SPECIAL PAIRS AND AUTOMORPHISMS OF CENTRELESS GROUPS

PHILIPP LÜCKE

Dedicated to Rüdiger Göbel on the occasion of his seventieth birthday.

Abstract. Let G be a group, A be a subset of the domain of G and $\mathcal{L}_A$ be the first-order language of group theory expanded by constant symbols for elements in A. We call the pair $\langle G, A\rangle$ *special* if every element g of G is uniquely determined by the set $\mathsf{qft}_{G,A}(g)$ consisting of all $\mathcal{L}_A$-terms $\mathrm{t}(v)$ with one free variable and $\mathrm{t}^G(g) = 1_G$. The pair $\langle G, A\rangle$ is *strongly special* if $\mathsf{qft}_{G,A}(g) \subseteq \mathsf{qft}_{G,A}(h)$ implies $g = h$ for all $g, h \in G$. Special pairs were introduced by Itay Kaplan and Saharon Shelah to analyze automorphism towers of centreless groups. The purpose of this note is the further analysis of special pairs and their interaction with automorphism groups. This analysis will allow us to prove an absoluteness result for the first three stages of the automorphism tower of countable, centreless groups. Moreover, we develop methods that enable us to construct a variety of examples of such pairs, including special pairs that are not strongly special.

1. Introduction

We let $\mathcal{L}_{GT} = \langle *, {}^{-1}, \mathbb{1}\rangle$ denote the first-order language of group theory. Given a group G and a subset A of the domain of G, we define $\mathcal{L}_A$ to be the first-order language that extends $\mathcal{L}_{GT}$ by introducing a new constant symbol $\dot{g}$ for each element g of A. We regard G as an $\mathcal{L}_A$-model in the obvious way.

We define $\mathcal{T}_A$ to be the set of all $\mathcal{L}_A$-terms $t \equiv t(v)$ with exactly one free variable. If g is an element of the domain of G, then we define

$$\mathsf{qft}_{G,A}(g) = \{t(v) \in \mathcal{T}_A \mid G \models \text{“}t(g) = \mathbb{1}\text{”}\}$$

and call this set the *quantifier-free A-type of g.*

Definition 1.1. Given a group G and a subset A of the domain of G, the pair $\langle G, A\rangle$ is *special* if the function

$$\mathsf{qft}_{G,A} : G \longrightarrow \mathcal{P}(\mathcal{T}_A);\ g \longmapsto \mathsf{qft}_{G,A}(g)$$

is injective.

Special pairs were introduced by Itay Kaplan and Saharon Shelah in [KS09] to analyze automorphism towers of centreless groups. Given a special pair $\langle G, A\rangle$, this notion allows us to measure the complexity of the group G by interpreting it as a set of subsets of $\mathcal{T}_A$. For example, if A is countable, then we can easily identify subsets

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of $\mathcal{T}_A$ with elements of *Cantor space* ${}^{\omega}2$ (i.e. *reals*) and talk about the complexity of G in terms of *descriptive set theory* (i.e. as *definable sets of reals*). This approach is used in [KS09] to find new upper bounds for the heights of automorphism towers.

The aim of this note is to further investigate this notion and the following strengthening of it.

Definition 1.2. Given a group G and a subset A of the domain of G, we call the pair $\langle G, A\rangle$ *strongly special* if $\mathsf{qft}_{G,A}(g) \subseteq \mathsf{qft}_{G,A}(h)$ implies $g = h$ for all $g, h \in G$.

We outline the content of this note. In Section 2, we will introduce automorphism towers and quote the results developed in [KS09] that connect special pairs with automorphism towers. We will show that those results also hold for strongly special pairs. In the next section, we introduce concepts and results from the theory of *Polish groups* and derive helpful consequences for special pairs consisting of a Polish group and a countable subset of its domain. These results are applied in Section 4 to prove an absoluteness result for the first three stages of automorphism towers of countable, centreless groups. This result contrasts existing non-absoluteness results for the automorphism towers of certain uncountable groups (see [Tho98], [HT00] and [FL]). Section 5 introduces another way to construct strongly special pairs using groups of autohomeomorphisms of certain Hausdorff spaces. This construction relies on methods and results developed by Robert R. Kallman in [Kal86]. In the last section, we will use a result of Manfred Droste, Michèle Giraudet and Rüdiger Göbel to show that there are special pairs that are not strongly special.

Notations. Given a group G, we will also use the letter G to denote the domain of G. We denote applications of the group operation by $g \cdot h$ and we will abbreviate the term $g \cdot h \cdot g^{-1}$ by h^g. If A is a subset of the domain of G, then we let $\langle A\rangle$ denote the subgroup of G generated by A.

If f is a function, A is a subset of the domain of f and B is a subset of the range of f, then $f[A]$ is the pointwise image of A under f and $f^{-1}[B]$ denotes the preimage of B under f. Given functions f and g with $\operatorname{ran}(g) \subseteq \operatorname{dom}(f)$, we use $f \circ g$ to denote the corresponding composition of functions.

We let $\operatorname{Sym}(X)$ denote the *symmetric group* of a set X and $\operatorname{Alt}(X)$ denote the corresponding *alternating group* consisting of all finite even permutations of X. If $a, b \in X$, then $(a\ b)$ denotes the *transposition* of a and b.

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2. Automorphism towers

We start this section by introducing automorphism towers of centreless groups. An extensive account of all aspects of the automorphism tower problem can be found in Simon Thomas' forthcoming monograph [Tho].

Let G be a group with trivial centre. For each $g \in G$, the *inner automorphism corresponding to* g is the map

$$\iota_g : G \longrightarrow G;\ h \longmapsto h^g.$$

It is easy to see that the map

$$\iota_G : G \longrightarrow \mathrm{Aut}(G);\ g \longmapsto \iota_g$$

is an embedding that maps G onto the subgroup $\mathrm{Inn}(G)$ of all inner automorphisms of G. An easy computation shows that $\pi \circ \iota_g \circ \pi^{-1} = \iota_{\pi(g)}$ holds for all $g \in G$ and $\pi \in \mathrm{Aut}(G)$. This implies that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$ and $\mathrm{C}_{\mathrm{Aut}(G)}(\mathrm{Inn}(G)) = \{\mathrm{id}_G\}$. In particular, $\mathrm{Aut}(G)$ is also a group with trivial centre. By iterating this process, we construct the automorphism tower of G.

Definition 2.1. A sequence $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ of groups is *an automorphism tower of a centreless group* G if the following statements hold.

(1) $G = G_0$.
(2) If $\alpha \in \mathrm{On}$, then G_α is a normal subgroup of $G_{\alpha+1}$ and the induced homomorphism

$$\varphi_\alpha : G_{\alpha+1} \longrightarrow \mathrm{Aut}(G_\alpha);\ g \mapsto \iota_g \restriction G_\alpha$$

is an isomorphism.
(3) If $\alpha \in \mathrm{Lim}$, then $G_\alpha = \bigcup\{G_\beta \mid \beta < \alpha\}$.

In this definition, we replaced $\mathrm{Aut}(G_\alpha)$ by an isomorphic copy $G_{\alpha+1}$ that contains G_α as a normal subgroup. This allows us to take unions instead of direct limits at limit stages. Given $\alpha \in \mathrm{On}$, it is easy to see that the α-th group in an automorphism tower of some centreless group G is uniquely determined up to isomorphisms that induce the identity on G.

We are now ready to state the result from [KS09] that establishes a connection between automorphism towers and special pairs.

Theorem 2.2 ([KS09, Conclusion 3.10])**.** *Let $\langle G, A\rangle$ be a special pair with $\mathrm{C}_G(A) = \{1_G\}$ and $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ be an automorphism tower of G. If $\alpha \in \mathrm{On}$, then $\langle G_\alpha, A\rangle$ is a special pair and $\mathrm{C}_{G_\alpha}(A) = \{1_G\}$ holds.*

Corollary 2.3. *If $\langle G, A\rangle$ is a special pair with A infinite and $\mathrm{C}_G(A) = \{1_G\}$, then there is an $\alpha < \left(2^{|A|}\right)^+$ with $G_\alpha = G_{\alpha+1}$.*

Proof. Let $\nu = 2^{|A|}$ and assume, toward a contradiction, that $G_\alpha \neq G_{\alpha+1}$ holds for all $\alpha < \nu^+$. Pick a sequence $\langle g_\alpha \mid \alpha < \nu^+\rangle$ with $g_\alpha \in G_{\alpha+1} \setminus G_\alpha$. Given $\alpha < \nu^+$ and $\beta > \alpha$, it is easy to see that $\mathsf{qft}_{G_{\alpha+1},G}(g_\alpha) = \mathsf{qft}_{G_\beta,G}(g_\alpha)$ holds. By Theorem 2.2, $\langle \mathsf{qft}_{G_{\alpha+1},G}(g_\alpha) \mid \alpha < \nu^+\rangle$ is a sequence of pairwise distinct subsets of $\mathcal{T}_A$. But, $\mathcal{T}_A$ has cardinality $|A|$ and there are only ν-many subsets of $\mathcal{T}_A$, a contradiction. $\square$

The above result allows a short proof of Simon Thomas' *automorphism tower theorem.*

Corollary 2.4 ([Tho98, Theorem 1.3])**.** *If G is an infinite centreless group of cardinality κ, then there is an $\alpha < (2^\kappa)^+$ with $G_\alpha = G_{\alpha+1}$.*

Proof. Since $\dot{g} * v^{-1} \in \mathsf{qft}_{G,G}(g)$ holds for all $g \in G$, it is easy to see that $\langle G, G\rangle$ is a special pair with $\mathrm{C}_G(G) = \mathrm{Z}(G) = \{1_G\}$. $\square$

Note that, in the above situation, $G_\alpha = G_{\alpha+1}$ implies $G_\alpha = G_\beta$ for all $\beta \geq \alpha$. The automorphism tower theorem allows us to make the following definition.

Definition 2.5. If G is a centreless group, then we let $\tau(G)$ denote the least ordinal α such that $G_\alpha = G_{\alpha+1}$ holds whenever $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ is an automorphism tower of G.

In the remainder of this section, we will show that the statement of Theorem 2.2 still holds if we replace *special pair* by *strongly special pair*. We start by generalizing the following characterization of special pairs in terms of local homomorphisms to strongly special pairs.

Lemma 2.6 ([KS09, Remark 3.5 (1)])**.** *If G is a group and A is a subset of the domain of G, then the following statements are equivalent.*

(1) *$\langle G, A\rangle$ is a special pair.*
(2) *If $g \in G$ and $\varphi : \langle A \cup \{g\}\rangle \longrightarrow G$ is a group monomorphism with $\varphi \restriction A = \mathrm{id}_A$, then $\varphi(g) = g$.*

This characterization generalizes to strongly special pairs in the following way.

Lemma 2.7. *If G is a group and A is a subset of the domain of G, then the following statements are equivalent.*

(1) *$\langle G, A\rangle$ is a strongly special pair.*
(2) *If $g \in G$ and $\varphi : \langle A \cup \{g\}\rangle \longrightarrow G$ is a group homomorphism with $\varphi \restriction A = \mathrm{id}_A$, then $\varphi(g) = g$.*

Proof. Let $\langle G, A\rangle$ be a strongly special pair, $g \in G$ and $\varphi : \langle A \cup \{g\}\rangle \longrightarrow G$ be a group homomorphism with $\varphi \restriction A = \mathrm{id}_A$. An easy induction shows that $\mathrm{t}^G(g) \in \langle A \cup \{g\}\rangle$ and $\varphi(\mathrm{t}^G(g)) = \mathrm{t}^G(\varphi(g))$ hold for every term $\mathrm{t}(v) \in \mathcal{T}_A$. In particular, $\mathsf{qft}_{G,A}(g) \subseteq \mathsf{qft}_{G,A}(\varphi(g))$ and we can conclude $g = \varphi(g)$.

Assume that the second statement holds. Let $g_0, g_1 \in G$ with $\mathsf{qft}_{G,A}(g_0) \subseteq \mathsf{qft}_{G,A}(g_1)$. Pick $\mathrm{t}_0, \mathrm{t}_1 \in \mathcal{T}_A$ with $\mathrm{t}_0^G(g_0) = \mathrm{t}_1^G(g_0)$. Then $\mathrm{t}_0 * \mathrm{t}_1^{-1} \in \mathsf{qft}_{G,A}(g_0) \subseteq \mathsf{qft}_{G,A}(g_1)$ and $\mathrm{t}_0^G(g_1) = \mathrm{t}_1^G(g_1)$. Given $h \in \langle A \cup \{g_0\}\rangle$, there is a term $\mathrm{t}(v) \in \mathcal{T}_A$ with $\mathrm{t}^G(g_0) = h$ and, if we define $\varphi(h) = \mathrm{t}^G(g_1)$, then the above computations show that $\varphi(h)$ does not depend on the choice of t. Moreover, these computations directly imply that $\varphi : \langle A \cup \{g_0\}\rangle \longrightarrow G$ is a group homomorphism with $\varphi(g_0) = g_1$ and $\varphi \restriction A = \mathrm{id}_A$. By our assumption, we have $g_0 = g_1$. □

This characterization allows us to prove a version of [KS09, Claim 3.8] for strongly special pairs. Note that the proofs of the two statements are almost identical.

Lemma 2.8. *Let $\langle G, A\rangle$ be a strongly special pair and H be a group such that G is a normal subgroup of H and $\mathrm{C}_H(G) = \{1_G\}$. Then $\langle H, A\rangle$ is a strongly special pair.*

Proof. Let $h \in H$ and $\varphi : \langle A \cup \{h\}\rangle \longrightarrow H$ be a group homomorphism with $\varphi \restriction A = \mathrm{id}_A$. Pick $a \in A$. Then $a^h \in G$, $\varphi(a^h) = a^{\varphi(h)} \in G$ and, if we define $\psi = \varphi \restriction \langle A \cup \{a^h\}\rangle$, then $\psi : \langle A \cup \{a^h\}\rangle \longrightarrow G$ is a group homomorphism with $\psi \restriction A = \mathrm{id}_A$. By our assumption, we have $a^h = \psi(a^h) = a^{\varphi(h)}$. This argument shows $h \cdot \varphi(h^{-1}) \in \mathrm{C}_H(A)$.

Now fix $g \in G$ and define $\xi : \langle A \cup \{g\}\rangle \longrightarrow G$ by $\xi = \iota_{h\cdot\varphi(h^{-1})} \restriction \langle A \cup \{g\}\rangle$. By the above computations, we have $\xi \restriction A = \mathrm{id}_A$ and this means $g = \xi(g) = g^{h\cdot\varphi(h^{-1})}$. We can conclude $h \cdot \varphi(h^{-1}) \in \mathrm{C}_H(G) = \{1_G\}$ and $h = \varphi(h)$. □

We are now ready to prove the promised version of Theorem 2.2 for strongly special pairs. Again, the proofs of both results are almost identical.

Theorem 2.9. *Let $\langle G, A\rangle$ be a strongly special pair with $\mathrm{C}_G(A) = \{1_G\}$ and $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ be an automorphism tower of G. If $\alpha \in \mathrm{On}$, then $\langle G_\alpha, A\rangle$ is a strongly special pair.*

Proof. We prove the statement of the theorem by induction.

Assume $\langle G_\alpha, A\rangle$ is a strongly special pair. If $h \in \mathrm{C}_{G_{\alpha+1}}(G_\alpha)$, then $\iota_h \restriction G_\alpha = \mathrm{id}_{G_\alpha}$ and $h = 1_G$. Since G_α is a normal subgroup of $G_{\alpha+1}$, we can apply Lemma 2.8 to see that $\langle G_{\alpha+1}, A\rangle$ is also a strongly special pair.

Let α be a limit ordinal and assume that $\langle G_\beta, A\rangle$ is a strongly special pair for every $\beta < \alpha$. Given $g_0, g_1 \in G_\alpha$ with $\mathsf{qft}_{G_\alpha,A}(g_0) \subseteq \mathsf{qft}_{G_\alpha,A}(g_1)$, there is a $\beta < \alpha$ with $g_0, g_1 \in G_\beta$ and it is easy to see that $\mathsf{qft}_{G_\alpha,A}(g_i) = \mathsf{qft}_{G_\beta,A}(g_i)$. In particular, we have $g_0 = g_1$. □

3. Unique Polish group topologies

We introduce techniques from the theory of *Polish groups* that will be essential for the proof of the absoluteness result for the automorphism towers of countable, centreless groups mentioned in the introduction. Remember that a *topological group* is a pair $\langle G, \tau\rangle$ consisting of a group G and a topology τ on the domain of G such that the map $\left[\langle g, h\rangle \mapsto g \cdot h^{-1}\right]$ is continuous with respect to τ. We call a topological space $\langle X, \tau\rangle$ *Polish* if τ is induced by a complete metric on X and there is a countable subset of X that is dense in τ. Finally, we call a topological group $\langle G, \tau\rangle$ a *Polish group* if the corresponding topological space is Polish. In this case, we call τ a *Polish group topology* on G.

Proposition 3.1. *Let $\langle G, \tau\rangle$ be a topological group such that the corresponding topological space is a Hausdorff space. If $\mathrm{t} \in \mathcal{T}_G$, then the set $\{g \in G \mid \mathrm{t}^G(g) = 1_G\}$ is closed in τ.*

Proof. An easy induction on shows that the map

$$\xi_{\mathrm{t}} : G^n \longrightarrow G;\ \vec{g} \longmapsto \mathrm{t}^G(\vec{g})$$

is continuous with respect to τ for every $\mathcal{L}_G$-term $\mathrm{t} \equiv \mathrm{t}(v_0, \ldots, v_{n-1})$ with n free variables. Since τ is a Hausdorff space, we can conclude that the set

$$\{g \in G \mid \mathrm{t}^G(g) = 1_G\} = \xi_{\mathrm{t}}^{-1}[\{1_G\}]$$

is closed in τ for every $t \in \mathcal{T}_G$. □

Next, we consider Polish groups whose topology is completely determined by the algebraic structure of the group.

Definition 3.2. Let G be a group. We say that G has a *unique Polish group topology* if there is exactly one topology τ on the domain of G such that $\langle G, \tau\rangle$ is a Polish group.

We state a theorem of George W. Mackey that allows a nice characterization of groups with unique Polish group topologies. Remember that a measurable space $\langle X, \mathcal{S}\rangle$ is a *standard Borel space* if there is a Polish topology τ on X such that $\mathcal{S}$ is equal to the σ-algebra $\mathcal{B}(\tau)$ of all subsets of X that are Borel with respect to τ.

Theorem 3.3 ([Mac57, Theorem 3.3])**.** *Let $\langle X, \mathcal{S}_0\rangle$ and $\langle X, \mathcal{S}_1\rangle$ be standard Borel spaces. If there is a countable point-separating family*[1] *of subsets of X whose members are elements of both $\mathcal{S}_0$ and $\mathcal{S}_1$, then $\mathcal{S}_0 = \mathcal{S}_1$.*

Corollary 3.4. *The following statements are equivalent for a Polish group $\langle G, \tau\rangle$.*

1. *τ is the unique Polish group topology on G.*
2. *There is a countable point-separating family of subsets of the domain of G whose members are Borel with respect to any Polish group topology on G.*

Proof. If τ is the unique Polish group topology on G and B is a countable basis of τ, then B satisfies the above properties.

In the other direction, assume that $\mathcal{F}$ is a family of subsets with the above properties and $\bar{\tau}$ is a Polish group topologies on G. If we define $\mathcal{B}(\tau)$ and $\mathcal{B}(\bar{\tau})$ as above, then Theorem 3.3 and our assumptions imply $\mathcal{B}(\tau) = \mathcal{B}(\bar{\tau})$. Since Borel sets have the Baire Property (see [Kec95, Proposition 8.22]), the identity map on G is a Baire-measurable group homomorphism with respect to τ and $\bar{\tau}$. By [BK96, Theorem 1.2.6], it is continuous and open with respect to τ and $\bar{\tau}$. This shows $\tau = \bar{\tau}$. □

Proposition 3.5. *Let $\langle G, \tau\rangle$ be a Polish group. If there is a countable subset A of the domain of G such that $\langle G, A\rangle$ is a special pair, then τ is the unique Polish group topology on G.*

Proof. If $\mathrm{t} \equiv \mathrm{t}(v)$ is a term in $\mathcal{T}_A$, then we define $T^0_{\mathrm{t}} = \{g \in G \mid \mathrm{t}^G(g) = 1_G\}$ and $T^1_{\mathrm{t}} = \{g \in G \mid \mathrm{t}^G(g) \neq 1_G\}$. Let $\mathcal{F}$ denote the family consisting of all subsets of the domain of G of the form T^0_{t} or T^1_{t} for some $\mathrm{t} \in \mathcal{T}_A$. Then $\mathcal{F}$ is countable and separates points, because $\langle G, A\rangle$ is a special pair. If $\bar{\tau}$ is a Polish group topology on G, then all elements of $\mathcal{F}$ are contained in $\mathcal{B}(\bar{\tau})$ by Proposition 3.1. Corollary 3.4 implies that τ is the unique Polish group topology on G. □

Remark 3.6. The converse of the above implication is not true: Bojana Pejić and Paul Gartside showed that the group $\mathrm{SO}(3, \mathbb{R})$ has a unique Polish group topology (see [GP08, Theorem 11]) and there is no countable subset I of $\mathcal{T}_{\mathrm{SO}(3,\mathbb{R})}$ such that the family $\{T^i_{\mathrm{t}} \mid \mathrm{t} \in I,\ i < 2\}$ separates points (see [GP08, Lemma 12]).

We close this section by introducing a consequence of the existence of a unique Polish group topology that allows us to deduce the absoluteness result in the next section. This consequence is called *automatic continuity of automorphisms.*

Proposition 3.7. *Let G be a group with a unique Polish group topology. Then every group automorphism of G is continuous with respect to the unique Polish group topology on G.*

Proof. Let τ be the unique Polish group topology on G and assume, toward a contradiction, that there is an automorphism π of G that is not continuous with respect to τ. Define $\bar{\tau}$ to be the collection of all subsets of G of the form $\pi[U]$,

[1] We call a family $\mathcal{F}$ of subsets of X *separating* if for any pair $\langle x, y\rangle$ of distinct elements in X, there is an $F \in \mathcal{F}$ with $x \in F$ and $y \notin F$.

where U is open in τ. It is easy to check that $\bar{\tau}$ is a Polish group topology that is not equal to τ, a contradiction. $\square$

4. An absoluteness result for automorphism towers of countable centreless groups

The aim of this section is to prove the following theorem.

Theorem 4.1. *Let M be a transitive model of* ZFC[2]*, G be a centreless group that is an element of M, $\langle G_\alpha^M \mid \alpha \in \mathrm{On} \cap M\rangle$ be an automorphism tower of G in M and $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ be an automorphism tower of G. If G is countable in M, then there is an embedding $\pi : G_2^M \longrightarrow G_2$ with $\pi \restriction G = \mathrm{id}_G$.*

This theorem directly implies the following absoluteness result for automorphism towers of countable centreless groups.

Corollary 4.2. *Let M be a transitive model of* ZFC*, G be a centreless group that is an element of M and $\langle G_\alpha^M \mid \alpha \in \mathrm{On} \cap M\rangle$ be an automorphism tower of G in M. If G is countable in M and $G_1^M \neq G_2^M$, then $\tau(G) > 1$.*

Proof of the corollary from Theorem 4.1. Let $\pi : G_2^M \longrightarrow G_2$ be the embedding given by Theorem 4.1. It suffices to show that $\pi^{-1}[G_1] \subseteq G_1^M$ holds.

Let $h \in G_2^M$ with $\pi(h) \in G_1$. Given $g \in G$, we have $\iota_{\pi(h)}(g) \in G$ and therefore $\pi(\iota_{\pi(h)}(g)) = \iota_{\pi(h)}(g) = \iota_{\pi(h)}(\pi(g)) = \pi(\iota_h(g))$. Since π is an embedding, we can conclude that $\iota_h(g) = \iota_{\pi(h)}(g)$ holds for all $g \in G$ and hence $\iota_h \restriction G = \iota_{\pi(h)} \restriction G \in \mathrm{Aut}(G) \cap M$. By the definition of G_2^M, there is an $\bar{h} \in G_1^M$ with $\iota_{\bar{h}} \restriction G = \iota_h \restriction G$ and this shows $h^{-1} \cdot \bar{h} \in \mathrm{C}_{G_2^M}(G)$. An application of Theorem 2.2 in M yields $h = \bar{h} \in G_1^M$. $\square$

The above result should be compared with the following non-absoluteness result due to Simon Thomas.

Theorem 4.3 ([Tho98, Theorem 2.4])**.** *There exists a group G with $\tau(G) = 2$ and $1_{\mathbb{P}} \Vdash$ "$\tau(\check{G}) = 1$" for every notion of forcing $\mathbb{P}$ that adds a real.*

We outline how the results of Section 3 can be applied to analyze the first stages of the automorphism tower of a countable, centreless group. If $\mathcal{L}$ is a first-order language and $\mathcal{M}$ is an $\mathcal{L}$-model with domain ω, then $\mathrm{Aut}(\mathcal{M})$ is a subset of Baire space ${}^\omega\omega$ and the corresponding subspace topology induces a Polish group topology on $\mathrm{Aut}(\mathcal{M})$ (see [Kec95, Example 9.B 7]). If B is the family of subsets of $\mathrm{Aut}(\mathcal{M})$ of the form $\{\sigma \in \mathrm{Aut}(\mathcal{M}) \mid \pi \restriction X = \sigma \restriction X\}$ for some $\pi \in \mathrm{Aut}(\mathcal{M})$ and a finite subset X of ω, then B forms a countable basis of this group topology.

Let G be a countable group and $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ be an automorphism tower of G. Let B denote the family of all subsets of G_1 of the form $\{h \in G_1 \mid \iota_g \restriction X = \iota_h \restriction X\}$ for some $g \in G_1$ and a finite subset X of G. By the above remarks, B is a countable basis of a Polish group topology on G_1. Moreover, Theorem 2.2 and Proposition 3.5 imply that this is the unique Polish group topology on G_1 and $\iota_\pi \restriction G_1$ is continuous with respect to this topology for every $\pi \in G_2$ by Proposition 3.7.

The following folklore result is the last ingredient in our proof of Theorem 4.1. A proof of this statement can be found in [BK96, page 6].

[2]Note that M can be set-sized or even countable. In addition, we only need to assume that M is a transitive model of a "*suitable*" finite fragment of ZFC which enables us to run all the arguments of this section that take place inside of M.

Proposition 4.4. *Let $\langle G, \tau\rangle$ be a Polish group, H be a subgroup of G that is dense in τ and $\varphi : H \longrightarrow G$ be a group homomorphism that is continuous with respect to the subspace topology induced by τ on H and τ. Then there is a unique group homomorphism $\varphi^* : G \longrightarrow G$ that extends φ and is continuous with respect to τ.*

Proof of Theorem 4.1. Let M be a transitive model of ZFC, G be a centreless group with domain ω contained in M, $\langle G^M_\alpha \mid \alpha \in \mathrm{On} \cap M\rangle$ be an automorphism tower of G in M and $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ be an automorphism tower of G. Since every automorphism of G in M is an automorphism of G, we may replace G_1 by an isomorphic copy and assume that G^M_1 is a subgroup of G_1. We fix the following collections of sets.

(1) Let τ denote the unique Polish group topology on G_1.
(2) Let τ^M denote the unique Polish group topology on G^M_1 in M.
(3) Let $\bar{\tau}$ denote the subspace topology induced by τ on G^M_1.

Note that τ^M is contained in $\bar{\tau}$, because every basic open set in τ^M is an element of $\bar{\tau}$.

Remember that a *tree on* ω^n is a set T of n-tuples of finite sequences of natural numbers with the following properties.

(a) If $\langle t_0, \ldots, t_{n-1}\rangle \in T$, then $\mathrm{lh}(t_0) = \cdots = \mathrm{lh}(t_{n-1})$.
(b) If $\langle t_0, \ldots, t_{n-1}\rangle \in T$ and $m < \mathrm{lh}(t_0)$, then $\langle t_0 \restriction m, \ldots, t_{n-1} \restriction m\rangle \in T$.

Given a tree T on ω^n and $\vec{x} = \langle x_0, \ldots, x_{n-1}\rangle \in ({}^\omega\omega)^n$, we call $\vec{x}$ a *cofinal branch through* T if $\langle x_0 \restriction m, \ldots, x_{n-1} \restriction m\rangle \in T$ for every $m < \omega$.

Let $U = \{h \in G_1 \mid \iota_g \restriction X = \iota_h \restriction X\}$ be a nonempty basic open set in τ with $g \in G_1$ and X is a finite subset of ω. Then both X and $\iota_g \restriction X$ are elements of M and there is a tree T on $\omega \times \omega$ in M such that every cofinal branch through T is of the form $\langle x, y\rangle \in {}^\omega\omega \times {}^\omega\omega$ with $x, y \in \mathrm{Aut}(G)$, $y = x^{-1}$ and $\iota_g \restriction X \subseteq x$. It is easy to see that this property is absolute between transitive ZFC-models. Since U is nonempty, there is a cofinal branch through T and, by Mostowski's Absoluteness Theorem (see [Jec03, Theorem 25.4]), there is a branch through T that is an element of M. We can conclude $G^M_1 \cap U \neq \emptyset$. This argument shows that G^M_1 is dense in τ.

Fix $h \in G^M_2$. Let U be a basic open set in τ defined by $g \in G_1$ and $X \subset \omega$ as above. The above computations show that we may assume $g \in G^M_1$ and

$$U \cap G^M_1 = \{h \in G^M_1 \mid \iota_g \restriction X = \iota_h \restriction X\}$$

is a basic open set in τ^M. The subset

$$(\iota_h^{-1} \restriction G^M_1)[U] = (\iota_h^{-1} \restriction G^M_1)[G^M_1 \cap U]$$

is an element of τ^M, because $\iota_h \restriction G^M_1$ is continuous with respect to τ^M in M. By the above remarks, the subset is also an element of $\bar{\tau}$. This shows that the map $\iota_h \restriction G^M_1 : G^M_1 \longrightarrow G_1$ is a group homomorphism that is continuous with respect to $\bar{\tau}$ and τ. By Proposition 4.4, there is a unique group homomorphism $h^* : G_1 \longrightarrow G_1$ that extends $\iota_h \restriction G^M_1$ and is continuous with respect to τ.

For all $h \in G^M_2$, the map $(h^{-1})^* \circ h^*$ is the identity on the dense subset G^M_1 and is therefore the identity on G_1. This shows $h^* \in \mathrm{Aut}(G_1)$ with $(h^*)^{-1} = (h^{-1})^*$. We let $\pi(h)$ denote the unique element of G_2 with $h^* = \iota_{\pi(h)} \restriction G_1$. This means $\iota_{\pi(h)} \restriction G^M_1 = \iota_h \restriction G^M_1$ and π is injective. Moreover, if $g \in G^M_1 \subseteq G_1$, then $\iota_{\pi(g)} \restriction G = \iota_g \restriction G$ and this shows $g = \pi(g)$.

Given $h_0, h_1 \in G_2^M$, our definitions imply that $\iota_{\pi(h_0 \cdot h_1)}$ is equal to $\iota_{\pi(h_0)\cdot\pi(h_1)}$ on G_1^M and therefore on G_1. This shows $\pi(h_0 \cdot h_1) = \pi(h_0) \cdot \pi(h_1)$ holds for all $h_0, h_1 \in G_2^M$ and π is a group homomorphism. □

5. Groups of autohomeomorphism

In this section, we produce a variety of examples of strongly special pairs using certain group actions on Hausdorff spaces. Given a group G that consists of autohomeomorphisms of a Hausdorff space and satisfies a *locally movability condition*, we will construct a subset A of the domain of G such that $\langle G, A\rangle$ is strongly special pair and the cardinality of A is equal to the cardinality of a basis of the corresponding Hausdorff space.

Definition 5.1. Let G be a group and $\langle X, \tau\rangle$ be a Hausdorff space. We say that G *acts locally mixing on* $\langle X, \tau\rangle$ if the following statements hold.

1. G is a subgroup of the group $\mathcal{H}(\tau)$ of all autohomeomorphisms of $\langle X, \tau\rangle$.
2. If U is an element of τ and consists of more than one point, then there is a $g \in G \setminus \{1_G\}$ with $g \restriction (X \setminus U) = \mathrm{id}_{X\setminus U}$.

This condition also appears in the study of topological spaces that can be reconstructed from their autohomeomorphism groups (see [Rub89]).

We present some easy examples of autohomeomorphism groups acting locally mixing on the corresponding topological space. Given a topological space $\langle X, \tau\rangle$ and a subset A of X, we let $\bar{A}$ denote the closure of A with respect to τ, δA denote the boundary of A with respect to τ and τ_A denote the corresponding subspace topology on A induced by τ.

Proposition 5.2. *Let $\langle X, \tau\rangle$ be a Hausdorff space. Assume that for every subset U in τ with at least two points, there is a $V \subseteq U$ in τ such that $\bar{V} \subseteq U$ and $\langle \bar{V}, \tau_{\bar{V}}\rangle$ has a nontrivial autohomeomorphisms π with $\pi \restriction \delta V = \mathrm{id}_{\delta V}$. Then $\mathcal{H}(\tau)$ acts locally mixing on $\langle X, \tau\rangle$.*

Proof. Let U be an element of τ with more than one point. Pick V and π as above and define $\pi^* = \pi \cup \mathrm{id}_{X\setminus\bar{V}}$. We show that π^* is continuous with respect to τ in every $x \in X$.

If $x \in X \setminus \bar{V}$, then this statement is trivial, because $\pi^* \restriction (X \setminus \bar{V}) = \mathrm{id}_{X\setminus\bar{V}}$ and $X \setminus \bar{V}$ is open. Given $x \in \delta V$ and W_1 open in τ with $x = \pi^*(x) \in W_1$, there is $\tilde{W}_0$ in $\tau_{\bar{V}}$ with $x \in \tilde{W}_0$ and $\tilde{W}_0 \subseteq \pi^{-1}[\bar{V} \cap W_1]$. Pick W_0 in τ with $\tilde{W}_0 = \bar{V} \cap W_0$. Then $x \in W_0 \cap W_1$ and $W_0 \cap W_1 \subseteq {\pi^*}^{-1}[W_1]$. Finally, if $x \in V$ and W_1 is open in τ with $\pi^*(x) \in W_1$, then $\pi(x) = \pi^*(x) \in V \cap W_1$ and there is $\tilde{W}_0$ in $\tau_{\bar{V}}$ with $x \in \tilde{W}_0$ and $\tilde{W}_0 \subseteq \pi^{-1}[V \cap W_1]$. Pick W_0 in τ with $\tilde{W}_0 = \bar{V} \cap W_0$. Then $x \in V \cap W_0$ and $V \cap W_0 \subseteq {\pi^*}^{-1}[W_1]$. □

Example 5.3. Let $\langle X, \tau\rangle$ be an n-dimensional topological manifold. If U is an element of τ and $x \in U$, then there is a W in τ with $x \in W$ and $\langle W, \tau_W\rangle$ is homeomorphic to an open Euclidean n-ball. The preimage of $U \cap W$ under this homeomorphism is nonempty and therefore contains an open n-ball. This shows that there is a V in τ such that $\bar{V} \subseteq U \cap W \subseteq U$ and there is an homeomorphism of $\langle \bar{V}, \tau_{\bar{V}}\rangle$ and $[-1, 1]^n$ that maps δV onto the boundary of $[-1, 1]^n$ in $\mathbb{R}^n$. There are nontrivial autohomeomorphisms of $[-1, 1]^n$ that map its boundary in $\mathbb{R}^n$ onto itself and, by the above calculations, this shows that $\mathcal{H}(\tau)$ acts locally mixing on $\langle X, \tau\rangle$.

Example 5.4. Remember that a partial order $\mathbb{P} = \langle P, <_{\mathbb{P}} \rangle$ is a *tree* if the set $prec(p) = \{q \in P \mid q <_{\mathbb{P}} p\}$ is a well-ordered by $<_{\mathbb{P}}$ for every $p \in P$. Given a tree $\mathbb{T} = \langle T, <_{\mathbb{T}} \rangle$, we call a subset of T a *branch through* $\mathbb{T}$ if it is linearly ordered by $<_{\mathbb{T}}$ and downwards-closed. We let $[\mathbb{T}]$ denote the set of all maximal branches through T. Let $\tau_{\mathbb{T}}$ denote the topology on $[\mathbb{T}]$ generated by basic open sets of the form $U_t = \{b \in [\mathbb{T}] \mid t \in b\}$ with $t \in T$.

Let $\mathbb{T} = \langle T, <_{\mathbb{T}} \rangle$ be a tree with the property that for every $t \in T$ there is an automorphism π of $\mathbb{T}$ with $\pi(t) = t$ and $\pi(s) \neq s$ for some $s \in T$ with $t <_{\mathbb{T}} s$. We show that $\mathcal{H}(\tau_{\mathbb{T}})$ acts locally mixing on $\langle [\mathbb{T}], \tau_{\mathbb{T}} \rangle$. By Proposition 5.2, it suffices to show that the space $\langle U_t, (\tau_{\mathbb{T}})_{U_t} \rangle$ has a nontrivial autohomeomorphism for every $t \in T$, because

$$[\mathbb{T}] \setminus U_t = \bigcup \{U_s \mid s \textit{ and } t \textit{ are incompatible in } \mathbb{T}\}$$

and this shows that U_t is also closed in $\tau_{\mathbb{T}}$. If $t \in T$ and $\pi \in \mathrm{Aut}(\mathbb{T})$ with $\pi(t) = t$ and $\pi(s) \neq s$ for some $s \in T$ with $t <_{\mathbb{T}} s$, then we define $\pi^*(b) = \pi[b]$ for every $b \in U_t$. It is easy to check that $\pi^* : U_t \longrightarrow U_t$ is continuous with respect to $(\tau_{\mathbb{T}})_{U_t}$ and if $s \in b \in U_t$, then $\pi^*(b) \neq b$, because $\pi(s) <_{\mathbb{T}} s$ or $s <_{\mathbb{T}} \pi(s)$ would contradict the well-foundedness of $<_{\mathbb{T}}$ below s.

In particular, if α is an ordinal, X is a set with at least two elements and ${}^{<\alpha}X$ is the tree consisting of functions f with $\mathrm{dom}(f) \in \alpha$ and $\mathrm{ran}(f) \subseteq X$ ordered by inclusion, then $[{}^{<\alpha}X]$ can be identified with the set ${}^{\alpha}X$ of all functions from α to X and the group of autohomeomorphisms of the corresponding topological space acts locally mixing on it.

Example 5.5. Let $\mathbb{L} = \langle L, <_{\mathbb{L}} \rangle$ be a linear order without end-points that has a nontrivial automorphism and the property that every nonempty, open interval $(a, b) = \{l \in L \mid a <_{\mathbb{L}} l <_{\mathbb{L}} b\}$ is order-isomorphic to $\mathbb{L}$. If $\tau_{\mathbb{L}}$ denotes the order-topology on $\mathbb{L}$, then Proposition 5.2 directly implies that $\mathrm{Aut}(\mathbb{L})$ acts locally mixing on $\langle L, \tau_{\mathbb{L}} \rangle$. In particular, the group of order-preserving bijections of the rational numbers $\mathbb{Q}$ acts locally mixing on $\mathbb{Q}$ equipped with the order topology.

We use methods and computations from Robert R. Kallman's proof of [Kal86, Theorem 1.1] to derive the following result.

Theorem 5.6. *Let G be a group, $\langle X, \tau \rangle$ be a Hausdorff space and* B *be a basis of τ. If G acts locally mixing on $\langle X, \tau \rangle$ and $\langle X, \tau \rangle$ does not have exactly two isolated points, then there is a subset A of the domain of G of cardinality $|\mathrm{B}| + \aleph_0$ such that $\langle G, A \rangle$ is a strongly special pair and $\mathrm{C}_G(A) = \{1_G\}$.*

For the rest of this section, we fix a Hausdorff space $\langle X, \tau \rangle$, a basis B of τ and a group G that acts locally mixing on $\langle X, \tau \rangle$. Given $Y \subseteq X$, we define

$$\mathrm{Sub}_{\mathrm{B}}(Y) = \{U \in \mathrm{B} \mid U \subseteq Y, \ |U| > 1\}.$$

and define $\bar{Y}$ to be the closure of Y with respect to τ. Finally, we fix a sequence $\langle g_U \in G \setminus \{1_G\} \mid U \in \mathrm{Sub}_{\mathrm{B}}(X) \rangle$ such that $g_U \upharpoonright (X \setminus U) = \mathrm{id}_{X \setminus U}$ holds for all $U \in \mathrm{Sub}_{\mathrm{B}}(X)$.

In the following, we adopt the arguments of [Kal86, Section 2] to our setting to prove Theorem 5.6.

Lemma 5.7. *Let U be open in τ such that U contains either no points isolated in τ or more than two points isolated in τ. The following statements are equivalent for all $h \in G$.*

(1) $h \restriction \bar{U} = \mathrm{id}_{\bar{U}}$.
(2) $g_{U'}^h = g_{U'}$ *holds for all* $U' \in \mathrm{Sub}_{\mathrm{B}}(U)$.

Proof. Assume $h \restriction \bar{U} = \mathrm{id}_{\bar{U}}$ and fix $U' \in \mathrm{Sub}_{\mathrm{B}}(U)$. Then $h \circ g_{U'} = g_{U'} \circ h$ holds, because we have $g_{U'} \restriction (X \setminus \bar{U}) = \mathrm{id}_{\bar{U}}$.

Now, assume that $g_{U'}^h = g_{U'}$ holds for all $U' \in \mathrm{Sub}_{\mathrm{B}}(U)$. By the continuity of h, it suffices to show $h \restriction U = \mathrm{id}_U$. Let I_U denote the set of all points in U that are isolated in τ. We start by showing $h \restriction I_U = \mathrm{id}_{I_U}$. If U contains no isolated points, then this is trivial. We may therefore assume $|I_U| > 2$.

Assume, toward a contradiction, that there is an $a \in I_U$ with $h(a) \neq a$. We can find distinct $b_0, b_1 \in I_U$ with $a \notin \{b_0, b_1\}$. Then $\{a, b_i\} \in \mathrm{Sub}_{\mathrm{B}}(U)$ and $g_{\{a,b_i\}} = (a\ b_i)$. Our first assumption yields $(a\ b_i)^h = (a\ b_i)$ and this implies $h[\{a, b_i\}] = \{a, b_i\}$. We can conclude $b_0 = h(a) = b_1$, a contradiction. This shows $h \restriction I_U = \mathrm{id}_{I_U}$.

Assume, toward a contradiction, that there is an $x \in U$ with $h(x) \neq x$. Since x is not isolated in τ and $\langle X, \tau \rangle$ is a Hausdorff space, we can find $V \in \mathrm{Sub}_{\mathrm{B}}(U)$ with $V \cap h[V] = \emptyset$. If $y \in V$ with $g_V(y) \neq y$, then $g_V^h = g_V$, $g_V(h(y)) = h(y)$ and therefore $h(y) = (g_V \circ h)(y) = (h \circ g_V)(y) \neq h(y)$, a contradiction. $\square$

Set $A = \{g_U \mid U \in \mathrm{Sub}_{\mathrm{B}}(X)\}$ and, for all $U, V \in \mathrm{Sub}_{\mathrm{B}}(X)$, we define

$$\mathrm{t}_{U,V}(v) \equiv v * \dot{g}_U * v^{-1} * \dot{g}_V * v * \dot{g}_U^{-1} * v^{-1} * \dot{g}_V^{-1} \in \mathcal{T}_A.$$

Lemma 5.8. *Let U and V be open subsets in τ. Assume that both U and $X \setminus \bar{V}$ contain either no points isolated in τ or more than two points isolated in τ. Then the following statements are equivalent for all $h \in G$.*

(1) $\mathrm{t}_{U',V'}^G(h) = 1_G$ *for all* $U' \in \mathrm{Sub}_{\mathrm{B}}(U)$ *and* $V' \in \mathrm{Sub}_{\mathrm{B}}(X \setminus \bar{V})$.
(2) $h[\bar{U}] \subseteq \bar{V}$.

Proof. The first statement is equivalent to $g_{U'}^h \circ g_{V'} = g_{V'} \circ g_{U'}^h$ for all $U' \in \mathrm{Sub}_{\mathrm{B}}(U)$ and $V' \in \mathrm{Sub}_{\mathrm{B}}(X \setminus \bar{V})$. By Lemma 5.7, this is equivalent to $g_{U'}^h \restriction (X \setminus \bar{V}) = \mathrm{id}_{X \setminus \bar{V}}$ for all $U' \in \mathrm{Sub}_{\mathrm{B}}(U)$ and we can reformulate this to

$(1)^*$ $(g_{U'} \circ h^{-1}) \restriction (X \setminus \bar{V}) = h^{-1} \restriction (X \setminus \bar{V})$ for all $U' \in \mathrm{Sub}_{\mathrm{B}}(U)$.

By our assumptions, the set of all points which are moved by some $g_{U'}$ with $U' \in \mathrm{Sub}_{\mathrm{B}}(U)$ is dense in U with respect to τ. This shows that $(1)^*$ is equivalent to $U \cap h^{-1}[X \setminus \bar{V}] = \emptyset$. This statement holds if and only if $h[U] \subseteq \bar{V}$ and this is equivalent to the second statement of the lemma. $\square$

Proof of Theorem 5.6. We may assume that B is closed under finite unions. By our assumptions, there are not exactly two points in X which are isolated in τ. If there is exactly one point $x_0 \in X$ which is isolated in τ, then it is easy to check that there is a group isomorphic to G that acts locally mixing on $\langle X \setminus \{x_0\}, \tau^* \rangle$, where τ^* is the subspace topology induced by τ. We may therefore assume that there are either no points isolated in τ or more than two.

Pick $g_0, g_1 \in G$ with $\mathsf{qft}_{G,A}(g_0) \subseteq \mathsf{qft}_{G,A}(g_1)$ and assume, toward a contradiction, that $g_0 \neq g_1$ holds. Then $U = \{x \in X \mid g_0(x) \neq g_1(x)\}$ is nonempty and open in τ. Let I_U denote the set of all points in U that are isolated in τ.

First, assume that there is an $x \in U \setminus I_U$. We can find disjoint subsets V_0 and V_1 in B such that $g_i(x) \in V_i$ for $i < 2$ and $X \setminus \bar{V}_0$ contains either no points isolated in τ or more than two. Now we can find $U' \in \mathrm{B}$ with $x \in U'$, $g_i[U'] \subseteq V_i$ and U' contains either no points isolated in τ or more than two. This means $g_0[\bar{U}'] \subseteq \bar{V}_0$ and we can apply Lemma 5.8 to conclude $\mathrm{t}_{U'',V'} \in \mathsf{qft}_{G,A}(g_0) \subseteq \mathsf{qft}_{G,A}(g_1)$ for all

$U'' \in \mathrm{Sub}_{\mathrm{B}}(U')$ and $V' \in \mathrm{Sub}_{\mathrm{B}}(X \setminus \bar{V}_0)$. Another application of the lemma yields $g_1[\bar{U}'] \subseteq \bar{V}_0$ and this means $g_1(x) \in \bar{V}_0 \subseteq X \setminus V_1$, a contradiction.

This shows $I_U = U \neq \emptyset$. Pick $x \in I_U$. By the above assumptions, we can find distinct $y_0, y_1 \in X$ isolated in τ with $x \notin \{y_0, y_1\}$. For all $i < 2$, we have $\{x, y_i\}, \{g_0(x), g_0(y_i)\} \in \mathrm{B}$, $g_{\{x,y_i\}} = (x \ y_i)$ and

$$g_{\{x,y_i\}}^{g_0} = (g_0(x) \ g_0(y_i)) = g_{\{g_0(x),g_0(y_i)\}}.$$

The above equalities allow us to conclude

$$v * \dot{g}_{\{x,y_i\}} * v^{-1} * \dot{g}_{\{g_0(x),g_0(y_i)\}} \in \mathsf{qft}_{G,A}(g_0) \subseteq \mathsf{qft}_{G,A}(g_1).$$

In particular, $g_1[\{x, y_i\}] = \{g_0(x), g_0(y_i)\}$ and this shows $g_1(x) = g_0(y_i)$, because $g_1(x) \neq g_0(x)$. We can conclude $g_0(y_0) = g_1(x) = g_0(y_1)$ and therefore $y_0 = y_1$, a contradiction.

If $h \in \mathrm{C}_G(A)$, then $g_U^h = g_U$ holds for all $U \in \mathrm{Sub}_{\mathrm{B}}(X)$. By our assumptions and the above remark, we can apply Lemma 5.7 to conclude $h = \mathrm{id}_X = 1_G$. □

6. Special pairs that are not strongly special

This section contains the construction of special pairs that are not strongly special using simple groups as building blocks. A theorem of Manfred Droste, Michèle Giraudet and Rüdiger Göbel will allow us to prove the following result.

Theorem 6.1. *If κ is an uncountable regular cardinal, then there is a special pair $\langle G, A\rangle$ such that G has cardinality 2^κ, A has cardinality κ, $\mathrm{C}_G(A) = \{1_G\}$ and $\langle G, A\rangle$ is not strongly special.*

We start with a simple statement about normal subgroups of automorphism groups of centreless groups.

Proposition 6.2. *Let G be a centreless group and N be a normal subgroup of $\mathrm{Aut}(G)$. Then $N \neq \{\mathrm{id}_G\}$ if and only if $\mathrm{Inn}(G) \cap N \neq \{\mathrm{id}_G\}$.*

Proof. Assume $\mathrm{Inn}(G) \cap N = \{\mathrm{id}_G\}$. Given $\pi \in N$, we have

$$\iota_{\pi(g)\cdot g^{-1}} = \pi \circ \iota_g \circ \pi^{-1} \circ \iota_g^{-1} \in \mathrm{Inn}(G) \cap N$$

and therefore $\pi(g) = g$ for all $g \in G$. This shows $N = \{\mathrm{id}_G\}$. □

In the proof of Theorem 6.1, we start by constructing a special pair $\langle G, A\rangle$ with $|G| = |A|$ that is not strongly special. The following proposition will allow us to replace G by a group of higher cardinality.

Proposition 6.3. *Let G and H be groups, A be a subset of the domain of G and $A^* = A \times \{1_H\} \cup \{1_G\} \times H \subseteq G \times H$.*

1. *If $\langle G, A\rangle$ is a special pair and $\mathrm{Z}(H) = \{1_H\}$, then $\langle G \times H, A^*\rangle$ is a special pair.*
2. *If $\langle G, A\rangle$ is not a strongly special pair, then $\langle G \times H, A^*\rangle$ is not a strongly special pair.*

Proof. (1) Assume that $\mathrm{Z}(H) = \{1_H\}$ holds, $\langle g_*, h_*\rangle \in G \times H$ and

$$\varphi : \langle A^* \cup \{\langle g_*, h_*\rangle\}\rangle \longrightarrow G \times H$$

is a monomorphism with $\varphi \restriction A^* = \mathrm{id}_{A^*}$ and $\varphi(\langle g_*, h_*\rangle) \neq \langle g_*, h_*\rangle$. Then $\langle k, 1_H\rangle \in \mathrm{dom}(\varphi)$ for every $k \in \langle A \cup \{g_*\}\rangle$ and $\varphi(\langle g_*, 1_H\rangle) \neq \langle g_*, 1_H\rangle$. Let $p_H : G \times H \longrightarrow H$ denote the canonical projection and define

$$\xi : \langle A \cup \{g_*\}\rangle \longrightarrow H;\ k \longmapsto (p_H \circ \varphi)(\langle k, 1_H\rangle).$$

Given $k \in \langle A \cup \{g_*\}\rangle$ and $h \in H$, we have

$$\begin{aligned}\xi(k) \cdot h &= (p_H \circ \varphi)(\langle k, 1_H\rangle) \cdot (p_H \circ \varphi)(\langle 1_G, h\rangle) = (p_H \circ \varphi)(\langle k, h\rangle)\\ &= (p_H \circ \varphi)(\langle 1_G, h\rangle) \cdot (p_H \circ \varphi)(\langle k, 1_H\rangle) = h \cdot \xi(k)\end{aligned}$$

and this shows $\mathrm{ran}(\xi) \subseteq \mathrm{Z}(H) = \{1_H\}$. We get a function $\bar{\varphi} : \langle A \cup \{g_*\}\rangle \longrightarrow G$ with $\varphi(\langle k, 1_H\rangle) = \langle \bar{\varphi}(k), 1_H\rangle$ for all $k \in \langle A \cup \{g_*\}\rangle$. By our assumptions, $\bar{\varphi}$ is a monomorphism, $\bar{\varphi} \restriction A = \mathrm{id}_A$ and $\bar{\varphi}(g_*) \neq g_*$. This shows that $\langle G, A\rangle$ is not a special pair.

(2) Assume $g_* \in G$ and $\bar{\varphi} : \langle A \cup \{g\}\rangle \longrightarrow G$ is a homomorphism with $\bar{\varphi} \restriction A = \mathrm{id}_A$ and $\bar{\varphi}(g_*) \neq g_*$. If $\langle k, h\rangle \in \langle A^* \cup \{\langle g_*, 1_H\rangle\}\rangle$, then $k \in \langle A \cup \{g_*\}\rangle$ and we can define

$$\varphi : \langle A^* \cup \{\langle g_*, 1_H\rangle\}\rangle \longrightarrow G \times H;\ \langle k, h\rangle \longmapsto \langle \bar{\varphi}(k), h\rangle.$$

Then $\langle G \times H, A^*\rangle$ is not a strongly special pair, because φ is a homomorphism with $\varphi \restriction A^* = \mathrm{id}_{A^*}$ and $\varphi(\langle g_*, 1_H\rangle) \neq \langle g_*, 1_H\rangle$. □

For the remainder of this section, we fix simple non-abelian groups H, S and a homomorphism $\mathfrak{c} : \mathrm{Aut}(S) \longrightarrow \mathrm{Aut}(H)$ with $\mathrm{Inn}(H) \subseteq \mathrm{ran}(\mathfrak{c})$. Define

$$G = H \rtimes_{\mathfrak{c}} \mathrm{Aut}(S)$$

and $A = \{1_H\} \times \mathrm{Aut}(S)$.

Lemma 6.4. *The following statements are equivalent.*

(1) *There is an isomorphism $\Psi : H \longrightarrow S$ with $\mathfrak{c}(\pi) = \Psi^{-1} \circ \pi \circ \Psi$ for all $\pi \in \mathrm{Aut}(S)$.*

(2) *$\langle G, A\rangle$ is not a special pair.*

Proof. Assume (1) holds. Define

$$\phi : G \longrightarrow G;\ \langle h, \pi\rangle \longmapsto \langle h^{-1}, \iota_{\Psi(h)} \circ \pi\rangle.$$

Clearly, ϕ is injective and $\phi \restriction A = \mathrm{id}_A$. If $\langle h^{-1}, \iota_{\Psi(h)} \circ \pi\rangle = \langle h, \pi\rangle$ holds with $h \in H$ and $\pi \in \mathrm{Aut}(S)$, then $\iota_{\Psi(h)} = \mathrm{id}_S$ and this means $h = 1_H$. This shows $\phi \neq \mathrm{id}_G$. Given $\langle h_0, \pi_0\rangle, \langle h_1, \pi_1\rangle \in G$, we have

$$\begin{aligned}\phi(\langle h_0, \pi_0\rangle \cdot \langle h_1, \pi_1\rangle) &= \phi(\langle h_0 \cdot \mathfrak{c}(\pi_0)(h_1), \pi_0 \circ \pi_1\rangle)\\ &= \langle \mathfrak{c}(\pi_0)(h_1^{-1}) \cdot h_0^{-1}, \iota_{\Psi(h_0 \cdot \mathfrak{c}(\pi_0)(h_1))} \circ \pi_0 \circ \pi_1\rangle\\ &= \langle h_0^{-1} \cdot \mathfrak{c}(\pi_0)(h_1^{-1})^{h_0}, \iota_{\Psi(h_0)} \circ \iota_{(\pi_0 \circ \Psi)(h_1)} \circ \pi_0 \circ \pi_1\rangle\\ &= \langle h_0^{-1} \cdot (\iota_{h_0} \circ \mathfrak{c}(\pi_0))(h_1^{-1}), \iota_{\Psi(h_0)} \circ \iota_{\Psi(h_1)}^{\pi_0} \circ \pi_0 \circ \pi_1\rangle\\ &= \langle h_0^{-1} \cdot \mathfrak{c}(\iota_{\Psi(h_0)} \circ \pi_0)(h_1^{-1}), \iota_{\Psi(h_0)} \circ \pi_0 \circ \iota_{\Psi(h_1)} \circ \pi_1\rangle\\ &= \langle h_0^{-1}, \iota_{\Psi(h_0)} \circ \pi_0\rangle \cdot \langle h_1^{-1}, \iota_{\Psi(h_1)} \circ \pi_1\rangle\\ &= \phi(\langle h_0, \pi_0\rangle) \cdot \phi(\langle h_1, \pi_1\rangle),\end{aligned}$$

because our assumption implies that $\mathfrak{c}(\iota_{\Psi(h)}) = \iota_h$ holds for all $h \in H$. This computation shows that ϕ is a group monomorphism and $\langle G, A\rangle$ is not a special pair by Lemma 2.6.

In the other direction, assume that $\langle G, A\rangle$ is not a special pair. By Lemma 2.6, there is a $g_* = \langle h_*, \pi_*\rangle \in G$ and a monomorphism $\phi : \langle A \cup \{g_*\}\rangle \longrightarrow G$ with $\phi \restriction A = \mathrm{id}_A$ and $\phi(g_*) \neq g_*$. This implies $h_* \neq 1_H$, $\langle h_*, \mathrm{id}_S\rangle \in \mathrm{dom}(\phi)$ and $\phi(\langle h_*, \mathrm{id}_S\rangle) \neq \langle h_*, \mathrm{id}_S\rangle$.

Let $N = \{h \in H \mid \langle h, \mathrm{id}_S\rangle \in \mathrm{dom}(\phi)\}$. If $h \in N$ and $k \in H$, then $\iota_k = \mathfrak{c}(\pi)$ for some $\pi \in \mathrm{Aut}(S)$,

$$\langle h^k, \mathrm{id}_S\rangle = \langle \mathfrak{c}(\pi)(h), \mathrm{id}_S\rangle = \langle 1_H, \pi\rangle \cdot \langle h, \mathrm{id}_S\rangle \cdot \langle 1_H, \pi^{-1}\rangle = \langle h, \mathrm{id}_S\rangle^{\langle 1_H,\pi\rangle} \in \mathrm{dom}(\phi)$$

and $h^k \in N$. This shows that N is a normal subgroup of H and therefore $N = H$, because $1_H \neq h_* \in N$.

Let $p_{\mathrm{Aut}(S)} : G \longrightarrow \mathrm{Aut}(S)$ denote the canonical projection map and define

$$\bar{\Psi} : H \longrightarrow \mathrm{Aut}(S);\ h \longmapsto (p_{\mathrm{Aut}(S)} \circ \phi)(\langle h, \mathrm{id}_S\rangle).$$

Assume, toward a contradiction, that $\ker(\bar{\Psi}) = H$. This assumption gives us a map $\xi : H \longrightarrow H$ with $\phi(\langle h, \mathrm{id}_S\rangle) = \langle \xi(h), \mathrm{id}_S\rangle$ for all $h \in H$. By our assumptions, ξ is a monomorphism. If $h, k \in H$ and $\pi \in \mathrm{Aut}(S)$ with $\mathfrak{c}(\pi) = \iota_k$, then

$$\phi(\langle h^k, \mathrm{id}_S\rangle) = \phi(\langle h, \mathrm{id}_S\rangle^{\langle 1_H,\pi\rangle}) = \phi(\langle h, \mathrm{id}_S\rangle)^{\langle 1_H,\pi\rangle} = \langle \xi(h)^k, \mathrm{id}_S\rangle,$$

and $\xi(h)^k = \xi(h^k) \in \mathrm{ran}(\xi)$. This shows that $\mathrm{ran}(\xi)$ is a normal subgroup of H. Since ϕ is injective and H is nontrivial, we can conclude that $H = \mathrm{ran}(\xi)$ and ξ is a nontrivial automorphism of H. Pick $h \in H$ and $\pi \in \mathrm{Aut}(S)$ with $\mathfrak{c}(\pi) = \iota_h$. If $k \in H$, then

$$\begin{aligned}
\langle k^{\xi(h)}, \pi\rangle &= \langle k^{\xi(h)}, \mathrm{id}_S\rangle \cdot \langle 1_H, \pi\rangle = \phi(\langle \xi^{-1}(k)^h, \mathrm{id}_S\rangle) \cdot \phi(\langle 1_H, \pi\rangle) \\
&= \phi(\langle \mathfrak{c}(\pi)(\xi^{-1}(k)), \pi\rangle) = \phi(\langle 1_H, \pi\rangle) \cdot \phi(\langle \xi^{-1}(k), \mathrm{id}_S\rangle) = \langle 1_H, \pi\rangle \cdot \langle k, \mathrm{id}_S\rangle \\
&= \langle \mathfrak{c}(\pi)(k), \pi\rangle = \langle k^h, \pi\rangle
\end{aligned}$$

and therefore $h^{-1} \cdot \xi(h) \in \mathrm{Z}(H) = \{1_H\}$. This shows $\xi = \mathrm{id}_H$, a contradiction.

By the above computations, $\bar{\Psi} : H \longrightarrow \mathrm{Aut}(S)$ is a monomorphism. If $\pi \in \mathrm{Aut}(S)$ and $h, k \in H$ with $\phi(\langle h, \mathrm{id}_S\rangle) = \langle k, \bar{\Psi}(h)\rangle$, then

$$(\star) \qquad \langle \mathfrak{c}(\pi)(k), \bar{\Psi}(h)^\pi\rangle = \langle k, \bar{\Psi}(h)\rangle^{\langle 1_H,\pi\rangle} = \phi(\langle h, \mathrm{id}_s\rangle^{\langle 1_H,\pi\rangle}) = \phi(\langle \mathfrak{c}(\pi)(h), \mathrm{id}_S\rangle)$$

and therefore $\bar{\Psi}(h)^\pi = \bar{\Psi}(\mathfrak{c}(\pi)(h)) \in \mathrm{ran}(\bar{\Psi})$. This shows that $\mathrm{ran}(\bar{\Psi})$ is a nontrivial normal subgroup of $\mathrm{Aut}(S)$. By Proposition 6.2, we have $\mathrm{Inn}(S) \cap \mathrm{ran}(\bar{\Psi}) \neq \{\mathrm{id}_S\}$ and this implies $\mathrm{Inn}(S) = \mathrm{Inn}(S) \cap \mathrm{ran}(\bar{\Psi}) = \mathrm{ran}(\bar{\Psi})$, because both $\mathrm{Inn}(S)$ and $\mathrm{ran}(\bar{\Psi})$ are simple groups. We have shown that $\bar{\Psi} : H \longrightarrow \mathrm{Inn}(S)$ is an isomorphism.

Define $\Psi : H \longrightarrow S$ to be the isomorphism $\iota_S^{-1} \circ \bar{\Psi}$. Given $\pi \in \mathrm{Aut}(S)$ and $h \in H$, the equalities in $(\star)$ show $\bar{\Psi}(\mathfrak{c}(\pi)(h)) = \bar{\Psi}(h)^\pi$ and this implies

$$\mathfrak{c}(\pi)(h) = \bar{\Psi}^{-1}(\bar{\Psi}(h)^\pi) = \bar{\Psi}^{-1}\left(\iota^\pi_{\Psi(h)}\right) = (\Psi^{-1} \circ \iota_S^{-1})\left(\iota_{(\pi\circ\Psi)(h)}\right) = (\Psi^{-1} \circ \pi \circ \Psi)(h).$$

This equality shows that Ψ is an isomorphism with the desired properties. □

Corollary 6.5. *If $\langle G, A\rangle$ is not a special pair, then $\mathfrak{c}$ is injective.* □

Proposition 6.6. *$\langle G, A\rangle$ is not a strongly special pair.*

Proof. Define

$$\varphi : G \longrightarrow G;\ \langle h, \pi\rangle \longmapsto \langle 1_H, \pi\rangle.$$

Then φ is a group homomorphism with $\varphi \restriction A = \mathrm{id}_A$ and $\varphi(\langle h, \mathrm{id}_S\rangle) \neq \langle h, \mathrm{id}_S\rangle$ for all $h \in H \setminus \{1_H\} \neq \emptyset$. By Lemma 2.7, this implies the statement of the proposition. □

We finish this note by stating the coding result mentioned above and proving Theorem 6.1.

Theorem 6.7 ([DGG01, Corollary 4.7])**.** *Let κ be an uncountable regular cardinal and G be a group of cardinality at most κ. Then there exists a simple group S of cardinality κ such that G is isomorphic to* $\mathrm{Aut}(S)/\mathrm{Inn}(S)$.

Proof of Theorem 6.1. Let κ be a regular uncountable cardinal. It is well-known that the group $\mathrm{Alt}(\kappa)$ is a simple, non-abelian group of cardinality κ. By Theorem 6.7, there is a simple group S of cardinality κ such that there is an isomorphism $\xi : \mathrm{Aut}(S)/\mathrm{Inn}(S) \longrightarrow \mathrm{Alt}(\kappa)$. If we define

$$\mathfrak{c} : \mathrm{Aut}(S) \longrightarrow \mathrm{Aut}(\mathrm{Alt}(\kappa));\ \pi \longmapsto \iota_{\xi(\pi\mathrm{Inn}(S))},$$

then $\mathfrak{c}$ is a non-injective group homomorphism with $\mathrm{Inn}(\mathrm{Alt}(\kappa)) \subseteq \mathrm{ran}(\mathfrak{c})$. We set $\bar{G} = \mathrm{Alt}(\kappa) \rtimes_{\mathfrak{c}} \mathrm{Aut}(S)$ and $\bar{A} = \{\mathrm{id}_\kappa\} \times \mathrm{Aut}(S)$. Since both S and $\mathrm{Aut}(S)/\mathrm{Inn}(S)$ have cardinality κ, $\mathrm{Aut}(S)$ has the same cardinality and $\bar{G}$ is a group of cardinality κ. Corollary 6.5 implies that $\langle \bar{G}, \bar{A}\rangle$ is a special pair and Proposition 6.6 shows that it is not strongly special.

Pick $\langle h, \pi\rangle \in \mathrm{C}_{\bar{G}}(\bar{A})$. Given $\sigma \in \mathrm{Aut}(S)$, we have

$$\langle h, \pi\rangle = \langle h, \pi\rangle^{\langle \mathrm{id}_\kappa, \sigma\rangle} = \langle \mathfrak{c}(\sigma)(h), \pi^\sigma\rangle$$

and this implies $\pi \in \mathrm{Z}(\mathrm{Aut}(S)) = \{\mathrm{id}_S\}$. If $k \in \mathrm{Alt}(\kappa)$ and $\sigma \in \mathrm{Aut}(S)$ with $\mathfrak{c}(\sigma) = \iota_k$, then

$$\langle h, \mathrm{id}_S\rangle = \langle h, \mathrm{id}_S\rangle^{\langle \mathrm{id}_\kappa, \sigma\rangle} = \langle \mathfrak{c}(\sigma)(h), \mathrm{id}_S\rangle = \langle h^k, \mathrm{id}_S\rangle$$

and hence $h \in \mathrm{Z}(\mathrm{Alt}(\kappa)) = \{\mathrm{id}_\kappa\}$.

Define $G = \bar{G} \times \mathrm{Alt}(\kappa)$ and $A = \bar{A} \times \{\mathrm{id}_\kappa\} \cup \{1_{\bar{G}}\} \times \mathrm{Alt}(\kappa)$. By Proposition 6.3, $\langle G, A\rangle$ is a special pair that is not strongly special. Moreover, it is easy to see that both G and A have cardinality κ and $\mathrm{C}_G(A) = \mathrm{C}_{\bar{G}}(\bar{A}) \times \mathrm{Z}(\mathrm{Alt}(\kappa)) = \{\langle 1_{\bar{G}}, \mathrm{id}_\kappa\rangle\}$.

Let $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ be an automorphism tower of G. Then G_1 has cardinality 2^κ, because the automorphism group of $\mathrm{Alt}(\kappa)$ is isomorphic to the group $\mathrm{Sym}(\kappa)$ of all permutations of κ and every automorphism of $\mathrm{Alt}(\kappa)$ induces a unique automorphism of G. By Theorem 2.2, $\langle G_1, A\rangle$ is a special pair with $\mathrm{C}_{G_1}(A) = \{1_G\}$. Finally, $\langle G_1, A\rangle$ is not a strongly special pair, because otherwise $\langle G, A\rangle$ would be a strongly special pair. □

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Institut für Mathematische Logik und Grundlagenforschung, Fachbereich Mathematik und Informatik, Universität Münster, Einsteinstrasse 62, 48149 Münster, Germany

E-mail address: `philipp.luecke@uni-muenster.de`