Iteratively changing the heights of automorphism towers

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Abstract

We extend the results of Hamkins and Thomas concerning the malleability of automorphism tower heights of groups by forcing. We show that any reasonable sequence of ordinals can be realized as the automorphism tower heights of a certain group in consecutive forcing extensions or ground models, as desired. For example, it is possible to increase the height of the automorphism tower by passing to a forcing extension, then increase it further by passing to a ground model, and then decrease it by passing to a further forcing extension, etc., transfinitely. We make sense of the limit models occurring in such a sequence of models. At limit stages, the automorphism tower height will always be 1.

1 Introduction

If G is a centerless group, then there is a natural embedding

$$\iota_G : G \longrightarrow \mathrm{Aut}(G); g \mapsto \iota_g := \left[h \mapsto h^g := g \circ h \circ g^{-1}\right]$$

that maps G to the subgroup $\mathrm{Inn}(G)$ of inner automorphisms of G. An easy computation shows that $\pi \circ \iota_g \circ \pi^{-1} = \iota_{\pi(g)}$ holds for all $g \in G$ and $\pi \in \mathrm{Aut}(G)$. Hence $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$, $\mathrm{C}_{\mathrm{Aut}(G)}(\mathrm{Inn}(G)) = \{\mathrm{id}_G\}$ and $\mathrm{Aut}(G)$ is also a group with trivial center.

By iterating this process, we inductively construct the automorphism tower of G.

Definition 1.1. A sequence $\langle G^\alpha \mid \alpha \in \mathrm{On}\rangle$ of groups is *the automorphism tower of a group G* if the following statements hold.

1. $G^0 = G$.
2. G^α is a normal subgroup of $G^{\alpha+1}$ and the induced homomorphism

$$\varphi_\alpha : G^{\alpha+1} \longrightarrow \mathrm{Aut}(G^\alpha); g \mapsto \iota_g \restriction G^\alpha$$

is an isomorphism.

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3. $G^\lambda = \bigcup\{G^\alpha \mid \alpha < \lambda\}$, if $\lambda \in \mathrm{Lim}$.

In this definition, we took $G^{\alpha+1}$ to be an isomorphic image of $\mathrm{Aut}(G^\alpha)$ of which G^α is a normal subgroup. This enables us to take unions at limits. Without this isomorphic correction, we would have to take direct limits at limit stages, and could have let $G^{\alpha+1}$ be $\mathrm{Aut}(G^\alpha)$, as it is sometimes done in the literature.

This definition implies that the center of G is trivial and we can construct such a tower for each centerless group by induction. It is easy to show that each group G^α is uniquely determined up to an isomorphism which is the identity on G, and therefore we can speak of *the* α-th group G^α in the automorphism tower of a centerless group G.

We say that the automorphism tower of a centerless group G terminates after α steps, if $G^\alpha = G^{\alpha+1}$ and therefore $G^\alpha = G^\beta$ for all $\beta \geq \alpha$. Simon Thomas' elegant proof of the following theorem uses Fodor's Lemma to show that every infinite centerless group has a terminating automorphism tower.

Theorem 1.2 ([Tho98])**.** *The automorphism tower of every infinite centerless group of cardinality κ terminates in fewer than $(2^\kappa)^+$ many steps.*

Definition 1.3. If G is a centerless group, then $\tau(G)$ denotes the least ordinal α such that $G^\alpha = G^{\alpha+1}$. $\tau(G)$ is called the height of the automorphism tower of G.

Although the definition of automorphism towers is purely algebraic, it has a set-theoretic essence, since there are groups whose automorphism tower heights depend on the model of set theory in which they are computed. In [Tho98], Simon Thomas constructs a centerless, complete (i.e. $\tau(G) = 0$) group G and a c.c.c. forcing $\mathbb{P}$ such that $\mathbb{P} \Vdash$ "$\tau(G) = 1$". In the other direction, he also constructs a centerless group H such that $\tau(G) = 2$ and $\mathbb{Q} \Vdash$ "$\tau(H) = 1$" for every forcing $\mathbb{Q}$ that adjoins a new real.

Let M, N be transitive models of ZFC with $M \subseteq N$ and $G \in M$ be a centerless group. By the above, the height of the automorphism tower of G computed in M, $\tau(G)_M$, can be higher or smaller than the height computed in N, $\tau(G)_N$. This leads to the natural question whether the value of $\tau(G)_M$ places any constraints on the value of $\tau(G)_N$, and vice versa. Obviously, $\tau(G)_N = 0$ implies $\tau(G)_M = 0$. The following result by Joel Hamkins and Simon Thomas suggests that this is the only provable implication that holds for all centerless groups in the above situation. In short, the theorem states that the existence of centerless groups *whose automorphism towers are highly malleable by forcing* is consistent with the axioms of ZFC.

Theorem 1.4 ([HT00])**.** *It is consistent that for every infinite cardinal κ and every ordinal $\alpha < \kappa$, there exists a centerless group G with the following properties.*

1. *$\tau(G) = \alpha$,*

2. *If β is any ordinal such that $0 < \beta < \kappa$, then there exists a notion of forcing $\mathbb{P}_\beta$, which preserves cofinalities and cardinalities, such that $\mathbb{P}_\beta \Vdash$ "$\tau(G) = \beta$".*

The proof of this theorem splits into an algebraic and a set-theoretic part. The following definition features the key concept of both parts of the proof. The terminology is taken from [FH08].

Definition 1.5. Let κ be a cardinal, $\langle \Gamma_\alpha \mid \alpha < \kappa \rangle$ be a sequence of rigid graphs and E be an equivalence relation on κ. We say that a forcing notion $\mathbb{P}$ *is able to realize* E, if $\mathbb{P}$ forces that all Γ_α are rigid and, that for all $\beta, \gamma < \kappa$, $\Gamma_\beta \cong \Gamma_\gamma \Leftrightarrow \beta E \gamma$.

The following theorem sums up the results of the set-theoretic part of the proof.

Theorem 1.6 ([HT00]). *It is consistent that for every regular cardinal $\kappa \geq \omega$, there exists a sequence $\langle \Gamma_\alpha \mid \alpha < \kappa^+ \rangle$ of pairwise nonisomorphic connected rigid graphs with the following property: Whenever E is an equivalence relation on κ^+, there exists a notion of forcing $\mathbb{P}_E$ with the following properties:*

1. *$\mathbb{P}_E$ preserves cardinals and cofinalities and adds no new κ-sequences,*
2. *$\mathbb{P}_E$ is able to realize E.*

The algebraic part of the proof then shows that the conclusions of Theorem 1.4 are a consequence of this theorem. Since we are going to adopt the techniques developed in these proofs, the next section contains an overview of the the construction of the groups in the algebraic part of the proof.

The consistency result of the former theorem is obtained by a class-sized forcing over a model of $\mathsf{ZFC} + \mathsf{GCH}$. In [FH08], Joel Hamkins and the first author showed that the conclusions of this theorem also hold in the constructible universe L. They deduce these conclusions from combinatorial principles that hold in L and that we will introduce presently.

Definition 1.7. Let κ be a cardinal and let Cof_κ denote the set $\{\alpha < \kappa^+ \mid \mathrm{cf}(\alpha) = \kappa\}$.

Then $\diamondsuit_{\kappa^+}(\mathrm{Cof}_\kappa)$ is the assertion that there is a sequence $\vec{D} = \{D_\alpha \mid \alpha \in \mathrm{Cof}_\kappa\}$ such that for any $A \subset \kappa^+$ the set $\{\alpha \in \mathrm{Cof}_\kappa | A \cap \alpha = D_\alpha\}$ is stationary in κ^+.

In L, the hypotheses that $2^{<\kappa} = \kappa$ and $\diamondsuit_{\kappa^+}(\mathrm{Cof}_\kappa)$ are known to hold for every regular cardinal κ. Note that $\diamondsuit_{\kappa^+}(\mathrm{Cof}_\kappa)$ implies that κ is regular, for otherwise Cof_κ is empty.

For the remainder of this paper, we fix a cardinal κ that satisfies the following assumption.

Assumption 1.8. κ is a regular, uncountable cardinal such that $2^{<\kappa} = \kappa$ and $\diamondsuit_{\kappa^+}(\mathrm{Cof}_\kappa)$ holds.

Definition 1.9. Let E be an equivalence relation on κ. If $\gamma < \kappa$, then we let $[\gamma]_E$ denote the E-equivalence class of γ. We call E *bounded*, if there is some $\bar{\kappa} < \kappa$ such that $[\gamma]_E = \{\gamma\}$ for all $\gamma \in [\bar{\kappa}, \kappa)$.

Now we are ready to formulate the statement of the theorem mentioned above.

Theorem 1.10 ([FH08], under Assumption 1.8). *There are sequences $\vec{T} = \langle T_\alpha \mid \alpha < \kappa \rangle$ and $\vec{C} = \langle C_{\alpha,\beta} \mid \alpha < \beta < \kappa \rangle$ of κ^+-Souslin trees with the following property: Whenever E is a bounded equivalence relation on κ, the full support product forcing*

$$C_E := \prod_{\substack{\gamma<\kappa \\ \gamma \neq \min[\gamma]_E}} C_{\min[\gamma]_E,\gamma}$$

has the following properties.

1. *C_E preserves cardinals and cofinalities and adds no new κ-sequences.*
2. *C_E is able to realize E.*

The aim of this paper is to show that this theorem already implies the existence of groups whose automorphism tower is even more malleable by forcing than those of the groups mentioned in Theorem 1.4. It gives rise to groups whose automorphism tower heights can be changed multiple times to any non-zero height by passing from one model of set-theory to another, either by always going to a forcing extension, by always passing to a ground model, or by mixing these possibilities. In fact, for the given cardinal κ, we will use Assumption 1.8 to construct a *single* complete group $\mathfrak{G} = \mathfrak{G}_\kappa$ the height of whose automorphism tower can be changed in each of these ways, repeatedly.

Let us now formulate precisely the three ways in which the height of the automorphism tower of $\mathfrak{G}$ can be changed repeatedly. The first main result, Theorem 3.10, addresses the possibility of passing from models to larger and larger forcing extensions in each step:

Theorem (Under Assumption 1.8)**.** *For every function $s : \kappa \longrightarrow (\kappa \setminus \{0\})$, there is a sequence of partial orders $\langle \mathbb{P}^s_\gamma \mid 0 < \gamma < \kappa \rangle$, such that the following statements hold for each $0 < \alpha < \kappa$.*

1. *$\mathbb{P}^s_\alpha$ preserves cardinals and cofinalities and adds no new κ-sequences.*
2. *$\mathbb{P}^s_{\alpha+1} \Vdash$ "$\tau(\mathfrak{G}) = s(\alpha)$".*
3. *If α is a limit ordinal, then $\mathbb{P}^s_\alpha \Vdash$ "$\tau(\mathfrak{G}) = 1$".*
4. *If $\beta < \alpha$, then $\mathbb{P}^s_\alpha$ extends $\mathbb{P}^s_\beta$.*

Moreover, if $t : \kappa \longrightarrow \kappa \setminus \{0\}$, and $s \restriction \gamma = t \restriction \gamma$ for some $0 < \gamma < \kappa$, then $\mathbb{P}^s_\gamma = \mathbb{P}^t_\gamma$.

The next main theorem addresses the possibility of producing a model with the property that the height of the automorphism tower of $\mathfrak{G}$ can be changed by passing to smaller and smaller ground models.

Theorem (Under Assumption 1.8)**.** *For every ordinal $\lambda < \kappa$, there is a notion of forcing $\mathbb{Q}_\lambda$ with the following properties.*

1. *$\mathbb{Q}_\lambda$ preserves cardinals and cofinalities and adds no new κ-sequences.*
2. *$\mathbb{Q}_\lambda \Vdash$ "$\tau(\mathfrak{G}) = 1$".*
3. *In every $\mathbb{Q}_\lambda$-generic forcing extension the following holds.*

   For every sequence $s : \lambda \longrightarrow (\lambda \setminus \{0\})$ there exists a decreasing sequence of ground models *$\langle M^s_\alpha \mid 0 < \alpha < \lambda \rangle$ such that for all $0 < \alpha < \lambda$ the following statements hold.*

   (a) $M^s_{\alpha+1} \models$ "$\tau(\mathfrak{G}) = s(\alpha)$".

   (b) If α is a limit ordinal, then $M^s_\alpha \models$ "$\tau(\mathfrak{G}) = 1$".

   Moreover, if $t : \lambda \longrightarrow (\lambda \setminus \{0\})$, then $s(\alpha) = t(\alpha)$ implies $M^s_{\alpha+1} = M^s_{\alpha+1}$ for all $\alpha < \lambda$ and $M^s_\nu = M^t_\nu$ for all limit ordinals $\nu < \lambda$.

This is Theorem 4.1, proven in Section 4.

Next, the possibilities of passing to a ground model or to a forcing extension can be mixed. In order to make sense of models that are reached by unboundedly often passing to a forcing extension and unboundedly often passing to a ground model, we need a suitable notion of limit. We make this precise and prove in Theorem 5.2, vaguely speaking, that all patterns can be realized, provided that the set of $\alpha < \kappa$ at which one passes to a forcing extension contains a club.

Finally, the last section shows that the lightface Closed Maximality Principle at the successor of a cardinal κ such that $2^{<\kappa} = \kappa$ implies the existence of a centerless group satisfying the statements mentioned in 2. and 3. of Theorem 4.1 with $\lambda = \kappa^+$ in V.

2 Preliminaries

As it is very difficult to compute the automorphism tower of a given group, we will use a technique developed by Simon Thomas which enables us to construct examples of automorphism towers of a given height without the need of computing each automorphism group. The so called *Normalizer Tower Technique* was developed in [Tho85].

Definition 2.1. If H is a subgroup of the group G, then *the normalizer tower* $\langle \mathrm{N}_G^{\alpha}(H) \mid \alpha \in \mathrm{On}\rangle$ *of H in G* is defined inductively as follows.

1. $\mathrm{N}_G^0(H) = H$,
2. $\mathrm{N}_G^{\alpha+1}(H) = \mathrm{N}_G(\mathrm{N}_G^{\alpha}(H)) = \{g \in G \mid g\,\mathrm{N}_G^{\alpha}(H)\,g^{-1} = \mathrm{N}_G^{\alpha}(H)\}$,
3. $\mathrm{N}_G^{\lambda}(H) = \bigcup\{\mathrm{N}_G^{\alpha}(H) \mid \alpha < \lambda\}$, if $\lambda \in \mathrm{Lim}$.

An easy cardinality argument shows that for each group G of cardinality κ and each subgroup H of G there is an $\alpha < \kappa^+$ such that $\mathrm{N}_G^{\alpha}(H) = \mathrm{N}_G^{\alpha+1}(H)$. The normalizer length $\tau_G^{nlg}(H)$ of H in G is the least such α.

The following theorem reduces the problem of manipulating automorphism towers to the problem of manipulating normalizer towers in automorphism groups of first-order structures.

Theorem 2.2 ([Tho85])**.** *Let $\mathcal{M}$ be a structure for the first-order language L and let H be a subgroup of $Aut(\mathcal{M})$. Then there exists a centerless group G such that the statement*

$$\tau(G) = \tau^{nlg}_{Aut(\mathcal{M})}(H)$$

holds and is upwards-absolute between transitive models of ZFC.

We will now summarize the results that we need in order to construct structures whose automorphism groups can be changed by forcing.

We call a pair (G, Ω) a *permutation group*, if G is a subgroup of $\mathrm{Sym}(\Omega)$. Given a family $\langle (G_i, \Omega_i) \mid \in I\rangle$ of permutation groups, the *direct product* of the family is defined to be the permutation group

$$\prod_{i\in I}(G_i, \Omega_i) = \left(\prod_{i\in I} G_i, \bigsqcup_{i\in I}\Omega_i\right),$$

where the direct product of groups acts on the disjoint union of sets in the obvious manner. We say that two permutation groups (G, Ω) and (H, Δ) are *isomorphic*, if there is a bijection $f : \Omega \to \Delta$ such that the induced isomorphism $f^* : \mathrm{Sym}(\Omega) \to \mathrm{Sym}(\Delta), \sigma \mapsto f \circ \sigma \circ f^{-1}$ maps G onto H. We write $(H_0, \Omega_0) \times (H_1, \Omega_1)$ instead of $\prod_{i<2}(H_i, \Omega_i)$ and $\tau^{nlg}(H, \Omega)$ instead of $\tau^{nlg}_{\mathrm{Sym}(\Omega)}(H)$.

For each ordinal α, we inductively define permutation groups $(H_\alpha, \Delta_\alpha)$ and $(F_\alpha, \Delta_\alpha)$ in the following way.

1. $\Delta_0 = \{\emptyset\}$ and $H_0 = F_0 = \{\mathrm{id}_{\Delta_0}\}$,
2. If $\alpha > 0$, then we define

$$\begin{aligned}(H_\alpha, \Delta_\alpha) &= (H_0, \Delta_0) \times \prod_{\beta<\alpha}(F_\beta, \Delta_\beta),\\ F_\alpha &= \mathrm{N}^{\alpha}_{\mathrm{Sym}(\Delta_\alpha)}(H_\alpha).\end{aligned}$$

Note that the second clause directly implies

$$(H_\alpha, \Delta_\alpha) \cong (H_\beta, \Delta_\beta) \times \prod_{\beta \leq \gamma < \alpha} (F_\gamma, \Delta_\gamma)$$

for all $\beta < \alpha$. In order to keep our calculation clear, we also define

$$(H^*_\alpha, \Delta^*_\alpha) = (H_\alpha, \Delta_\alpha) \times (F_1, \Delta_1) \times (F_1, \Delta_1)$$

for $\alpha > 1$.

An easy induction shows $\max(\{\omega, \overline{\overline{\alpha}}\})$ is an upper bound for the cardinality of Δ_α and this means that the definitions of $(H_\alpha, \Delta_\alpha)$ and $(F_\alpha, \Delta_\alpha)$ are absolute between models with the same α-sequences of ordinals, because the symmetric group of Δ_β is the same in those models for all $\beta \leq \alpha$.

These permutation groups are the first ingredient in our construction. The following theorem summarizes their important properties deduced in the algebraic part of [HT00].

Theorem 2.3. *For each ordinal α, the following statements hold.*

1. $\tau^{nlg}(H_\alpha, \Delta_\alpha) = \alpha$.
2. $\tau^{nlg}(F_\alpha, \Delta_\alpha) = 0$.
3. *If $\alpha > 1$, then* $\tau^{nlg}(H^*_\alpha, \Delta^*_\alpha) = 1$.

Proof. The first statement is [HT00, Lemma 2.10] and the second statement follows directly from the first, together with the definition of F_α. The third statement is [HT00, Lemma 2.14] with $\beta = 1$. □

The trees $\langle C_\alpha \mid \alpha < \kappa \rangle$ and $\langle T_\alpha \mid \alpha < \kappa \rangle$ constructed in 1.10 are the second ingredient in our construction. By coding the trees T_α into connected graphs[1] (see [Tho, Theorem 4.1.8]), we see that under Assumption 1.8 there exists a sequence $\langle \Gamma_\alpha \mid \alpha < \kappa \rangle$ of rigid, pairwise non-isomorphic connected graphs such that for every bounded equivalence relation E on κ the notion of forcing C_E mentioned in Theorem 1.10 is able to realize E.

If E and F are equivalence relations on κ, then we define

$$E \preceq F \Leftrightarrow E \subseteq F \wedge (\forall \alpha < \kappa)([\alpha]_E \neq \{\alpha\} \rightarrow \min[\alpha]_E = \min[\alpha]_F).$$

Note that, as the notation suggests, $\preceq$ is a reflexive, transitive relation. Moreover, by checking the definition of the forcing C_E in Theorem 1.10, we arrive at the following observation.

Observation 2.4 (Under Assumption 1.8)**.** *If $E \preceq F$ are bounded equivalence relations on κ, then the forcing C_F extends C_E, in the strong sense that there is a partial order $\mathbb{Q}$ such that $C_F \cong C_E \times \mathbb{Q}$.* □

The following construction allows us to combine the two ingredients.

If $\langle \Gamma_i = (X_i, E_i) \mid i \in I \rangle$ is a family of graphs, then we define the *direct sum* of the family to be the graph

$$\bigoplus_{i \in I} \Gamma_i = \left(\bigsqcup_{i \in I} X_i, \bigsqcup_{i \in I} E_i \right)$$

obtained by taking the disjoint unions of the sets of vertices and edges, respectively.

[1]By a graph (without further qualification), we mean a nondirected graph.

We call a pair (G, Γ) a *graph permutation group*, if Γ is a graph and G is a subgroup of $\mathrm{Aut}(\Gamma)$. As above, if a $\langle (G_i, \Gamma_i) \mid i \in I \rangle$ is a family of graph permutation groups, then we define the *direct product* of the family to be the graph permutation groups

$$\prod_{i \in I} (G_i, \Gamma_i) = \left(\prod_{i \in I} G_i, \bigoplus_{i \in I} \Gamma_i \right),$$

where the product of groups acts on the direct sum of graphs in the obvious way. We say that two graph permutation groups are *isomorphic*, if there is an isomorphism of the underlying graphs such that the induced isomorphism of automorphism groups maps the subgroups correctly. Again, we write $(G_0, \Gamma_0) \times (G_1, \Gamma_1)$ instead of $\prod_{i<2}(G_i, \Gamma_i)$ and $\tau^{nlg}(G, \Gamma)$ instead of $\tau^{nlg}_{\mathrm{Aut}(\Gamma)}(G)$.

If Ω is a set and Γ is a graph, then we define

$$\mathcal{G}_\Omega(\Gamma) = \bigoplus_{x \in \Omega} \Gamma$$

to be the graph obtained by replacing each element of Ω by a copy of Γ. We can embed $\mathrm{Sym}(\Omega)$ into $\mathrm{Aut}(\mathcal{G}_\Omega(\Gamma))$ in a natural way and, if Γ is connected and rigid, then it is not hard to show that this embedding is an isomorphism.

If (G, Ω) is a permutation group, then we get a new graph permutation group $(G(\Gamma), \mathcal{G}_\Omega(\Gamma))$, where $G(\Gamma)$ is the image of G under the above embedding of $\mathrm{Sym}(\Omega)$ into $\mathrm{Aut}(\mathcal{G}_\Omega(\Gamma))$.

In the following lemma, we list facts about graph permutation groups used in the algebraic part of [HT00]. They will play an important role in our later constructions, because they will enable us to compute normalizer towers in products of graph permutation groups.

Lemma 2.5. *If $\vec{\Gamma} = \langle \Gamma_i \mid i \in I \rangle$ is a sequence of connected rigid graphs and $\langle (G_i, \Omega_i) \mid i \in I \rangle$ is a sequence of permutation groups, then then following statements hold for all $i_0 \in I$.*

1. $\tau^{nlg}\left(G_{i_0}(\Gamma_{i_0}), \mathcal{G}_{\Omega_{i_0}}(\Gamma_{i_0})\right) = \tau^{nlg}(G_{i_0}, \Omega_{i_0})$.

2. *If $\vec{\Gamma}$ consists of pairwise non-isomorphic graphs, $\tau^{nlg}(G_{i_0}, \Omega_{i_0}) \geq 1$ and $\tau^{nlg}(G_j, \Omega_j) \leq 1$ holds for all $j \in I \setminus \{i_0\}$, then*

$$\tau^{nlg}\left(\prod_{i \in I} (G_i(\Gamma_i), \mathcal{G}_{\Omega_i}(\Gamma_i)) \right) = \tau^{nlg}(G_{i_0}, \Omega_{i_0}).$$

3. *If $\vec{\Gamma}$ consists of pairwise isomorphic graphs and $(G, \Omega) = \prod_{i \in I} (G_i, \Omega_i)$, then*

$$(G(\Gamma_{i_0}), \mathcal{G}_\Omega(\Gamma_{i_0})) \cong \prod_{i \in I} (\Gamma_i(G_i), \mathcal{G}_{\Omega_i}(\Gamma_i)).$$

Proof. By the assumption, the embedding of $\mathrm{Sym}(\Omega_i)$ into $\mathrm{Aut}(\mathcal{G}_{\Omega_i}(\Gamma_i))$ is an isomorphism and maps G onto $G(\Gamma_i)$. This proves the first statement.

The set of connected components of $\prod_{i \in I} (G_i(\Gamma_i), \mathcal{G}_{\Omega_i}(\Gamma_i))$ consists of a copy of Γ_i for each element of Ω_i and each $i \in I$. If all Γ_i's are pairwise non-isomorphic, then each subgraph of the form $\mathcal{G}_{\Omega_i}(\Gamma_i)$ is invariant under all automorphisms and therefore each automorphism of the graph is induced by an element of the group $\prod_{i \in I} \mathrm{Aut}(\mathcal{G}_{\Omega_i}(\Gamma_i))$ acting on the graph in the obvious way. By the rigidity of the Γ_i's, this means that the automorphism group of $\bigoplus_{i \in I} \mathcal{G}_{\Omega_i}(\Gamma_i)$ is isomorphic to $\prod_{i\in I}\mathrm{Sym}(\Omega_i)$ and this isomorphism sends $\prod_{i\in I} G_i(\Gamma_i)$ to $\prod_{i\in I} G_i$. An easy induction then shows

$$\mathrm{N}^{\alpha}_{\prod_{i\in I}\mathrm{Sym}(\Omega_i)}\left(\prod_{i\in I} G_i\right) \cong \mathrm{N}^{\alpha}_{\mathrm{Sym}(\Omega_{i_0})}(G_{i_0}) \times \prod_{j\in I\setminus\{i_0\}} \mathrm{N}^{1}_{\mathrm{Sym}(\Omega_j)}(G_j)$$

for all $\alpha > 0$ and, by the existence of the above isomorphism, this proves the second statement.

Each automorphism of $\bigoplus_{i\in I}\mathcal{G}_{\Omega_i}(\Gamma_i)$ that fixes a connected component setwise also fixes it pointwise by rigidity. This shows that the natural isomorphism between $\bigoplus_{i\in I}\mathcal{G}_{\Omega_i}(\Gamma_i)$ and $\bigoplus_{j\in I}\mathcal{G}_{\Omega_j}(\Gamma_{i_0})$ induced by the ismorphisms between Γ_{i_0} and the Γ_i's is also an ismorphism between the graph permutation groups $(G(\Gamma_{i_0}),\mathcal{G}_{\Omega}(\Gamma_{i_0}))$ and $\prod_{i\in I}(\Gamma_i(G_i),\mathcal{G}_{\Omega_i}(\Gamma_i))$. □

We now introduce the group $\mathfrak{G}$ which is the protagonist of the present article. Fix, once and for all, a sequence $\langle (G_\alpha,\Omega_\alpha) \mid \alpha<\kappa\rangle$ of permutation groups such that each (G_α,Ω_α) is of the form $(F_{\bar\alpha},\Delta_{\bar\alpha})$, for some $\bar\alpha<\kappa$, and such that for every $\beta<\kappa$, the set of $\delta<\kappa$ such that $(G_\delta,\Omega_\delta)=(F_\beta,\Delta_\beta)$ is unbounded in κ. So for example, using the Gödel pairing function, we could let $(G_\gamma,\Omega_\gamma)=(F_\alpha,\Delta_\alpha)$, if $\gamma=\prec\alpha,\beta\succ<\kappa$. We write $\mathcal{G}_\alpha(\Gamma)$ instead of $\mathcal{G}_{\Omega_\alpha}(\Gamma)$.

Definition 2.6. If $\vec{\Pi}=\langle \Pi_\alpha \mid \alpha<\kappa\rangle$ is a sequence of graphs, then we define

$$\mathcal{G}(\vec{\Pi}) = \prod_{\alpha<\kappa}\left(G_\alpha(\Pi_\alpha),\mathcal{G}_\alpha(\Pi_\alpha)\right).$$

As noted above, the definition of $\mathcal{G}(\vec{\Pi})$ is absolute between models with the same κ-sequences of ordinals that contain $\vec{\Pi}$.

Under Assumption 1.8, we also fix sequences $\vec{T}=\langle T_\alpha \mid \alpha<\kappa\rangle$ and $\vec{C}=\langle C_{\alpha,\beta} \mid \alpha<\beta<\kappa\rangle$ as in Theorem 1.10, as well as the corresponding sequence $\vec{\Gamma}=\langle \Gamma_\alpha \mid \alpha<\kappa\rangle$ of graphs coding the trees $\langle T_\alpha \mid \alpha<\kappa\rangle$.

Definition 2.7. Let $\mathfrak{G}=\mathfrak{G}_\kappa$ be the centerless group the existence of which is postulated in Theorem 2.2, with respect to $\mathcal{G}(\vec{\Gamma})$.

So by definition, $\tau(\mathfrak{G})=\tau^{nlg}(\mathcal{G}(\vec{\Gamma}))$ holds and is upwards-absolute. Hence we can change the height of the automorphism tower of $\mathfrak{G}$ by changing the height of the normalizer tower of $\mathcal{G}(\vec{\Gamma})$ in the corresponding symmetric group.

Since all Γ_α are rigid and pairwise non-isomorphic and $\tau^{nlg}(G_\alpha,\Omega_\alpha)=\tau^{nlg}(F_{\bar\alpha},\Delta_{\bar\alpha})=0$, we may use Theorem 2.3 and the second part of Lemma 2.5 to get the following statement.

Observation 2.8 (Under Assumption 1.8)**.** $\tau(\mathfrak{G})=\tau^{nlg}(\mathcal{G}(\vec{\Gamma}))=0$. □

3 Consecutive Forcing Extensions

To make the following constructions clearer, we introduce some vocabulary. We would like to remind the reader that we are working under Assumption 1.8, and that we have fixed the objects mentioned at the end of the previous section.

Definition 3.1. Let X be a subset of κ with monotone enumeration $\langle \gamma_\alpha \mid \alpha<\mathrm{otp}(X)\rangle$.

1. We call X *active* if the order type of X is of the form $\mathrm{otp}(X)=\beta+1>2$ and
   (a) For all $\alpha<\beta$, $(G_{\gamma_\alpha},\Omega_{\gamma_\alpha})=(F_\alpha,\Delta_\alpha)$.
   (b) $(G_{\gamma_\beta},\Omega_{\gamma_\beta})=(F_0,\Delta_0)$.

2. We call X *sealed* if the order type of X is of the form $\mathrm{otp}(X) = \beta + 3$, $X \cap (\gamma_\beta + 1)$ is active and $\left(G_{\gamma_{\beta+1}}, \Omega_{\gamma_{\beta+1}}\right) = \left(G_{\gamma_{\beta+2}}, \Omega_{\gamma_{\beta+2}}\right) = (F_1, \Delta_1)$.

3. If X is a sealed subset of κ with order type $\beta + 3$ and $1 < \bar{\beta} \leq \beta$, then $\{\gamma_\alpha | \alpha < \bar{\beta}\} \cup \{\gamma_\beta\}$ is *the active segment of X of order type $\bar{\beta} + 1$*.

4. We call X *trimmed*, $\mathrm{otp}(X) = 2$ and $(G_{\gamma_0}, \Omega_{\gamma_0}) = (G_{\gamma_1}, \Omega_{\gamma_1}) = (F_0, \Delta_0)$. If Y is an active subset of κ with monotone enumeration $\langle \delta_\alpha \mid \alpha < \beta + 1 \rangle$ or a sealed subset of κ with monotone enumeration $\langle \delta_\alpha \mid \alpha < \beta + 3 \rangle$, then $\{\delta_0, \delta_\beta\}$ is *the trimmed segment of Y*.

So the permutation groups associated to a sealed subset X of κ with monotone enumeration $\langle \gamma_\alpha \mid \alpha < \beta + 3 \rangle$ look as follows:

$$\begin{array}{cccccccc} (G_{\gamma_0}, \Omega_{\gamma_0}) & (G_{\gamma_1}, \Omega_{\gamma_1}), & (G_{\gamma_2}, \Omega_{\gamma_2}) & \ldots & (G_{\gamma_\beta}, \Omega_{\gamma_\beta}) & (G_{\gamma_{\beta+1}}, \Omega_{\gamma_{\beta+1}}) & (G_{\gamma_{\beta+2}}, \Omega_{\gamma_{\beta+2}}) \\ \| & \| & \| & & \| & \| & \| \\ (F_0, \Delta_0) & (F_1, \Delta_1) & (F_2, \Delta_2) & \ldots & (F_0, \Delta_0) & (F_1, \Delta_1) & (F_1, \Delta_1) \end{array}$$

Note that a sealed subset of κ must have order type at least 5. By definition, the following equation holds for the above set X.

$$\prod_{\delta \in X} (G_\delta, \Omega_\delta) = (F_0, \Delta_0) \times \left(\prod_{\alpha < \beta} (F_\alpha, \Delta_\alpha) \right) \times (F_1, \Delta_1) \times (F_1, \Delta_1) = \left(H^*_\beta, \Delta^*_\beta \right). \tag{1}$$

If $\bar{\beta} \leq \beta$ and Y is the active segment of X of order type $\bar{\beta} + 1$, then the following equation holds.

$$\prod_{\delta \in Y} (G_\delta, \Omega_\delta) = (F_0, \Delta_0) \times \prod_{\alpha < \bar{\beta}} (F_\alpha, \Delta_\alpha) = \left(H_{\bar{\beta}}, \Delta_{\bar{\beta}} \right). \tag{2}$$

Finally, if $Z = \{\xi_0, \xi_1\}$ is a trimmed subset of κ, then the following equation holds.

$$\prod_{\delta \in Z} (G_\delta, \Omega_\delta) = (F_0, \Delta_0) \times (F_0, \Delta_0) = (H_1, \Delta_1). \tag{3}$$

We extend the above definitions to equivalence relations on κ and show how we can use them to change the height of the automorphism tower of $\mathfrak{G}$.

Definition 3.2. Let E be a non-trivial equivalence relation on κ.

1. We call E *inactive*, if every non-trivial equivalence class is either a sealed or a trimmed subset of κ.

2. We call E *active*, if all non-trivial E-equivalence classes are either active, sealed or trimmed subsets of κ and there is a unique active E-equivalence class.

Lemma 3.3 (Under Assumption 1.8)**.** *If E is a bounded, inactive equivalence relation on κ, then $C_E \Vdash$ "$\tau(\mathfrak{G}) = 1$".*

Proof. We work in V^{C_E}. As noted after Definition 2.6, $\mathcal{G}(\vec{\Gamma}) = \prod_{\alpha<\kappa} (G_\alpha(\Gamma_\alpha), \mathcal{G}_\alpha(\Gamma_\alpha))$ still holds. Let S denote the set of all sealed E-equivalence classes, and for $c \in S$, let $\langle \gamma^c_\alpha \mid \alpha < \beta^c + 3 \rangle$ be the monotone enumeration of c. Define T to be the set of all trimmed E-equivalence classes and let $d = \{\xi_0^d, \xi_1^d\}$ for each $d \in T$. Finally, let N denote the union of all trivial E-equivalence classes. Using the third part of Lemma 2.5 and the equations (1) and (3), the following holds in V^{C_E}.

$$\begin{aligned}\mathcal{G}(\vec{\Gamma}) \cong \left(\prod_{\alpha\in N}(G_\alpha(\Gamma_\alpha), \mathcal{G}_\alpha(\Gamma_\alpha))\right) \times \left(\prod_{c\in S}\left(\prod_{\delta\in c}(G_\delta(\Gamma_{\gamma_0^c}), \mathcal{G}_\delta(\Gamma_{\gamma_0^c}))\right)\right)\\ \times \prod_{d\in T}\Big((G_{\xi_0^d}(\Gamma_{\xi_0^d}), \mathcal{G}_{\xi_0^d}(\Gamma_{\xi_0^d})) \times (G_{\xi_1^d}(\Gamma_{\xi_0^d}), \mathcal{G}_{\xi_1^d}(\Gamma_{\xi_0^d}))\Big)\\ \cong \left(\prod_{\alpha\in N}(G_\alpha(\Gamma_\alpha), \mathcal{G}_\alpha(\Gamma_\alpha))\right) \times \left(\prod_{c\in S}(H^*_{\beta^c}(\Gamma_{\gamma_0^c}), \mathcal{G}_{\Delta^*_{\beta^c}}(\Gamma_{\gamma_0^c}))\right) \times \prod_{d\in T}(H_1(\Gamma_{\xi_0^d}), \mathcal{G}_{\Delta_1}(\Gamma_{\xi_0^d})).\end{aligned}$$

By assumption, all graphs appearing in this product are rigid and pairwise non-isomorphic. The first part of Lemma 2.5 and Theorem 2.3 now yield:

1. For all $\alpha \in N$, $\tau^{nlg}\left(G_\alpha(\Gamma_\alpha), \mathcal{G}_\alpha(\Gamma_\alpha)\right) = \tau^{nlg}\left(G_\alpha, \Omega_\alpha\right) = 0$,
2. for all $c \in S$, $\tau^{nlg}(H^*_{\beta^c}(\Gamma_{\gamma_0^c}), \mathcal{G}_{\Delta^*_{\beta^c}}(\Gamma_{\gamma_0^c})) = \tau^{nlg}(H^*_{\beta^c}, \Delta^*_{\beta^c}) = 1$,
3. for all $t \in T$, $\tau^{nlg}(H_1(\Gamma_{\xi_0^d}), \mathcal{G}_{\Delta_1}(\Gamma_{\xi_0^d})) = \tau^{nlg}(H_1, \Delta_1) = 1$.

By definition, there is at least one non-trivial equivalence class and we can therefore apply the second part of Lemma 2.5 to see that $\tau(\mathfrak{G}) = \tau^{nlg}(\mathcal{G}(\vec{\Gamma})) = 1$ holds. □

Lemma 3.4 (Under Assumption 1.8). *Let E be a bounded, active equivalence relation on κ. If e is the unique active E-equivalence class, then $C_E \Vdash$ "$\tau(\mathfrak{G}) + 1 = \mathrm{otp}(e, <)$".*

Proof. We work in V^{C_E}. By the definition of active subsets, the monotone enumeration of e is of the form $\langle \gamma_\alpha \mid \alpha < \beta + 1\rangle$ for some $1 < \beta < \kappa$. Define N, S, T, γ^c_α and ξ^d_i as in the proof of Lemma 3.3. Using the third part of Lemma 2.5 and the equations (1)-(3), we get the following equalities.

$$\begin{aligned}\mathcal{G}(\vec{\Gamma}) \cong \left(\prod_{\alpha\in N}(G_\alpha(\Gamma_\alpha), \mathcal{G}_\alpha(\Gamma_\alpha))\right) \times \left(\prod_{c\in S}\left(\prod_{\delta\in c}(G_\delta(\Gamma_{\gamma_0^d}), \mathcal{G}_\delta(\Gamma_{\gamma_0^d}))\right)\right)\\ \times \left(\prod_{d\in T}\Big((G_{\xi_0^d}(\Gamma_{\xi_0^d}), \mathcal{G}_{\xi_0^d}(\Gamma_{\xi_0^d})) \times (G_{\xi_1^d}(\Gamma_{\xi_0^d}), \mathcal{G}_{\xi_1^d}(\Gamma_{\xi_0^d}))\Big)\right) \times \prod_{\delta\in e}(G_\delta(\Gamma_{\gamma_0}), \mathcal{G}_\delta(\Gamma_{\gamma_0}))\\ \cong \left(\prod_{\alpha\in N}(G_\alpha(\Gamma_\alpha), \mathcal{G}_\alpha(\Gamma_\alpha))\right) \times \left(\prod_{c\in S}(H^*_{\beta^c}(\Gamma_{\gamma_0^c}), \mathcal{G}_{\Delta^*_{\beta^c}}(\Gamma_{\gamma_0^c}))\right) \times \left(\prod_{d\in T}(H_1(\Gamma_{\xi_0^d}), \mathcal{G}_{\Delta_1}(\Gamma_{\xi_0^d}))\right)\\ \times (H_\beta(\Gamma_{\gamma_0}), \mathcal{G}_{\Delta_\beta}(\Gamma_{\gamma_0})).\end{aligned}$$

Again, all graphs in this products are rigid and pairwise non-isomorphic and

$$\tau^{nlg}\left(H_\beta(\Gamma_{\gamma_0}), \mathcal{G}_{\Delta_\beta}(\Gamma_{\gamma_0})\right) = \tau^{nlg}\left(H_\beta, \Delta_\beta\right) = \beta > 1.$$

By the second part of Lemma 2.5 and the computations made in the proof of Lemma 3.3,

$$\tau(\mathfrak{G}) + 1 = \tau^{nlg}(\mathcal{G}(\vec{\Gamma})) + 1 = \beta + 1 = \mathrm{otp}(e, <).$$

□

Next, we define a family of functions that allows us the construction of special bounded equivalence relations in our proofs of the theorems. Remember that for each $\alpha < \kappa$ the set $\{\beta < \kappa \mid (G_\beta, \Omega_\beta) = (F_\alpha, \Delta_\alpha)\}$ is unbounded in κ.

Lemma 3.5. *For each $s : \kappa \longrightarrow (\kappa \setminus \{0,1\})$, there exists a function $s^* : \kappa \to [\kappa]^{<\kappa}$ with the following properties.*

1. *If $\beta < \alpha$, then $s^*(\beta) \subseteq \min(s^*(\alpha))$.*
2. *For all $\alpha < \kappa$, $s^*(\alpha)$ is a sealed subset of κ with* $\mathrm{otp}(s^*(\alpha), <) = s(\alpha) + 3$.[2]

Proof. Assume $s^* \restriction \alpha$ is already defined, for some $\alpha < \kappa$. We define $s^*(\alpha) = \{\gamma^\alpha_\delta \mid \delta < s(\alpha) + 3\}$ where $\langle \gamma^\alpha_\delta \mid \delta < s(\alpha) + 3\rangle$ is defined as follows: γ^α_0 is the least $\nu < \kappa$ such that $\bigcup\{s^*(\beta) | \beta < \alpha\} \subseteq \nu$ and $(G_\nu, \Omega_\nu) = (F_0, \Delta_0)$. If $0 < \delta < s(\alpha)$ and $\langle \gamma^\alpha_\xi \mid \xi < \delta\rangle$ is already defined, then γ^α_δ is the least $\nu < \kappa$ such that $\nu > \sup(\{\gamma^\alpha_\xi | \xi < \delta\})$ and $(G_\nu, \Omega_\nu) = (F_\delta, \Delta_\delta)$. Finally, $\gamma^\alpha_{s(\alpha)}$ is the least $\nu < \kappa$ such that $\nu > \sup(\{\gamma^\alpha_\delta | \delta < s(\alpha)\})$ and $(G_\nu, \Omega_\nu) = (F_0, \Delta_0)$, $\gamma^\alpha_{s(\alpha)+1}$ is the least $\nu < \kappa$ such that $\nu > \gamma^\alpha_{s(\alpha)}$ and $(G_\nu, \Omega_\nu) = (F_1, \Delta_1)$, and $\gamma^\alpha_{s(\alpha)+2}$ is the least $\nu < \kappa$ such that $\nu > \gamma^\alpha_{s(\alpha)+1}$ and $(G_\nu, \Omega_\nu) = (F_1, \Delta_1)$. □

From now on, we fix an operator $s \mapsto s^*$ with the above properties. We may also assume that if $s, t : \kappa \longrightarrow (\kappa \setminus \{0,1\})$ are such that $s \restriction \gamma = t \restriction \gamma$, for some $\gamma < \kappa$, then $s^* \restriction \gamma = t^* \restriction \gamma$. For each $s : \kappa \longrightarrow (\kappa \setminus \{0,1\})$ and each $\alpha < \kappa$ we define a bounded, inactive equivalence relation E^s_α on κ by

$$\gamma E^s_\alpha \delta \Leftrightarrow \gamma = \delta \vee (\exists \beta < \alpha)\gamma, \delta \in s^*(\beta).$$

It is easy to see that $\alpha < \beta < \kappa$ implies $E^s_\alpha \preceq E^s_\beta$.

Definition 3.6. Let E be a bounded equivalence relation on κ. If E is active and e is the unique active E-equivalence classe, then we define $\mathrm{ht}(E)$ to be the unique ordinal α with $\mathrm{otp}(e, <) = \alpha + 1$. If E is inactive, then we define $\mathrm{ht}(E) = 1$.

As an illustration of the concepts introduced above, note the following observation which is a direct consequence of Lemmas 3.3 and 3.4.

Observation 3.7 (Under Assumption 1.8)**.** *If E is a bounded equivalence relation on κ and E is either active or inactive, then $C_E \Vdash$ "$\tau(\mathfrak{G}) = \mathrm{ht}(E)$".* □

Next, we want to analyze $\preceq$-ascending and -descending chains of equivalence relations.

Definition 3.8. Let $\vec{A} = \langle A_\alpha \mid \alpha < \beta\rangle$ be a sequence of sets. We say that $\vec{A}$ *converges*, if for every x there is an $\alpha < \beta$ such that either $x \in A_\gamma$ for all $\alpha \le \gamma < \beta$ or $x \notin A_\gamma$ for all $\alpha \le \gamma < \beta$. If $\vec{A}$ converges, then we define the limit of $\vec{A}$ to be the set

$$\lim_{\alpha \to \beta} A_\alpha = \bigcup_{\alpha < \beta} \bigcap_{\alpha \le \gamma < \beta} A_\gamma.$$

If $\beta = 0$ or $\beta = \alpha + 1$, then $\vec{A}$ automatically converges. Namely, $\lim_{\gamma \to 0} A_\gamma = \emptyset$, and $\lim_{\gamma \to \alpha+1} A_\gamma = A_\alpha$. Trivially, if $\vec{A}$ is increasing (in the inclusion relation), then $\vec{A}$ converges with limit $\bigcup_{\alpha<\beta} A_\alpha$, and if it is decreasing, then it converges with limit $\bigcap_{\alpha<\beta} A_\alpha$. It is easy to see that if $\vec{A}$ is a convergent sequence of equivalence relations on a set I, then $\lim \vec{A}$ is also an equivalence relation on I.

We will apply the following facts in the proofs of the first two main results. They follow directly from the above remarks and the transitivity of "$\preceq$".

[2]Remember that a sealed subset of κ must have order type at least 5. This is why we require $s(\alpha) > 1$ here.

Observation 3.9. *Let $\langle E_\alpha \mid \alpha < \kappa \rangle$ be a sequence of equivalence relations on κ.*

1. *If $E_\gamma \preceq E_\beta$ holds for all $\gamma < \beta < \kappa$, then $\langle E_\beta \mid \beta < \alpha \rangle$ converges for all $\alpha < \kappa$ and $\lim_{\beta \to \bar\alpha} E_\beta \preceq \lim_{\beta \to \alpha} E_\beta$ holds for all $\bar\alpha \leq \alpha < \kappa$.*

2. *If $E_\beta \preceq E_\gamma$ holds for all $\gamma < \beta < \kappa$, then $\langle E_\beta \mid \beta < \alpha \rangle$ converges for all $\alpha < \kappa$ and $\lim_{\beta \to \alpha} E_\beta \preceq \lim_{\beta \to \bar\alpha} E_\beta$ holds for all $\bar\alpha \leq \alpha < \kappa$.* □

We are now ready to apply our methods and constructions in order to prove the main theorem of this section:

Theorem 3.10 (Under Assumption 1.8)**.** *For every function $s : \kappa \longrightarrow \kappa \setminus \{0\}$, there is a sequence of partial orders $\langle \mathbb{P}^s_\gamma \mid 0 < \gamma < \kappa \rangle$, such that the following statements hold for each $0 < \alpha < \kappa$.*

1. *$\mathbb{P}^s_\alpha$ preserves cardinals and cofinalities and adds no new κ-sequences.*

2. *$\mathbb{P}^s_{\alpha+1} \Vdash$ "$\tau(\mathfrak{G}) = s(\alpha)$".*

3. *If α is a limit ordinal, then $\mathbb{P}^s_\alpha \Vdash$ "$\tau(\mathfrak{G}) = 1$".*

4. *If $\beta < \alpha$, then $\mathbb{P}^s_\alpha$ extends $\mathbb{P}^s_\beta$ (in the sense that $\mathbb{P}^s_\alpha \cong \mathbb{P}^s_\beta \times \mathbb{Q}$, for some poset $\mathbb{Q}$).*

Moreover, if $t : \kappa \longrightarrow \kappa \setminus \{0\}$, and $s \upharpoonright \gamma = t \upharpoonright \gamma$ for some $0 < \gamma < \kappa$, then $\mathbb{P}^s_\gamma = \mathbb{P}^t_\gamma$.

Proof. For a given $s : \kappa \longrightarrow (\kappa \setminus \{0\})$, let $s + 1$ be the function with domain κ defined by $s + 1(\alpha) = s(\alpha) + 1$, as usual. We construct a sequence $\langle E_\alpha \mid \alpha < \kappa \rangle$ of equivalence relations on κ by defining the nontrivial equivalence classes of each relation. For $\alpha < \kappa$, a subset $Z \subseteq \kappa$ is a nontrivial equivalence class of E_α iff one of the following conditions holds:

1. $Z = (s + 1)^*(\beta)$, for some $\beta < \alpha$,

2. $s(\alpha) = 1$ and $Z = (s + 1)^*(\alpha)$,

3. $s(\alpha) > 1$ and Z is the active segment of $(s + 1)^*(\alpha)$ of order type $s(\alpha) + 1$.

It is easy to check that the following claims hold for all $\alpha < \kappa$:

(1) *E_α is bounded and either active or inactive. Moreover,* $\mathrm{ht}(E_\alpha) = s(\alpha)$.

(2) *For all $\beta < \alpha$, $E_\beta \preceq E_\alpha$. In particular, $\langle E_\beta \mid \beta < \alpha \rangle$ converges and $E^*_\alpha = \lim_{\beta \to \alpha} E_\beta$ is a bounded equivalence relation on κ.*

For each $\alpha < \kappa$, we define $\mathbb{P}^s_\alpha = C_{E^*_\alpha}$. These forcings satisfy the first property of the theorem by the first statement of Theorem 1.10. By the first part of Observation 3.9, if $\beta < \alpha < \kappa$, then $E^*_\beta \preceq E^*_\alpha$ and we can use Observation 2.4 to see that the forcings satisfy the last property of the theorem.

Let $\alpha < \kappa$. We have $E^*_{\alpha+1} = E_\alpha$ and therefore $\mathbb{P}^s_{\alpha+1} = C_{E_\alpha} \Vdash \tau(\mathfrak{G}) = \mathrm{ht}(E_\alpha) = s(\alpha)$. If α is a limit ordinal, then it is not hard to show that $E^*_\alpha = \lim_{\beta \to \alpha} E_\beta = \bigcup_{\beta < \alpha} E_\beta = E^{s+1}_\alpha$ holds and this means $\mathbb{P}^s_\alpha = C_{E^{s+1}_\alpha} \Vdash \tau(\mathfrak{G}) = \mathrm{ht}(E^{s+1}_\alpha) = 1$, because E^{s+1}_α is an inactive bounded equivalence relation on κ.

Finally, if $s \upharpoonright \gamma = t \upharpoonright \gamma$ for $s, t \in {}^\kappa\kappa$ and $\gamma < \kappa$, then we also have $s^* \upharpoonright \gamma = t^* \upharpoonright \gamma$ and it is easy to check that the above construction yields the same equivalence relations E_δ for all $\delta < \gamma$. Since $E^*_\gamma = \lim_{\delta \to \gamma} E_\delta$, the resulting E^*_γ coincide, and therefore $\mathbb{P}^s_\gamma = \mathbb{P}^t_\gamma$. □

4 Consecutive Ground Models

In this section, we prove the second main result:

Theorem 4.1 (Under Assumption 1.8). *For every ordinal $\lambda < \kappa$, there is a notion of forcing $\mathbb{Q}_\lambda$ with the following properties.*

1. *$\mathbb{Q}_\lambda$ preserves cardinals and cofinalities and adds no new κ-sequences.*

2. *$\mathbb{Q}_\lambda \Vdash$ "$\tau(\mathfrak{G}) = 1$".*

3. *In every $\mathbb{Q}_\lambda$-generic forcing extension the following holds.*

   For every sequence $s : \lambda \longrightarrow (\lambda \setminus \{0\})$ there exists a decreasing sequence of ground models *$\langle M^s_\alpha \mid 0 < \alpha < \lambda\rangle$ such that for all $0 < \alpha < \lambda$ the following statements hold.*

   (a) $M^s_{\alpha+1} \models$ "$\tau(\mathfrak{G}) = s(\alpha)$".
   (b) If α is a limit ordinal, then $M^s_\alpha \models$ "$\tau(\mathfrak{G}) = 1$".

   Moreover, if $t : \lambda \longrightarrow (\lambda \setminus \{0\})$, then $s(\alpha) = t(\alpha)$ implies $M^s_{\alpha+1} = M^s_{\alpha+1}$ for all $\alpha < \lambda$ and $M^s_\nu = M^t_\nu$ for all limit ordinals $\nu < \lambda$.

Before proving this theorem, we would like to comment on the first order expressibility of its statement. It is by now a well-known fact that every ground model is uniformly definable in a parameter, see [Lav07]. Even this *fact*, though, may at first not seem to be first order expressible. But here is a simple way to state it: There is a first order formula $\varphi(x, y)$ in the language of set theory[3] such that the following is provable in ZFC:

$$(\forall \mathbb{P})(\forall z) \quad \left[(\mathbb{P} \text{ is a partial order and } z = \mathcal{P}(\overline{\overline{\mathbb{P}}}^+)) \implies \mathbb{P} \Vdash \check{\mathrm{V}} = \{x \mid \varphi(x, z)\}\right]$$

Vice versa, given a set z, it is a simple matter to check whether $\{x \mid \varphi(x, z)\}$ is a ZFC model of which the universe is a forcing extension. So point 2. of the theorem can be expressed by saying that for every sequence $s : \lambda \longrightarrow \lambda \setminus \{0\}$, there is a sequence $\langle z_\alpha \mid 0 < \alpha < \lambda\rangle$ of sets such that, for all $0 < \alpha < \lambda$, $M^s_\alpha := \{x \mid \varphi(x, z_\alpha)\}$ is a ground model and (a), (b) hold as stated. Formulating the additional requirement in 2. doesn't pose a problem either. So let's turn to the proof.

Proof of Theorem 4.1. Let $t : \kappa \longrightarrow \kappa$ denote the function with constant value $\lambda + 2$, and let t^* be the function given by Lemma 3.5. We define E to be the bounded, sealed equivalence relation E^t_λ on κ, i.e.

$$\mu E \eta \Leftrightarrow \mu = \eta \vee (\exists \alpha < \lambda)\mu, \eta \in t^*(\alpha).$$

Set $\mathbb{Q}_\lambda = C_E$. By Theorem 1.10 and Lemma 3.3, $\mathbb{Q}_\lambda$ satisfies the first and the second statement.

Let $\mathrm{V}[G]$ be a $\mathbb{Q}_\lambda$-generic extension and let $s : \lambda \longrightarrow (\lambda \setminus \{0\})$ be a sequence in $\mathrm{V}[G]$. By the above remark, s is already an element of V and we can make the following definitions there.

For $\alpha < \lambda$, we define an equivalence relation E_α on κ by specifying that $Z \subseteq \kappa$ is a nontrivial equivalence class of E_α iff one of the following conditions holds:

1. $Z = t^*(\beta)$, for some $\alpha < \beta < \lambda$,

2. $s(\alpha) = 1$ and $Z = t^*(\alpha)$,

[3]Of course, this existential quantification can be eliminated by writing down the formula φ explicitly, but the details of its definition are irrelevant for our purposes.

3. $s(\alpha) > 1$ and Z is the active segment of $t^*(\alpha)$ of order type $s(\alpha)+1$.

Again, the following claims are obvious for all $\alpha < \lambda$:

(1) *E_α is bounded and either active or inactive. Moreover,* $\mathrm{ht}(E_\alpha) = s(\alpha)$.

(2) *For all $\beta < \alpha$, $E_\alpha \preceq E_\beta$. In particular, $\langle E_\beta \mid \beta < \alpha\rangle$ converges and $E^*_\alpha = \lim_{\beta\to\alpha} E_\beta$ is a bounded equivalence relation on κ.*

For each $\alpha < \lambda$, we define $\mathbb{P}^s_\alpha = C_{E^*_\alpha}$ and $M^s_\alpha = \mathrm{V}[G \cap \mathbb{P}^s_\alpha]$. By the second part of Observation 3.9, if $\beta < \alpha < \lambda$, then $E^*_\alpha \preceq E^*_\beta$ and we can use Observation 2.4 to see that the sequence $\langle M^s_\alpha \mid \alpha < \lambda\rangle$ of ground models is decreasing.

Let $\alpha < \lambda$. We have $E^*_{\alpha+1} = E_\alpha$ and Observation 3.7 yields $\mathbb{P}^s_{\alpha+1} = C_{E_\alpha} \Vdash \tau(\mathfrak{G}) = \mathrm{ht}(E_\alpha) = s(\alpha)$. If α is a limit ordinal, then $E^*_\alpha = \lim_{\beta\to\alpha} E_\beta = \bigcap_{\beta<\alpha} E_\beta$, because the sequence $\langle E_\beta \mid \beta < \alpha\rangle$ is decreasing. As a result, the nontrivial equivalence classes of E^*_α are precisely the sets $\{t^*(\beta) \mid \alpha \leq \beta < \lambda\}$ and this shows that E^*_α is an inactive bounded equivalence relation on κ. By Observation 3.7, $\mathbb{P}^s_\alpha = C_{E^*_\alpha} \Vdash \tau(\mathfrak{G}) = \mathrm{ht}(E^*_\alpha) = 1$.

If $s(\alpha) = s'(\alpha)$ for some $s, s' : \lambda \longrightarrow (\lambda \setminus \{0\})$ and $\alpha < \lambda$, then the above construction produces the same equivalence relation E_α for both functions and therefore the same model $M_{\alpha+1} = \mathrm{V}[G \cap C_{E_\alpha}]$. Finally, by the above analysis, the equivalence relation $E^*_\nu = \lim_{\beta\to\nu} E_\beta$ is the same for all $s : \lambda \longrightarrow (\lambda \setminus \{0\})$ and limit ordinal $\nu < \lambda$. □

5 The Mix

In this section, we are producing models of set theory, where a given sequence of nonzero ordinals can be realized as the height of the automorphism tower of $\mathfrak{G}$ in consecutive models such that the next one is a forcing extension *or* a ground model of the previous one, as desired. There are some limitations on the possible patterns, and to formalize them precisely, we introduce the notion of a *realizable prescription.*

Definition 5.1. A function $s : \kappa \longrightarrow (\kappa \setminus \{0\}) \times 2$ is a *prescription* on κ. It is *realizable* if the set of $\alpha < \kappa$ such that $(s(\alpha))_1 = 0$ is not stationary, and if $(s(0))_1 = 1$.[4]

The interpretation is that the first coordinate of $s(\alpha)$ gives the desired height of the automorphism tower of $\mathfrak{G}$ in the $(\alpha+1)$-st model, and the second coordinate says whether the $(\alpha+1)$-st model should be a forcing extension or a ground model of the α-th model.

Theorem 5.2 (Under Assumption 1.8)**.** *For every realizable prescription s on κ, there is a sequence $\vec{E} := \langle E_\alpha \mid \alpha < \kappa\rangle$ of bounded equivalence relations on κ with the following properties:*

1. *For every limit $\lambda < \kappa$, $\vec{E}\restriction\lambda$ is convergent. For $\alpha < \kappa$, set:*

$$M_\alpha = \mathrm{V}^{C_{\lim_{\beta<\alpha} E_\beta}}.$$

2. *For every $\alpha < \kappa$, $\tau(\mathfrak{G})^{M_{\alpha+1}} = (s(\alpha))_0$.*

3. *If $\alpha < \kappa$ is a limit ordinal, then $\tau(\mathfrak{G})^{M_\alpha} = 1$. Of course, $M_0 = \mathrm{V}$, so $\tau(\mathfrak{G})^{M_0} = 0$.*

4. *For every $\alpha < \kappa$, the following is true:*

   (a) If $s(\alpha)_1 = 0$, then $M_{\alpha+1}$ is a ground model of M_α, and

[4] Here, we use the following notation for components of ordered pairs: $(\langle x, y\rangle)_0 = x$, $(\langle x, y\rangle)_1 = y$.

(b) *if $s(\alpha)_1 = 1$, then $M_{\alpha+1}$ is a forcing extension of M_α.*

Proof. Let a realizable prescription s be given. Let $C \subseteq \kappa$ be a club of α with $(s(\alpha))_1 = 1$, such that $0 \in C$. Let $f_C : \kappa \longrightarrow\!\!\!\!\rightarrow C$ be the monotone enumeration of C. Given $\beta < \kappa$, let $i(\beta)$ be that ordinal less than κ such that $\beta \in [f_C(i(\beta)), f_C(i(\beta)+1))$. Let t be the function with domain κ defined by setting $t(\alpha) = (s(\alpha))_0 + 1$.

For $\beta < \kappa$, we define an equivalence relation E_β on κ by specifying its nontrivial equivalence classes. Namely, X is a nontrivial equivalence class of E_β iff one of the following holds:

D.1. There is an $\alpha < \beta$ such that $(s(\alpha+1))_1 = 1$ and $X = t^*(\alpha)$.

D.2. There is an $\alpha < \beta$ such that $(s(\alpha+1))_1 = 0$ and X is the trimmed segment of $t^*(\alpha)$.

D.3. There is an $\alpha \in (\beta, f_C(i(\beta)+1))$ such that $(s(\alpha))_1 = 0$ and $X = t^*(\alpha)$.

D.4. $(s(\beta))_0 > 1$ and X is the active segment of $t^*(\beta)$ of order type $t(\beta)$ (which is $(s(\beta))_0 + 1$), *or* $(s(\beta))_0 = 1$ and $X = t^*(\beta)$.

This defines the sequence $\langle E_\beta \mid \beta < \kappa \rangle$ of equivalence relations. Obviously, each E_β is bounded. If E_β is active, then its active equivalence class is the active segment of $t^*(\beta)$ of order type $(s(\beta))_0+1$. In particular, in $M_{\beta+1} = \mathrm{V}^{C_{E_\beta}}$, $\tau(\mathfrak{G}) = (s(\beta))_0$. If E_β is not active, then $(s(\beta))_0 = 1$, E_β is inactive, and in $M_{\beta+1}$, $\tau(\mathfrak{G}) = (s(\beta))_0$, as well.

We have to show the sequence has the desired properties. To this end, we verify the following claims.

(1) *For every $\alpha \leq \kappa$, the sequence $\langle E_\beta \mid \beta < \alpha \rangle$ converges. Let E^*_α denote its limit.*

Proof of (1). Fix a limit ordinal $\alpha \leq \kappa$. Let $\gamma, \delta < \kappa$ be given. We have to find $\bar{\alpha} < \alpha$ such that either for all $\beta \in (\bar{\alpha}, \alpha)$, $\gamma E_\beta \delta$ holds, or for all $\beta \in (\bar{\alpha}, \alpha)$, $\gamma E_\beta \delta$ fails. This is trivial if $\gamma = \delta$, and it is also trivial if there is no $\mu < \alpha$ such that $\gamma E_\mu \delta$ holds. But if there is such a μ, then this means that $\gamma, \delta \in t^*(\xi)$, for some $\xi < f_C(i(\mu)+1)$ – this is easily confirmed by looking at the definition of E_μ above. If $\xi < \alpha$, then for all $\beta, \beta' \in (\xi, \alpha)$, $\gamma E_\beta \delta \iff \gamma E_{\beta'} \gamma$ (again, this is easily checked by referring to the clauses D.1-D.4 defining the equivalence relations), so we can let $\bar{\alpha} = \xi$. But if $\xi \geq \alpha$, then this means that $t^*(\xi)$ is a nontrivial equivalence class of E_μ due to condition D.3, so $\xi \in (\mu, f_C(i(\mu)+1)$. But then, for all $\beta \in [\mu, \alpha)$, $i(\beta) = i(\mu)$, and again, by D.3, $t^*(\xi)$ will be a nontrivial equivalence class of E_β. So in this case, we can set $\bar{\alpha} = \mu$. $\square_{(1)}$

It is also easy to see that in case α is a limit, E^*_α is inactive, so that in $\mathrm{V}^{C_{E^*_\alpha}}$, $\tau(\mathfrak{G}) = 1$.

(2) *For $\alpha < \kappa$ with $(s(\alpha))_1 = 0$, it follows that $C_{E_\alpha} \preceq C_{E^*_\alpha}$.*

Proof of (2). Note that if $(s(\alpha))_1 = 0$, then $\alpha \in (f_C(i(\alpha)), f_C(i(\alpha)+1)$, since $\alpha \notin C$. There are two cases to consider here.

The first case is that α is a limit ordinal. In that case, it follows that the only disagreement between E^*_α and E_α is that the α-th nontrivial equivalence class of E^*_α is $t^*(\alpha)$, while the α-th nontrivial equivalence class of E_α is the active segment of $t^*(\alpha)$ of order type $(s(\alpha))_0 + 1$. So $E_\alpha \preceq E^*_\alpha$.

The second case is that α is a successor ordinal, say $\alpha = \bar{\alpha}+1$. In this case, $E^*_\alpha = E_{\bar{\alpha}}$, and we have to show that $E_\alpha \preceq E_{\bar{\alpha}}$. Since $\alpha \in (f_C(i(\alpha)), f_C(i(\alpha)+1)$, it follows that the α-th nontrivial equivalence class of $E_{\bar{\alpha}}$ is $t^*(\alpha)$, while the α-th nontrivial equivalence class of E_α is the active segment of $t^*(\alpha)$ of order type $(s(\alpha))_0 + 1$ (using clause D.4. in the definition of E_α and clause D.3. in the definition of $E_{\bar{\alpha}}$). Moreover, the $\bar{\alpha}$-th nontrivial equivalence class of $E_{\bar{\alpha}}$ is the active segment of $t^*(\bar{\alpha})$ (by clause D.4. in the definition of $E_{\bar{\alpha}}$), and the $\bar{\alpha}$-th nontrivial equivalence class of E_α is the trimmed segment of $t^*(\bar{\alpha})$ (by clause D.2. in the definition of E_α). E_α and $E_{\bar{\alpha}}$ agree about the other nontrivial equivalence classes, so that it follows that $E_\alpha \preceq E_{\bar{\alpha}}$, as desired. $\Box_{(2)}$

(3) *If $\alpha < \kappa$ is such that $(s(\alpha))_1 = 1$, then $C_{E^*_\alpha} \preceq C_{E_\alpha}$.*

Proof of (3). As in the proof of claim (2), we distinguish two cases. The first case is that α is a limit ordinal. As before, E_α and E^*_α agree about the γ-th equivalence classes. The α-th equivalence class of E_α is the active segment of $t^*(\alpha)$ of order type $(s(\alpha))_0+1$, while for $\gamma \in t^*(\alpha)$, $\{\gamma\} = [\gamma]_{E^*_\alpha}$. E_α and E^*_α agree about the other nontrivial equivalence classes, which are of the form $t^*(\beta)$, for $\beta \in (\alpha, f_C(i(\alpha)+1))$. So $E^*_\alpha \preceq E_\alpha$, as claimed.

In the second case to consider, α is a successor ordinal, say $\alpha = \bar{\alpha}+1$. So $E^*_\alpha = E_{\bar{\alpha}}$, and we have to show that $E_{\bar{\alpha}} \preceq E_\alpha$. The α-th nontrivial equivalence class of E_α is the active segment of $t^*(\alpha)$ of order type $(s(\alpha))_0 + 1$ (using clause D.4. in the definition of E_α), and for $\gamma \in t^*(\alpha)$, $\{\gamma\} = [\gamma]_{E_{\bar{\alpha}}}$. The $\bar{\alpha}$-th nontrivial equivalence class of $E_{\bar{\alpha}}$ is the active segment of $t^*(\bar{\alpha})$ (by clause D.4. in the definition of $E_{\bar{\alpha}}$), and the $\bar{\alpha}$-th nontrivial equivalence class of E_α is $t^*(\bar{\alpha})$ (by clause D.1. in the definition of E_α). E_α and $E_{\bar{\alpha}}$ agree about the other nontrivial equivalence classes, so that it follows that $E_\alpha \preceq E_{\bar{\alpha}}$, as desired. $\Box_{(3)}$

This finishes the proof of the theorem. $\Box$

6 The effect of Closed Maximality Principles

It was shown in [Fuc08, Section 3.3] that Closed Maximality Principles imply the existence of groups with malleable automorphism tower heights. The *Lightface Closed Maximality Principle* at a regular cardinal λ, $\mathsf{MP}_{<\lambda-\text{closed}}(\{\lambda\})$, says that every statement about λ that can be forced by $<\lambda$-closed forcing in such a way that it stays true in further forcing extensions obtained by $<\lambda$-closed forcing is already true in the ground model. It is a scheme of first order statements involving λ as a parameter.

It might be viewed as a defect in the previous results of this article that in Theorems 4.1 and 5.2, one first has to pass to a forcing extension in order to be able to change the height of the automorphism tower of $\mathfrak{G}$ by passing to ground models. But of course, there is no way around it, if one just makes our assumption $\diamondsuit_{\kappa^+}(\mathrm{Cof}_\kappa)$+ $2^{<\kappa} = \kappa$. This assumption holds in L, and there is no way to pass to a proper ground model of L.

But it is one of the merits of Maximality Principles that they imply that there are many ground models. For example, the statement "*the universe is a nontrivial forcing extension of a ground model*" can be forced to be true, and once true, it stays true in further forcing extensions - see [Fuc08, Section 6] for the relevance of this observation. The hope is that we get groups for which we may realize a given sequence of ordinals as the automorphism tower heights in consecutive grounds, without being required to pass to a forcing extension in the first step. So let's replace $\diamondsuit_{\kappa^+}(\mathrm{Cof}_\kappa)$ in our assumption by $\mathsf{MP}_{<\kappa^+-\text{closed}}(\{\kappa^+\})$, meaning that our revised assumption now reads:

Assumption 6.1. κ is a regular, uncountable cardinal such that

1. $2^{<\kappa} = \kappa$
2. $\mathsf{MP}_{<\kappa^+-\text{closed}}(\{\kappa^+\})$ holds.

It has been shown in [Fuc08, Theorem 3.15] that from this assumption, we get a sequence of κ^+-Souslin trees which is able to realize *any* equivalence relation on κ^+, not just any *bounded* one. Here is the version of Theorem 4.1 using Maximality Principles.

Theorem 6.2 (Under Assumption 6.1)**.** *There is a group* $\mathfrak{H}$ *such that*

1. $\tau(\mathfrak{H}) = 1$.
2. *For every function* $s : \kappa^+ \longrightarrow (\kappa^+ \setminus \{0\})$ *there exists a* decreasing sequence of ground models $\langle M^s_\alpha \mid 0 < \alpha < \kappa^+\rangle$ *such that for all* $0 < \alpha < \kappa^+$ *the following statements hold.*
   (a) $M^s_{\alpha+1} \models$ "$\tau(\mathfrak{H}) = s(\alpha)$".
   (b) *If* α *is a limit ordinal, then* $M^s_\alpha \models$ "$\tau(\mathfrak{H}) = 1$".

   Moreover, if $t : \kappa^+ \longrightarrow (\kappa^+ \setminus \{0\})$*, then* $s(\alpha) = t(\alpha)$ *implies* $M^s_{\alpha+1} = M^s_{\alpha+1}$ *for all* $\alpha < \kappa^+$ *and* $M^s_\nu = M^t_\nu$ *for all limit ordinals* $\nu < \kappa^+$.

Proof. We adopt the proof of [HT00, Theorem 3.1]. The argument works as follows: First force with the $<\kappa^+$-closed partial order $\mathbb{Q}$ to add the sequence $\vec{T}$ of Souslin trees. $\mathbb{Q}$ consists of conditions $q = \langle t^q_\alpha \mid \alpha < \kappa^+\rangle$ such that for all but κ many α, $t^q_\alpha = \emptyset$, and for all α, t^q_α is an initial segment of the α-th Souslin tree to be added. The ordering is the obvious one - the forcing can be viewed as a product of the Jech partial order to add a Souslin tree. The sequence $\vec{T}$ will consist of rigid, mutually nonisomorphic κ^+-Souslin trees that are able to realize every equivalence relation on κ^+.

Now, in a second step, we force to a model where $\vec{T}$ realizes a "maximal" equivalence relation E on κ^+. This will be a model from which we can pass down to consecutive grounds in order to realize the desired patterns of automorphism tower heights. We follow the construction in Theorem 4.1, but this time, we don't need to be as careful as before, since we can realize every equivalence relation. On the other hand, we have to set things up a little differently, since our equivalence classes will have to have order type κ^+. Since this is a limit ordinal, the notion of a "*sealed*" equivalence class has to be changed slightly.

The "*maximal*" equivalence relation E on κ^+ has the equivalence classes $\{C_\alpha \mid \alpha < \kappa^+\}$, where

$$C_\alpha = \{\prec\alpha,\nu\succ \mid \nu < \kappa^+\}.$$

So E has κ^+ many equivalence classes each of which has order type κ^+. We'll define a group $\mathfrak{H}$ so that in a slightly changed sense, E is sealed with respect to $\mathfrak{H}$. Thus, we define permutation groups $\langle P_\alpha \mid \alpha < \kappa^+\rangle$ so that

- $P_{\prec\alpha,0\succ} = (F_0, \Delta_0)$,
- $P_{\prec\alpha,1\succ} = (F_1, \Delta_1)$,
- $P_{\prec\alpha,2\succ} = (F_1, \Delta_1)$,
- $P_{\prec\alpha,3+\beta\succ} = (F_\beta, \Delta_\beta)$, for $\beta < \kappa^+$.

Let $\mathfrak{H}$ be defined relative to $\langle P_\alpha \mid \alpha < \kappa^+\rangle$ and $\vec{T}$ like $\mathfrak{G}$ was defined relative to $\langle (G_\alpha, \Omega_\alpha) \mid \alpha < \kappa\rangle$ and the sequence of Souslin trees we worked with before.

Now, in V$[\vec{T}]$, let $\mathbb{P}_E$ be the following variant of the usual forcing to realize E: It consists of sequences of the form $p = \langle p_{\prec\alpha,\nu\succ} \mid \alpha,\nu < \lambda_p,\ \nu \neq 3\rangle$, where $\lambda_p < \kappa^+$ and for each $\alpha,\nu < \lambda_p$, $\nu \neq 3$, there is a $\xi < \kappa^+$ such that $p_{\prec\alpha,\nu\succ}$ is an isomorphism between $T_{\prec\alpha,3\succ}|(\xi+1)$ and $T_{\prec\alpha,\nu\succ}|(\xi+1)$, the restrictions of these trees to levels less than or equal to ξ. The ordering is by "*pointwise inclusion*". The reason for choosing $\prec\alpha,3\succ$ instead of $\min(C_\alpha)$ is that we want the forcing that realizes the equivalence relation with nontrivial equivalence classes of the form $C_\alpha \setminus (\prec\alpha,3\succ + 1)$ to be completely contained in $\mathbb{P}_E$. For the active segment of C_α of order type $\gamma < \kappa^+$ will now be the set $\{\prec\alpha, 3+\nu\succ \mid \nu < \gamma\}$.

Assuming that $2^{<\kappa} = \kappa$ and $2^\kappa = \kappa^+$ it was shown in [HT00, Lemma 3.12] that $\mathbb{P}_E$ realizes E. These assumptions are implied by our current working Assumption 6.1 ($\mathsf{MP}_{<\kappa^+-\mathrm{closed}}(\{\kappa^+\})$ implies that $2^\kappa = \kappa^+$, see [Fuc08, Section 3]). Moreover, $\mathbb{Q} * \dot{\mathbb{P}}_E$ has a dense suborder that is $<\kappa^+$-closed, as was shown in [HT00, Proof of Lemma 3.12]. So, letting I be $\mathbb{P}_E$-generic over $\mathrm{V}[\vec{T}]$, what needs to be shown now is that the statements 1. and 2. of the theorem are $<\kappa^+$-closed-necessary in $\mathrm{V}[\vec{T}][I]$. For then, the existence of a group satisfying the statements 1. and 2. is $<\kappa^+$-closed-forceably necessary. The only parameter occurring in these statements is κ^+, so that it follows by $\mathsf{MP}_{<\kappa^+-\mathrm{closed}}(\{\kappa^+\})$ that they are true, finishing the proof.

So let $\mathbb{P} \in \mathrm{V}[\vec{T}][I]$ be $<\kappa^+$-closed, and let G be $\mathbb{P}$-generic over $\mathrm{V}[\vec{T}][I]$. First, note that each T_α is a rigid κ^+-Souslin tree in $\mathrm{V}[\vec{T}][I][G]$. This is because this is the case in $\mathrm{V}[\vec{T}][I]$, and that property of T_α is $<\kappa^+$-closed-necessary (since it is $\Pi^1_1(H_{\kappa^+})$ in T_α - see [Fuc08, Lemma 3.5]). More generally, forcing with $\mathbb{P}$ won't add any isomorphisms between κ^+-Souslin trees, for the same reason. So in $\mathrm{V}[\vec{T}][I][G]$, $\vec{T}$ continues to realize E. In particular, statement 1. holds, since every equivalence class of E is "*sealed*", in the obviously modified sense, and the proof of Lemma 3.3 still works in this situation.

The proof of statement 2. is very similar to the proof of Theorem 4.1: Given a function $s : \kappa^+ \longrightarrow (\kappa^+ \setminus \{0\})$, define a sequence of equivalence relations $\langle E_\alpha \mid \alpha < \kappa^+ \rangle$ by specifying that Z is a nontrivial equivalence class of E_α if and only if one of the following holds:

- $Z = C_\beta$, for some $\beta \in (\alpha, \kappa^+)$,
- $s(\alpha) = 1$ and $Z = C_\alpha$,
- $s(\alpha) > 1$ and $Z = \{\prec\alpha, 3+\nu\succ \mid \nu < s(\alpha)\}$

Letting $E^*_\alpha = \lim_{\beta \to \alpha} E_\beta$, the desired sequence of ground models will be given by $M_\alpha = \mathrm{V}[\vec{T}][\mathbb{P}_{E^*_\alpha} \cap I]$, for $\alpha > 0$, and, of course, $M_0 = \mathrm{V}[\vec{T}][I][G]$. So in passing from M_0 to M_1, all of the forcing $\mathbb{P}$ is undone. The verifications that this sequence of models has the desired properties work as before, with the necessary, straightforward modifications in the proofs of Lemma 3.3 and 3.4 caused by the change of the notion of "*active*" and "*sealed*" equivalence classes. □

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