GROUPS WITH UNBOUNDED POTENTIAL AUTOMORPHISM TOWER HEIGHTS

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Abstract. We show that it is consistent with the axioms of ZFC that there exists an infinite centreless group G with the property that for every ordinal α there is a notion of forcing $\mathbb{P}$ that preserves cardinalities and cofinalities and forces the automorphism tower of G to be taller than α.

1. Introduction

We start by giving a brief introduction to the so-called *automorphism tower problem*[1]. Let G be a group with trivial centre. For each $g \in G$, the map

$$\iota_g : G \longrightarrow G;\ h \mapsto g \circ h \circ g^{-1}$$

is an automorphism of G and is called the *inner automorphism corresponding to g*. Clearly, $\iota_g = \mathrm{id}_G$ if and only if $g = \mathbb{1}_G$. The map

$$\iota_G : G \longrightarrow \mathrm{Aut}(G);\ g \mapsto \iota_g$$

is an embedding of groups that maps G onto the subgroup $\mathrm{Inn}(G)$ of all inner automorphisms of G. An easy computation shows that $\pi \circ \iota_g \circ \pi^{-1} = \iota_{\pi(g)}$ holds for all $g \in G$ and $\pi \in \mathrm{Aut}(G)$. This shows that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$ and $\mathrm{Aut}(G)$ is also a group with trivial centre. By iterating this process, we inductively construct the automorphism tower of G.

Definition 1.1. A sequence $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ of groups is *the automorphism tower of a centreless group G* if the following statements hold.

(1) $G_0 = G$.
(2) For all $\alpha \in \mathrm{On}$, G_α is a normal subgroup of $G_{\alpha+1}$ and the induced homomorphism

$$\varphi_\alpha : G_{\alpha+1} \longrightarrow \mathrm{Aut}(G_\alpha);\ g \mapsto \iota_g \restriction G_\alpha$$

is an isomorphism.
(3) For all $\alpha \in \mathrm{Lim}$, $G_\alpha = \bigcup\{G_\beta \mid \beta < \alpha\}$.

In this definition, we replaced $\mathrm{Aut}(G_\alpha)$ by an isomorphic copy $G_{\alpha+1}$ that contains G_α as a normal subgroup. This allows us to take unions at limit stages. Without this isomorphic correction, we would have to take direct limits at limit stages. By induction, we can construct such a tower for each centreless group and it is easy to show that each group G_α in this tower is uniquely determined up to an

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[1] An extensive account of all aspects of the automorphism tower problem can be found in Simon Thomas' forthcoming monograph [Tho].

isomorphism which is the identity on G. We can therefore speak of *the* α-th group G_α in the automorphism tower of a centreless group G.

It is natural to ask whether the automorphism tower of a centreless group eventually *terminates*, in the sense that there is an $\alpha \in \mathrm{On}$ with $G_\alpha = G_{\alpha+1}$ and therefore $G_\alpha = G_\beta$ for all $\beta \geq \alpha$. A classical result of Helmut Wielandt (see [Wie39, page 212]) says that the automorphism tower of a finite centreless group terminates after finitely many steps. In [Tho85] and [Tho98], Simon Thomas uses Fodor's Lemma to show that every infinite centreless group has a terminating automorphism tower by proving the following result.

Theorem 1.2 (Tho98, Theorem 1.3)**.** *If G is an infinite centreless group of cardinality κ, then there is an $\alpha < (2^\kappa)^+$ with $G_\alpha = G_{\alpha+1}$.*

This result allows us to make the following definitions.

Definition 1.3. Given a centreless group G, we let $\tau(G)$ denote the least ordinal α satisfying $G_\alpha = G_{\alpha+1}$ and call this ordinal the *height of the automorphism tower of G*. For every infinite cardinal κ, we define

$$\tau_\kappa = \mathrm{lub}\{\tau(G) \mid G \textit{ is a centreless group of cardinality } \kappa\}.$$

There are only 2^κ-many centreless groups of infinite cardinality κ and this shows that Simon Thomas' result implies $\tau_\kappa < (2^\kappa)^+$ for all infinite cardinals κ. The following result of Winfried Just, Saharon Shelah and Simon Thomas shows that $(2^\kappa)^+$ is the best *upper cardinal bound* for τ_κ provable in ZFC for uncountable regular κ.

Theorem 1.4 (JST99, Theorem 1.4)**.** *Assume* (GCH)*. Let κ be a regular uncountable cardinal, ν be a cardinal with $\kappa < \mathrm{cof}(\nu)$ and α be an ordinal with $\alpha < \nu^+$. Then there exists a $<\kappa$-closed partial order $\mathbb{P}$ that satisfies the κ^+-chain condition and the following statements are true in every $\mathbb{P}$-generic extension of* V*.*

(1) $2^\kappa = \nu$.
(2) *There exists a centreless group G of cardinality κ such that $\tau(G) = \alpha$.*

Note that the following problem is still open.

Problem 1.5. *Find a model $\langle M, \in_M\rangle$ of* ZFC *and an infinite cardinal κ in M such that it is possible to compute*[2] *the exact value of τ_κ in M.*

One of the reasons why it is so difficult to compute the value of τ_κ is that although the definition of automorphism towers is purely algebraic, there can be groups whose automorphism tower heights depend on the model of set theory in which they are computed. Therefore, you always have to take into account the set-theoretic background in which the computation of τ_κ takes place. This shows that the automorphism tower construction contains a *set-theoretic essence* (this formulation is due to Joel David Hamkins, see [Ham02]). We give a short overview on results concerning the existence of such groups and continue by introducing a new class of groups whose automorphism towers are highly malleable by forcing.

In [Tho98], Simon Thomas constructs a centreless group G with $\tau(G) = 0$ and a partial order $\mathbb{P}$ that satisfies the countable chain condition and $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\tau(\check{G}) \geq 1$".

[2]By "*compute*", we mean *set-theoretic* characterizations of τ_κ. Examples of such characterizations are $\langle M, \in_M\rangle \models$ "$\tau_\kappa = \kappa^+$" or $\langle M, \in_M\rangle \models$ "$\tau_\kappa = 2^\kappa$".

In the other direction, he also constructs a centreless group H with $\tau(H) = 2$ and $\mathbb{1}_{\mathbb{Q}} \Vdash_{\mathbb{Q}}$ "$\tau(\check{H}) = 1$" for every notion of forcing $\mathbb{Q}$ that adds a new real.

Let G be an infinite centreless group, $\mathbb{P}$ be a partial order and F be $\mathbb{P}$-generic over V. By the results mentioned above, the height of the automorphism tower of G computed in V, $\tau(G)^{\mathrm{V}}$, can be higher or smaller than the height computed in V$[F]$, $\tau(G)^{\mathrm{V}[F]}$. It is natural to ask whether the value of $\tau(G)^{\mathrm{V}}$ places any constraints on the value of $\tau(G)^{\mathrm{V}[F]}$, and vice versa. Obviously, $\tau(G)^{\mathrm{V}[F]} = 0$ implies $\tau(G)^{\mathrm{V}} = 0$. The following result by Joel David Hamkins and Simon Thomas suggests that this is the only implication provable in ZFC that holds for all centreless groups in the above situation.

Theorem 1.6 (HT00, Theorem 1.4)**.** *It is consistent with the axioms of* ZFC *that for every infinite cardinal κ and every ordinal $\alpha < \kappa$, there exists a centreless group G with the following properties.*

(1) $\tau(G) = \alpha$.
(2) *Given $0 < \beta < \kappa$, there exists a partial order $\mathbb{P}$, which preserves cofinalities and cardinalities, such that* $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\tau(\check{G}) = \check{\beta}$".

In [FH08], Gunter Fuchs and Joel David Hamkins showed that Gödel's constructible universe L is a model of the above statement. Using techniques developed in [HT00] and [FH08], Gunter Fuchs and the author extended the above results by constructing ZFC-models containing groups whose automorphism tower height can changed again and again by passing to another model of set theory.

Theorem 1.7 (FL10)**.** *It is consistent with the axioms of* ZFC *that for every infinite cardinal κ there is a centreless group G with $\tau(G) = 0$ and the property that for every function $s : \kappa \longrightarrow (\kappa \setminus \{0\})$ there is a sequence $\langle \mathbb{P}_\alpha \mid 0 < \alpha < \kappa\rangle$ of partial orders such that the following statements hold.*

(1) *For all $0 < \alpha < \kappa$, $\mathbb{P}_\alpha$ preserves cardinalities and cofinalities.*
(2) *For all $0 < \alpha < \beta < \kappa$, there is a partial order $\mathbb{Q}$ with $\mathbb{P}_\beta = \mathbb{P}_\alpha \times \mathbb{Q}$.*
(3) *For all $\alpha < \kappa$, we have* $\mathbb{1}_{\mathbb{P}_{\alpha+1}} \Vdash_{\mathbb{P}_{\alpha+1}}$ "$\tau(\check{G}) = \check{s}(\check{\alpha})$".
(4) *If $0 < \alpha < \kappa$ is a limit ordinal, then* $\mathbb{1}_{\mathbb{P}_\alpha} \Vdash_{\mathbb{P}_\alpha}$ "$\tau(\check{G}) = 1$".

Again, L is a model of this statement. In another direction, [FL10] also shows how to construct a model of ZFC that contains an infinite centreless group whose automorphism tower height can be changed again and again by passing to smaller and smaller inner models.

Next, we introduce another class of groups whose automorphism tower height can be changed drastically by passing to forcing extensions of the ground model. We state the main result of this note.

Theorem 1.8. *It is consistent with the axioms of* ZFC *that there is a centreless group G of cardinality $\aleph_1$ with the property that for every ordinal α there is a partial order $\mathbb{P}$ satisfying the following statements.*

(1) *$\mathbb{P}$ is σ-distributive and satisfies the $\aleph_2$-chain condition.*
(2) $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\tau(\check{G}) \geq \check{\alpha}$".

In the above situation, we say that G has *unbounded potential automorphism tower heights*. The following section contains the proof of this result. We close this introduction with questions motivated by this result.

Question 1.9. *Is* L *a model of the statement of Theorem 1.8?*

Given a partial order $\mathbb{P}$, consider the following property.

($*$) $\mathbb{P}$ is σ-closed and satisfies the $\aleph_2$-chain condition.

Question 1.10. *Is it consistent with the axioms of* ZFC *that there is a centreless group G of cardinality $\aleph_1$ with the property that for every ordinal α there is a partial order $\mathbb{P}$ that satisfies ($*$) and $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\tau(\check{G}) \geq \check{\alpha}$ "?*

If this question has a positive answer, then it is natural to ask the following question.

Question 1.11. *Is it provable in* ZFC *that there is a partial order $\mathbb{P}$ that satisfies ($*$) and forces the statement of Question 1.10 to hold true in every $\mathbb{P}$-generic extension of the ground model?*

A negative answer to this questions opens the possibility of finding a solution to Problem 1.5 using an iterated forcing construction. This solution would require an analysis of the absoluteness properties of the long automorphism towers added by Theorem 1.4 under σ-closed forcing and the use of techniques similar to the ones used in the next section.

2. Constructing the Model

In this section, we construct a ZFC-model containing a group with unbounded potential automorphism tower heights. This construction is motivated by Joel David Hamkins' alternative proof of the consistency of the *Maximality Principle* (see [Ham03, page 533]). We combine those techniques with Theorem 1.4 and some basic forcing results. For completeness, we provide proofs of these results.

Let $\mathcal{L}_{\in,\dot{\delta}}$ be the first-order language extending the language $\mathcal{L}_{\in}$ of set theory by a constant symbol $\dot{\delta}$. Given an $\mathcal{L}_{\in}$-formula $\varphi(v_0,\dots,v_{n-1})$, we let $\varphi^{\mathrm{V}_{\dot{\delta}}}(v_0,\dots,v_{n-1})$ denote the $\mathcal{L}_{\in,\dot{\delta}}$-formula

$$(\exists x) \ \left[\varphi^x(v_0,\dots,v_{n-1}) \ \wedge \ (\forall y)\left(y \in x \leftrightarrow \mathrm{rnk}(y) < \dot{\delta}\right)\right],$$

where $\varphi^x(v_0,\dots,v_{n-1})$ is the usual relativization of $\varphi(v_0,\dots,v_{n-1})$ to x. We let "$\mathrm{V}_{\dot{\delta}} \prec \mathrm{V}$" denote the $\mathcal{L}_{\in,\dot{\delta}}$ - theory of all axioms of the form

$$(\forall x_0,\dots,x_{n-1}) \ [(\mathrm{rnk}(x_0),\dots,\mathrm{rnk}(x_{n-1}) < \dot{\delta} \wedge \varphi(x_0,\dots,x_{n-1})) \longrightarrow \varphi^{\mathrm{V}_{\dot{\delta}}}(x_0,\dots,x_{n-1})],$$

where $\varphi(v_0,\dots,v_{n-1})$ is an n-ary $\mathcal{L}_{\in}$-formula.

Proposition 2.1. *If* ZF *is consistent, then so is the theory* ZFC + (GCH) + *"$\mathrm{V}_{\dot{\delta}} \prec \mathrm{V}$ ".*

Proof. The consistency of ZF implies the existence of a model $\langle \mathrm{V}, \in\rangle$ of ZFC + (GCH). Given a finite fragment F of "$\mathrm{V}_{\dot{\delta}} \prec \mathrm{V}$", we can apply Levy's Reflection Principle (see [Mos09, Theorem 8C.4]) to find an element δ of V with

$$\langle \mathrm{V}, \in, \delta\rangle \models \mathrm{F} + \text{"}\dot{\delta} \textit{ is a strong limit cardinal}\text{"}.$$

By the Compactness Theorem, this shows that the theory ZFC+(GCH)+"$\mathrm{V}_{\dot{\delta}} \prec \mathrm{V}$" is consistent. □

Proposition 2.2. *If $\langle \mathrm{V}, \in, \delta\rangle$ is a model of* ZFC + *"$\mathrm{V}_{\dot{\delta}} \prec \mathrm{V}$ ", then δ is a strong limit cardinal in* V*.*

Proof. By elementarity, we have $\langle \mathrm{V}_\delta, \in\rangle \models (\mathrm{Inf})$ and this implies $\omega < \delta$. Further applications of elementarity yield $2^\kappa < \delta$ for every cardinal $\kappa < \delta$. □

If $\mathrm{V}[G]$ is a generic extension of the ground model and α is an ordinal, then we also write $\mathrm{V}[G]_\alpha$ to denote the set of all elements x of $\mathrm{V}[G]$ with $\mathrm{rnk}(x)^{\mathrm{V}[G]} < \alpha$. Given a model $\langle \mathrm{V}, \in, \delta\rangle$ of ZFC + "$\mathrm{V}_{\dot\delta} \prec \mathrm{V}$ ", we investigate forcing extensions of $\langle \mathrm{V}, \in\rangle$ by partial orders contained in V_δ.

Proposition 2.3. *Let δ be a cardinal and $\mathbb{P} \in \mathrm{V}_\delta$ be a partial order. If G is $\mathbb{P}$-generic over* V*, then*

$$\mathrm{V}[G]_\delta = \{\dot x^G \in \mathrm{V}[G] \mid \dot x \in \mathrm{V}^{\mathbb{P}} \cap \mathrm{V}_\delta\}.$$

Proof. Let G be $\mathbb{P}$-generic over V. An easy induction shows that $\mathrm{rnk}(\dot x^G) \leq \mathrm{rnk}(\dot x)$ holds for each name $\dot x \in \mathrm{V}^{\mathbb{P}}$.

We define a sequence $\langle \dot v_\alpha \in \mathrm{V}^{\mathbb{P}} \mid \alpha < \delta\rangle$ of names in the following way.

(1) $\dot v_0 = \emptyset$.
(2) For all $\alpha < \delta$, we define $\dot v_{\alpha+1} = \mathcal{P}\left(\mathrm{dom}(\dot v_\alpha) \times \mathbb{P}\right) \times \{\mathbb{1}_{\mathbb{P}}\}$.
(3) If $\alpha \in \mathrm{Lim} \cap \delta$, then we define $\dot v_\alpha = \bigcup\{\dot v_\beta \mid \beta < \alpha\}$.

If $\mathrm{rnk}(\mathbb{P}) = \alpha_0 < \delta$, then another easy induction shows that $\mathrm{rnk}(\dot v_\alpha) < \alpha_0 + \alpha + \omega$ and

$$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \text{“}(\forall x)\ [\mathrm{rnk}(x) < \check\alpha \longrightarrow x \in \dot v_\alpha]\text{”}$$

holds for all $\alpha < \delta$ (the proof of [Kun80, Theorem 4.2, page 201] contains the details of the successor case).

Now, suppose $\dot x \in \mathrm{V}^{\mathbb{P}}$ with $\mathrm{rnk}(\dot x^G)^{\mathrm{V}[G]} < \delta$. By the above constructions, there is an $\alpha < \delta$ and a condition $p \in G$ with $p \Vdash_{\mathbb{P}}$ "$\dot x \subseteq \dot v_\alpha$ ". If we define

$$\dot y = \{\langle \dot z, r\rangle \in \mathrm{V}^{\mathbb{P}} \times \mathbb{P} \mid (\exists q \in \mathbb{P})\ [r \leq_{\mathbb{P}} p, q \wedge \langle \dot z, q\rangle \in \dot v_\alpha \wedge r \Vdash_{\mathbb{P}} \text{“}\dot z \in \dot x\text{”}]\} \in \mathrm{V}^{\mathbb{P}},$$

then $\mathrm{rnk}(\dot y) < \delta$ and $p \Vdash_{\mathbb{P}}$ "$\dot x = \dot y$ ". □

Lemma 2.4. *Let $\langle \mathrm{V}, \in, \delta\rangle$ be a model of* ZFC + *"$\mathrm{V}_{\dot\delta} \prec \mathrm{V}$ " and $\mathbb{P} \in \mathrm{V}_\delta$ be a partial order. If G is $\mathbb{P}$-generic over* V*, then $\langle \mathrm{V}[G], \in, \delta\rangle$ is a model of* ZFC + *"$\mathrm{V}_{\dot\delta} \prec \mathrm{V}$ ".*

Proof. By our assumptions, $\langle \mathrm{V}_\delta, \in\rangle$ is a transitive ZFC-model, G is $\mathbb{P}$-generic over V_δ, $G \in \mathrm{V}[G]_\delta$, $\mathrm{V}_\delta^{\mathbb{P}} = \mathrm{V}^{\mathbb{P}} \cap \mathrm{V}_\delta$ and Proposition 2.3 shows $\mathrm{V}_\delta[G] = \mathrm{V}[G]_\delta$.

Let $\varphi(v_0, \ldots, v_{n-1})$ be an n-ary $\mathcal{L}_\in$-formula and $x_0, \ldots, x_{n-1} \in \mathrm{V}[G]_\delta$. Fix $\dot x_0, \ldots, \dot x_{n-1} \in \mathrm{V}_\delta^{\mathbb{P}}$ with $x_i = \dot x_i^G$ for all $i < n$. If $\langle \mathrm{V}[G], \in\rangle \models \varphi(x_0, \ldots, x_{n-1})$, then there is a condition $p \in G$ with $\langle \mathrm{V}, \in\rangle \models [p \Vdash_{\mathbb{P}} \varphi(\dot x_0, \ldots, \dot x_{n-1})]$. All parameters of this statement are elements of V_δ and we can conclude $\langle \mathrm{V}_\delta, \in\rangle \models [p \Vdash_{\mathbb{P}} \varphi(\dot x_0, \ldots, \dot x_{n-1})]$. An application of the Forcing Theorem in V_δ gives us $\langle \mathrm{V}[G]_\delta, \in\rangle \models \varphi(x_0, \ldots, x_{n-1})$. □

In the following, we state and prove standard results that show how a generic extension of the ground model by a *big* partial order can be factored into a two-step iteration with the property that the intermediate model contains a certain *small* set from the original forcing extension and is a generic extension of the ground model by a *small* partial order. Given a boolean algebra $\mathbb{B}$, we let $\mathbb{B}^*$ denote the partial order with domain $\mathbb{B} \setminus \{0_{\mathbb{B}}\}$ ordered by the restriction of $\leq_{\mathbb{B}}$ to this set.

Lemma 2.5. *Let κ be an infinite cardinal, $\mathbb{B}$ be a complete boolean algebra and $\dot x \in \mathrm{V}^{\mathbb{B}^*}$ with $\mathbb{1}_{\mathbb{B}} \Vdash_{\mathbb{B}^*}$ "$\dot x \subseteq \check\kappa$ ". Then there is a κ-generated complete subalgebra $\mathbb{C}$ of $\mathbb{B}$ in* V *and names $\dot{\mathbb{D}}, \dot y \in \mathrm{V}^{\mathbb{C}^*}$ with the following properties.*

(1) $\mathbb{1}_{\mathbb{C}} \Vdash_{\mathbb{C}^*}$ "*$\dot{\mathbb{D}}$ is a partial order*" *and there is a dense embedding* $i : \mathbb{B}^* \longrightarrow \mathbb{C}^* * \dot{\mathbb{D}}$ *such that* $i(c) = \langle c, \dot{c}\rangle$ *with* $c \Vdash_{\mathbb{C}^*}$ "$\dot{c} = \check{c}$" *for all* $c \in \mathbb{C}^*$.

(2) *If* $G_0 * G_1$ *is* $(\mathbb{C}^* * \dot{\mathbb{D}})$*-generic over* V *and* G *is the preimage of* $G_0 * G_1$ *under* i*, then* $\dot{x}^G = \dot{y}^{G_0} \in \mathrm{V}[G_0]$.

Proof. Given $\alpha < \kappa$, set $\mathbb{B}_\alpha = \{b \in \mathbb{B}^* \mid b \Vdash_{\mathbb{B}^*} \text{“}\check{\alpha} \in \dot{x}\text{”}\}$ and $b_\alpha = \sup_{\mathbb{B}} \mathbb{B}_\alpha$. Let $\mathbb{C}$ be the complete subalgebra of $\mathbb{B}$ generated by the set $\{b_\alpha \mid \alpha < \kappa\}$ and define

$$\dot{y} = \{\langle \check{\alpha}, b_\alpha\rangle \in \mathrm{V}^{\mathbb{C}^*} \times \mathbb{C}^* \mid \alpha < \kappa,\ b_\alpha \neq 0_{\mathbb{B}}\} \in \mathrm{V}^{\mathbb{C}^*}.$$

Let G be $\mathbb{B}^*$-generic over V. If $\alpha \in \dot{x}^G$, then there is a $b \in G$ with $b \Vdash_{\mathbb{B}^*}$ "$\check{\alpha} \in \dot{x}$" and this shows $b_\alpha \in G$ and $\alpha \in \dot{y}^{G \cap \mathbb{C}^*}$. The other direction follows directly from the fact that $b_\alpha \in \mathbb{B}_\alpha$ holds for all $\alpha < \kappa$.

There is a canonical $\mathbb{C}^*$-name $\dot{\mathbb{D}}$ with the property that, whenever G_0 is $\mathbb{C}^*$-generic over V, then $\dot{\mathbb{D}}^{G_0}$ is the partial order whose domain is the set

$$\{b \in \mathbb{B}^* \mid (\forall c \in G_0)\ b \parallel_{\mathbb{B}^*} c\}$$

ordered by the restriction of $\leq_{\mathbb{B}}$ to this domain.

If $b \in \mathbb{B}^*$ and G is $\mathbb{B}^*$-generic over V with $b \in G$, then $b \in \dot{\mathbb{D}}^{G \cap \mathbb{C}^*}$ and there is a $c \in G \cap \mathbb{C}^*$ with $c \Vdash_{\mathbb{C}^*}$ "$\check{b} \in \dot{\mathbb{D}}$". This shows that the function

$$i_0 : \mathbb{B}^* \longrightarrow \mathbb{C}^*;\ b \mapsto \sup_{\mathbb{B}}\{c \in \mathbb{C}^* \mid c \Vdash_{\mathbb{C}^*} \text{“}\check{b} \in \dot{\mathbb{D}}\text{”}\}$$

is well-defined. Pick a function $i_1 : \mathbb{B}^* \longrightarrow \mathrm{V}^{\mathbb{C}^*}$ with $\mathbb{1}_{\mathbb{C}} \Vdash_{\mathbb{C}^*}$ "$i_1(b) \in \dot{\mathbb{D}}$" and $i_0(b) \Vdash_{\mathbb{C}^*}$ "$\check{b} = i_1(b)$" for all $b \in \mathbb{B}^*$. Define $i : \mathbb{B}^* \longrightarrow \mathbb{C}^* * \dot{\mathbb{D}}$ by setting $i(b) = \langle i_0(b), i_1(b)\rangle$.

Given $c, c' \in \mathbb{C}^*$, it is easy to see that $c' \Vdash_{\mathbb{C}^*}$ "$\check{c} \in \dot{\mathbb{D}}$" is equivalent to $c' \leq_{\mathbb{C}} c$. This shows that $i_0(c) = c$ holds for all $c \in \mathbb{C}^*$. We show that i is a dense embedding.

Let $b_0, b_1 \in \mathbb{B}^*$ with $b_0 \leq_{\mathbb{B}} b_1$. Given $c \in \mathbb{C}^*$, if $c \Vdash_{\mathbb{C}^*}$ "$\check{b}_0 \in \dot{\mathbb{D}}$", then $c \Vdash_{\mathbb{C}^*}$ "$\check{b}_1 \in \dot{\mathbb{D}}$". This shows that $i_0(b_0) \leq_{\mathbb{C}^* * \dot{\mathbb{D}}} i_0(b_1)$ holds and hence $i(b_0) \leq_{\mathbb{C}^* * \dot{\mathbb{D}}} i(b_1)$. Next, fix $a_0, a_1 \in \mathbb{B}^*$ with $a_0 \perp_{\mathbb{B}^*} a_1$. Assume, toward a contradiction, that there is a $\langle c, \dot{d}\rangle \in \mathbb{C}^* * \dot{\mathbb{D}}$ with $\langle c, \dot{d}\rangle \leq_{\mathbb{C}^* * \dot{\mathbb{D}}} i(a_0), i(a_1)$. We can find a $0_{\mathbb{C}} <_{\mathbb{C}} c_* \leq_{\mathbb{C}} c$ and a condition $d \in \mathbb{B}^*$ with $c_* \Vdash_{\mathbb{C}^*}$ "$\dot{d} = \check{d}$". This means $c_* \Vdash_{\mathbb{C}^*}$ "$\check{d} \leq_{\dot{\mathbb{D}}} \check{a}_0, \check{a}_1$" and therefore $0_{\mathbb{B}} <_{\mathbb{B}} d \leq_{\mathbb{B}} a_0, a_1$, a contradiction. Finally, fix $\langle c, \dot{d}\rangle \in \mathbb{C}^* * \dot{\mathbb{D}}$. As above, there are $0_{\mathbb{C}} <_{\mathbb{C}} c_* \leq_{\mathbb{C}} c$ and $d \in \mathbb{B}^*$ with $c_* \Vdash_{\mathbb{C}^*}$ "$\dot{d} = \check{b}$". Since $c_* \Vdash_{\mathbb{C}^*}$ "$\check{c}_* \parallel_{\check{\mathbb{B}}^*} \check{d}$", there is a condition $d_* \in \mathbb{B}^*$ with $d_* \leq_{\mathbb{B}} c_*, d$. By the above computations, $i_0(d_*) \leq_{\mathbb{C}} i_0(c_*) = c_* \leq_{\mathbb{C}} c$ and $i_0(d_*) \Vdash_{\mathbb{C}^*}$ "$i_1(d_*) \leq_{\dot{\mathbb{D}}} \dot{d}$". This means $i(d_*) \leq_{\mathbb{C}^* * \dot{\mathbb{D}}} \langle c, \dot{d}\rangle$ and i is a dense embedding.

If $G_0 * G_1$ is $(\mathbb{C}^* * \dot{\mathbb{D}})$-generic over V and G is the preimage of $G_0 * G_1$ under i, then both G_0 and $G \cap \mathbb{C}^*$ are $\mathbb{C}^*$-generic over V. Since $i_0 \restriction \mathbb{C}^* = \mathrm{id}_{\mathbb{C}^*}$, it follows that $G \cap \mathbb{C}^* \subseteq G_0$ and the maximality of generic filters yields $G \cap \mathbb{C}^* = G_0$. By the above calculations, $\dot{x}^G = \dot{y}^{G \cap \mathbb{C}^*} = \dot{y}^{G_0} \in \mathrm{V}[G_0]$. □

Given an infinite cardinal κ and a set X, we let $[X]^{<\kappa}$ denote the set of all subsets of X of cardinality less than κ. If $\mathbb{B}$ is a boolean algebra and $\mathbb{C}$ is a subalgebra of $\mathbb{B}$, then $\mathbb{C}$ is called $<\kappa$-complete in $\mathbb{B}$ if $\inf_{\mathbb{B}} X \in \mathbb{C}$ for all $X \in [\mathbb{C}]^{<\kappa}$.

Proposition 2.6. *Let* κ *be an infinite regular cardinal,* $\mathbb{B}$ *be a complete boolean algebra that satisfies the* κ*-chain condition and* $\mathbb{C}$ *be a subalgebra of* $\mathbb{B}$*. If* $\mathbb{C}$ *is* $<\kappa$*-complete in* $\mathbb{B}$*, then* $\mathbb{C}$ *is a complete subalgebra of* $\mathbb{B}$*.*

Proof. Assume, toward a contradiction, that $\mathbb{C}$ is not a complete subalgebra of $\mathbb{B}$ and let ν be the least cardinal such that there is a sequence $\langle c_\alpha \in \mathbb{C} \mid \alpha < \nu\rangle$ with $\inf_{\mathbb{B}}\{c_\alpha \mid \alpha < \nu\} \notin \mathbb{C}$. By our assumption, $\nu \geq \kappa$ and it is easy to see that ν is a regular cardinal. Given $\alpha < \nu$, we define $b_\alpha = \inf_{\mathbb{B}}\{c_\beta \mid \beta < \alpha\}$. Our assumptions imply $0_{\mathbb{B}} \neq b_\alpha \in \mathbb{C}$ and $b_\beta \leq_{\mathbb{B}} b_\alpha$ for all $\alpha \leq \beta < \nu$. Moreover, $\inf_{\mathbb{B}}\{b_\alpha \mid \alpha < \nu\} = \inf_{\mathbb{B}}\{c_\alpha \mid \alpha < \nu\} \notin \mathbb{C}$. If we define $a_\alpha = b_\alpha - b_{\alpha+1}$ for all $\alpha < \nu$, then the set $A = \{a_\alpha \in \mathbb{B} \mid \alpha < \nu, a_\alpha \neq 0_{\mathbb{B}}\}$ is an anti-chain in $\mathbb{B}$ and has cardinality less than κ. This means that there is an $\alpha < \nu$ with $a_\beta = 0_{\mathbb{B}}$ for all $\alpha \leq \beta < \nu$ and an easy induction shows that this implies $b_{\alpha+1} = b_\beta$ for all $\alpha < \beta < \nu$. We can conclude $\inf_{\mathbb{B}}\{b_\alpha \mid \alpha < \nu\} = b_{\alpha+1} \in \mathbb{C}$, a contradiction. □

Lemma 2.7. *Let κ be an infinite cardinal, $\mathbb{B}$ be a complete boolean algebra that satisfies the κ^+-chain condition and C be a subset of $\mathbb{B}$ of cardinality at most κ. If $\mathbb{C}$ is the complete subalgebra of $\mathbb{B}$ generated by C, then $\mathbb{C}$ has cardinality at most 2^κ.*

Proof. It suffices to construct a complete subalgebra $\mathbb{C}^+$ of $\mathbb{B}$ that contains C and has cardinality at most 2^κ. We define an ascending sequence $\langle \mathbb{C}_\alpha \mid \alpha < \kappa^+\rangle$ of subalgebras of $\mathbb{B}$ in the following way.

(1) $\mathbb{C}_0$ is the subalgebra of $\mathbb{B}$ generated by C.
(2) If $\alpha \in \kappa^+ \cap \mathrm{Lim}$, then $\mathbb{C}_\alpha = \bigcup\{\mathbb{C}_\beta \mid \beta < \alpha\}$.
(3) $\mathbb{C}_{\alpha+1}$ is the subalgebra of $\mathbb{B}$ generated by the set $\{\inf_{\mathbb{B}} X \mid X \in [\mathbb{C}_\alpha]^{<\kappa^+}\}$ for all $\alpha < \kappa^+$.

An easy induction shows that the subalgebra $\mathbb{C}_\alpha$ has cardinality at most 2^κ for all $\alpha < \kappa^+$ and this shows that the subalgebra $\mathbb{C}^+ = \bigcup\{\mathbb{C}_\alpha \mid \alpha < \kappa^+\}$ also has cardinality at most 2^κ. We show that $\mathbb{C}^+$ is a complete subalgebra of $\mathbb{B}$. By Proposition 2.6, it suffices to show that $\mathbb{C}^+$ is $<\kappa^+$-complete in $\mathbb{B}$. If $X \in [\mathbb{C}^+]^{<\kappa^+}$, then there is an $\alpha < \kappa^+$ with $X \subseteq \mathbb{C}_\alpha$. But this means $\inf_{\mathbb{B}} X \in \mathbb{C}_{\alpha+1} \subseteq \mathbb{C}^+$. □

We are now ready to prove our main result.

Proof of Theorem 1.8. By Proposition 2.1, the consistency of ZFC gives us a model $\langle \mathrm{V}, \in, \delta\rangle$ of ZFC + (GCH) + "$\mathrm{V}_{\dot\delta} \prec \mathrm{V}$". Work in V and apply Theorem 1.4 with $\kappa = \omega_1$ and $\nu = \alpha = \delta^+$ to produce a partial order $\mathbb{P}$ with the above properties. By [Kun80, Lemma 3.3, page 63], there is a complete boolean algebra $\mathbb{B}$ and a dense embedding $d : \mathbb{P} \longrightarrow \mathbb{B}^*$.

We can find a name $\dot{x} \in \mathrm{V}^{\mathbb{B}^*}$ with the property that, whenever H is $\mathbb{B}^*$-generic over V, then $\dot{x}^H \subseteq \omega_1$ and there is a centreless group $G \in \mathrm{V}[H]$ with domain ω_1, $\tau(G)^{\mathrm{V}[H]} = \delta$ and

$$\alpha \circ_G \beta = \gamma \iff \prec\prec\alpha, \beta\succ, \gamma\succ \in \dot{x}^H \tag{1}$$

for all $\alpha, \beta, \gamma < \omega_1$.[3] Apply Lemma 2.5 with ω_1, $\mathbb{B}$ and $\dot{x}$ to find $\mathbb{C}$, $\dot{\mathbb{D}}$, i and $\dot{y}$ with the above properties. Since $\mathbb{B}$ satisfies the $\aleph_2$-chain condition, we can apply Lemma 2.7 to see that $\mathbb{C}$ has cardinality at most $2^{\aleph_1} < \delta$. Let $\mathbb{P}_0 \in \mathrm{V}_\delta$ be a partial order isomorphic to $\mathbb{C}^*$ in V. Our construct ensures that $\mathbb{P}_0$ is σ-distributive and satisfies the $\aleph_2$-chain condition. Let F be $\mathbb{P}_0$-generic over V. By Lemma 2, $\langle \mathrm{V}[F], \in, \delta\rangle$ is a model of ZFC + "$\mathrm{V}_{\dot\delta} \prec \mathrm{V}$". We will show that $\langle \mathrm{V}[F], \in\rangle$ contains a group with unbounded automorphism tower heights.

[3] We let $\prec\cdot,\cdot\succ : \mathrm{On} \times \mathrm{On} \longrightarrow \mathrm{On}$ denote the Gödel pairing function.

Let $H_0 \in \mathrm{V}[F]$ be the image of F under the isomorphism between $\mathbb{P}_0$ and $\mathbb{C}^*$. We have $\mathrm{V}[F] = \mathrm{V}[H_0]$. Let H_1 be $\dot{\mathbb{D}}^{H_0}$-generic over $\mathrm{V}[H_0]$ and H be the preimage of $H_0 * H_1$ under i. In $\mathrm{V}[H] = \mathrm{V}[H_0][H_1]$, there is a centreless group G with domain ω_1 and $\tau^{\mathrm{V}[H]} = \delta$ whose group operation is coded by $\dot{x}^H$ as in (1). The above construction yields $\dot{x}^H \in \mathrm{V}[H_0]$ and this means $G \in \mathrm{V}[H_0]_\delta = \mathrm{V}[F]_\delta$. In $\mathrm{V}[H_0]$, $\dot{\mathbb{D}}^{H_0}$ is a σ-distributive partial order that satisfies the $\aleph_2$-chain condition and we have

$$\langle \mathrm{V}[H_0], \in\rangle \models \left[\mathbb{1}_{\dot{\mathbb{D}}^{H_0}} \Vdash_{\dot{\mathbb{D}}^{H_0}} \text{“}\tau(\check{G}) > \check{\delta}\text{”}\right]. \tag{2}$$

Let $\varphi(v_0, \dots, v_4)$ be the $\mathcal{L}_\in$-formula that is the conjunction of the following statements:

(1) “*v_0 is a σ-distributive partial order satisfying the $\aleph_2$-chain condition*”,
(2) “*v_1 is a centreless group and v_2 is the canonical v_0-name for v_1*”,
(3) “*v_3 is an ordinal and v_4 is the canonical v_0-name for v_3*”,
(4) $\mathbb{1}_{v_0} \Vdash_{v_0}$ “$\tau(v_2) \geq v_4$”.

In order to show that G is a group with unbounded potential automorphism tower heights in $\langle \mathrm{V}[F], \in\rangle$, it clearly suffices to show

$$\langle \mathrm{V}[F], \in\rangle \models (\forall \alpha \in \mathrm{On})(\exists x, y, z)\ \varphi(x, G, y, \alpha, z). \tag{3}$$

Given $\alpha < \delta$, the above (2) implies

$$\langle \mathrm{V}[F], \in\rangle \models (\exists x, y, z)\ \varphi(x, G, y, \alpha, z)$$

and all parameters of this statement are contained in $\mathrm{V}[F]_\delta$. By elementarity, we have

$$\langle \mathrm{V}[F]_\delta, \in\rangle \models (\forall \alpha \in \mathrm{On})(\exists x, y, z)\ \varphi(x, G, y, \alpha, z).$$

and another application of elementarity shows that (3) holds. □

References

[FH08] Gunter Fuchs and Joel David Hamkins. Changing the heights of automorphism towers by forcing with Souslin trees over L. *J. Symbolic Logic*, 73(2):614–633, 2008.

[FL10] Gunter Fuchs and Philipp Lücke. Iteratively changing the heights of automorphism towers. Submitted, 2010.

[Ham02] Joel David Hamkins. How tall is the automorphism tower of a group? In *Logic and algebra*, volume 302 of *Contemp. Math.*, pages 49–57. Amer. Math. Soc., Providence, RI, 2002.

[Ham03] Joel David Hamkins. A simple maximality principle. *J. Symbolic Logic*, 68(2):527–550, 2003.

[HT00] Joel David Hamkins and Simon Thomas. Changing the heights of automorphism towers. *Ann. Pure Appl. Logic*, 102(1-2):139–157, 2000.

[JST99] Winfried Just, Saharon Shelah, and Simon Thomas. The automorphism tower problem revisited. *Adv. Math.*, 148(2):243–265, 1999.

[Kun80] Kenneth Kunen. *Set theory*, volume 102 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1980. An introduction to independence proofs.

[Mos09] Yiannis N. Moschovakis. *Descriptive set theory*, volume 155 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2009.

[Tho] Simon Thomas. *The Automorphism Tower Problem.* To appear.

[Tho85] Simon Thomas. The automorphism tower problem. *Proc. Amer. Math. Soc.*, 95(2):166–168, 1985.

[Tho98] Simon Thomas. The automorphism tower problem. II. *Israel J. Math.*, 103:93–109, 1998.

[Wie39] Helmut Wielandt. Eine Verallgemeinerung der invarianten Untergruppen. *Math. Z.*, 45(1):209–244, 1939.

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