm{V}_{\lambda+1}$ is an I2-embedding, then the following statements hold in an inner model:

- There is an I2-embedding $i : \mathrm{V}_{\lambda+1} \longrightarrow \mathrm{V}_{\lambda+1}$.
- There is a subset of ${}^{\lambda}2$ without the perfect set property that is definable by a Σ_1-formula with parameters in $\mathcal{P}(\lambda)$.

Separating rank-into-rank axioms through their descriptive consequences

The above results raise the possibility of separating rank-into-rank axioms through their descriptive consequences.

More specifically, they motivate the question whether we can canonically assign parameter sets to rank-into-rank axioms in a way that ...

- ... the given axiom implies that all sets definable by a Σ_1-formula with parameters from the given set have the perfect set property.
- ... weaker axiom, if consistent, do not imply this regularity property.

Recent work with Vincenzo Dimonte reveals that this is indeed possible for I1-, I2- and I3-embeddings, and unveils a canonical generalized descriptive set theory in the presence of rank-into-rank axioms.

Theorem (Dimonte–L.)

If $j : \mathrm{V}_{\lambda+1} \longrightarrow \mathrm{V}_{\lambda+1}$ is an I1-embedding, then every subset of ${}^{\lambda}2$ that is definable by a Σ_1-formula with parameters in $\mathrm{V}_{\lambda+1}$ has the perfect set property.

Theorem (Dimonte–L.)

If there is an I3-embedding, then there is a cardinal λ such that the following statements hold in an inner model of $\mathbf{ZFC}$ of a forcing extension of V:

- There is an I3-embedding $j : \mathrm{V}_{\lambda} \longrightarrow \mathrm{V}_{\lambda}$.
- There is a subset of ${}^{\lambda}2$ without the perfect set property that is definable by a Σ_1-formula with parameter V_{λ}.

Thank you for listening!