

Forcing lightface definable well-orders without the GCH

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Abstract

For any given uncountable cardinal κ with $\kappa^{<\kappa} = \kappa$, we present a forcing that is $<\kappa$ -directed closed, has the κ^+ -cc and introduces a lightface definable well-order of $H(\kappa^+)$. We use this to define a global iteration that adds such a well-order for all such κ simultaneously and is capable of preserving the existence of many large cardinals in the universe.

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1 Introduction

If κ is an infinite cardinal, a lightface definable wellorder of $H(\kappa^+)$ is a well-order of $H(\kappa^+)$ that is definable over $\langle H(\kappa^+), \in \rangle$ without parameters. In [2] and [3], Sy Friedman and the first author show that given any uncountable cardinal κ that satisfies $\kappa^{<\kappa} = \kappa$ (note that this implies that κ is regular) and $2^\kappa = \kappa^+$, there is a $<\kappa$ -directed closed, κ^+ -cc partial order of size 2^κ which yields a lightface definable well-order of $H(\kappa^+)^{V[G]}$ whenever G is generic for that forcing.¹ They use this to define a class sized iteration which, assuming the GCH, introduces a lightface definable well-order of $H(\kappa^+)$ for every uncountable cardinal κ , preserving the GCH and all cofinalities, and show that whenever κ is λ -supercompact for λ regular, then the λ -supercompactness of κ is preserved by the iteration. Moreover they show that introducing those well-orders by a variant of the above class sized iteration also allows for preserving many instances of n -hugeness.

We generalize those results to a non-GCH context as follows. First we show that even if $2^\kappa > \kappa^+$, there is a very nice forcing to introduce a lightface definable well-order of $H(\kappa^+)$. The key new ingredient will be a new coding forcing (that we call *Club Coding*) which will be introduced in Section 3.

Theorem 1.1. *Suppose κ is an uncountable cardinal with $\kappa^{<\kappa} = \kappa$. Then there is a partial order \mathcal{Q} with the following properties.*

¹If $\kappa = \omega$, strong large cardinal assumptions imply that no definable wellorder of $H(\omega_1)$ exists, even if we allow for the use of parameters (see [12]).

1. The partial order \mathcal{Q} is a subset of $H(\kappa^+)$, has a $<\kappa$ -directed closed dense subset and satisfies the κ^+ -chain condition.
2. Forcing with \mathcal{Q} introduces a lightface definable well-order of $H(\kappa^+)$.

Using the properties of these *single-step forcings*, it is straightforward (see Section 6) to iterate this forcing for all uncountable cardinals κ that satisfy $\kappa^{<\kappa} = \kappa$, over a model of the *Singular Cardinal Hypothesis* (SCH), and obtain the following result.

Theorem 1.2. *Assume that the SCH holds. There is a ZFC-preserving class forcing \mathbb{P} with the following properties.*

1. Forcing with \mathbb{P} preserves all cofinalities and the continuum function (i.e. the value of 2^κ for every cardinal κ).
2. Forcing with \mathbb{P} introduces a lightface definable well-order of $H(\kappa^+)$ whenever κ is an uncountable cardinal with $\kappa^{<\kappa} = \kappa$.

The role of the SCH in the above is very similar to the situation in [6]. We refer the reader to the first chapter of that paper (or also to [7]) for a more detailed discussion. Note that it is possible to force the SCH to hold using a class-sized iteration that preserves the cofinality of all regular cardinals κ such that there is no singular strong limit cardinal λ with $\lambda^+ < \kappa \leq 2^\lambda$ and the value of 2^κ for all cardinals κ such that there is no singular strong limit cardinal λ with $2^\lambda > \lambda^+$ and $\lambda^+ \leq \kappa \leq 2^\lambda$.

In Section 7, we will show that forcing with \mathbb{P} allows for various forms of large cardinal preservation. Supercompactness preservation seems to be a difficult issue in a non-GCH setting and we only obtain a partial result (originating from [2]) that relies on instances of the GCH to hold. Using *parser* iterations, one may use supercompactness preservation arguments for a non-GCH context that were developed in [7] and given a simplified presentation in a somewhat different context in [6]. We give a sample result in Section 8. Back in Section 7, we also present stronger results on large cardinal preservation for other types of large cardinals: hyperstrong and n -superstrong cardinals for $2 \leq n \leq \omega$.

2 More on related Results

In this short section, we want to comment on the results of this paper and their relationship with other recent results on introducing locally lightface definable well-orders by forcing. In [6], Sy Friedman together with the second and third authors provides a class sized iteration that introduces a lightface definable well-order of $H(\kappa^+)$ whenever κ is inaccessible (see Section 8 of the present article for the exact statement of their theorem). It is fairly simple to introduce a lightface definable well-order of $H(\kappa^+)$ for a single inaccessible cardinal κ , so that paper is mainly concerned with finding a sufficiently uniform way of building a class sized forcing that does this for all inaccessibles κ and allows for large cardinal preservation. In the present article, we improve on this by providing a much more *well-behaved* forcing to introduce a lightface definable well-order of $H(\kappa^+)$ that works both for (suitable) successor cardinals and for inaccessibles. This actually allows us to give (as a sample result) a different proof (which we will only hint towards) of the main result of [6] in Section 8.

Given an uncountable cardinal κ that satisfies $\kappa^{<\kappa} = \kappa$ and such that $\lambda^{<\lambda} < \kappa$ for every $\lambda < \kappa$, under additional *anti-large cardinal hypotheses*² it is possible to use the methods developed by the second and third authors in [9] and introduce a Σ_1 -definable well-order of $H(\kappa^+)$ that only uses κ as parameter (and is thus Σ_3 -definable over $H(\kappa^+)$ without parameters) by $<\kappa$ -closed forcing that has the κ^+ -cc and has size 2^κ . In particular, this shows that it is consistent to have 2^κ large in the presence of a lightface definable well-order of $H(\kappa^+)$. While the complexity of the well-orders introduced by the forcing provided in the present article is certainly higher than Σ_3 , our forcings work for a larger class of cardinals and they lend themselves well to large cardinal preservation (see Section 7). Most importantly however, we do not need to assume any kind of anti-large cardinal hypothesis to hold.

3 Club Coding (relative to a stationary set)

In this section we will introduce a coding forcing that could be seen as combining ideas from Solovay's *almost disjoint coding* ([11]) and the *canonical function coding* introduced by Sy Friedman and the first author in [2] and [3]. Although we will never explicitly make use of this coding forcing, it will be woven into our main forcing construction in Section 5 and some of our later proofs will be variations of the arguments given in this section. Moreover the forcing itself might prove to be interesting (in fact it has already been made use of in [8] and [9]). We will call the coding we want to introduce *club coding (relative to a stationary set)*. Given an uncountable cardinal κ that satisfies $\kappa^{<\kappa} = \kappa$, we will present a notion of forcing with nice properties that will allow us to make a subset of $H(\kappa^+)$ definable by a generically added subset of κ . Under the above assumptions on κ , both the almost disjoint coding forcing at κ and canonical function coding at κ are capable of making a subset of $H(\kappa^+)$ definable by a generically added subset of κ , however canonical function coding requires the additional assumption that $2^\kappa = \kappa^+$ and almost disjoint coding does not possess the crucial property (for our present purposes) that we will verify for club coding in Lemma 3.7 (see the paragraph following its proof).

Throughout this section we fix a regular uncountable cardinal κ with $\kappa = \kappa^{<\kappa}$, a stationary set $S \subseteq \kappa \cap \text{cof}(\omega)$, and a subset A of ${}^\kappa\kappa$. We will first recall the definition of almost disjoint coding at κ (see [9] for a more detailed account and a collection of its basic properties).

Definition 3.1. Assume that $\vec{s} = \langle s_\alpha \mid \alpha < \kappa \rangle$ is an enumeration of ${}^{<\kappa}\kappa$ with the property that every element of ${}^{<\kappa}\kappa$ is enumerated κ -many times. We define a partial order $\mathbb{Q}(A)$ by the following clauses.

- A condition in $\mathbb{Q}(A)$ is a pair $p = \langle t_p, a_p \rangle$ with $t_p: \alpha_p \rightarrow 2$ for some $\alpha_p < \kappa$ and $a_p \in [A]^{<\kappa}$.
- We have $q \leq_{\mathbb{Q}(A)} p$ if and only if $t_p \subseteq t_q$, $a_p \subseteq a_q$ and

$$s_\beta \subseteq x \longrightarrow t(\beta) = 0$$

²A sufficient condition is that fat stationary subsets of κ in \mathbf{L} remain fat stationary in \mathbf{V} - note this implies that 0^\sharp does not exist. However it is also shown in that paper that weaker conditions, that allow for the existence of measurable cardinals, suffice. See also Question 9.3 of the present paper and the paragraph preceding it.

for every $x \in a_p$ and $\alpha_p \leq \beta < \alpha_q$.

Now we want to introduce the definition of club coding (relative to a stationary set S). The important differences when compared to the almost disjoint coding forcing are that the enumeration of ${}^{<\kappa}\kappa$ is added generically and (and that's the main point) whenever $x \in A$, this is reflected correctly only on a club (relative to S) and not (as is the case with the almost disjoint coding) on a final segment of the generically added coding subset of κ (if G is generic for either the almost disjoint coding forcing $\mathbb{Q}(A)$ or the club coding forcing $\mathbb{Q}^*(A, S)$, this coding subset of κ is equal to $\bigcup_{p \in G} t_p$).

Definition 3.2. We define $\mathbb{Q}^*(A, S)$ to be the partial order whose conditions are tuples

$$p = \langle s_p, t_p, \langle c_x^p \mid x \in a_p \rangle \rangle$$

such that the following statements hold for some successor ordinal $\gamma_p < \kappa$.

- $s_p : \gamma_p \rightarrow {}^{<\kappa}\kappa$, $t_p : \gamma_p \rightarrow 2$ and $a_p \in [A]^{<\kappa}$.
- If $x \in a_p$, then c_x^p is a closed subset of γ_p and

$$s_p(\alpha) \subseteq x \rightarrow t_p(\alpha) = 0$$

for all $\alpha \in c_x^p \cap S$.

We define $q \leq p$ to hold if $s_p = s_q \upharpoonright \gamma_p$, $t_p = t_q \upharpoonright \gamma_p$, $a_p \subseteq a_q$ and $c_x^p = c_x^q \cap \gamma_p$ for every $x \in a_p$.

Lemma 3.3. *The partial order $\mathbb{Q}^*(A, S)$ is $<\kappa$ -closed, κ^+ -Knaster and has cardinality at most 2^κ .*

Proof. Let $\lambda < \kappa$ and $\langle p_\alpha \mid \alpha < \lambda \rangle$ be a descending sequence in $\mathbb{Q}^*(A, S)$. If there is an $\bar{\alpha} < \lambda$ with $\gamma_{p_\alpha} = \gamma_{p_{\bar{\alpha}}}$ for all $\bar{\alpha} \leq \alpha < \lambda$, then

$$p = \left\langle s_{p_{\bar{\alpha}}}, t_{p_{\bar{\alpha}}}, \left\langle \bigcup_{x \in a_{p_\alpha}} c_x^{p_\alpha} \mid x \in \bigcup_{\alpha < \lambda} a_{p_\alpha} \right\rangle \right\rangle$$

is a condition in $\mathbb{Q}^*(A, S)$ with $p \leq p_\alpha$ for all $\alpha < \lambda$. Now define $\gamma = \sup_{\alpha < \lambda} \gamma_{p_\alpha}$ and assume that $\gamma > \gamma_{p_\alpha}$ for all $\alpha < \lambda$. Define

- $s = \{(\gamma, \emptyset)\} \cup \bigcup \{s_{p_\alpha} \mid \alpha < \lambda\}$.
- $t = \{(\gamma, 0)\} \cup \bigcup \{t_{p_\alpha} \mid \alpha < \lambda\}$.
- $a = \bigcup \{a_{p_\alpha} \mid \alpha < \lambda\}$.
- $c_x = \{\gamma\} \cup \bigcup \{c_x^{p_\alpha} \mid \alpha < \lambda, x \in a_{p_\alpha}\}$ for all $x \in a$.
- $p = \langle s, t, \langle c_x \mid x \in a \rangle \rangle$.

Then p is a condition in $\mathbb{Q}^*(A, S)$ with $p \leq p_\alpha$ for all $\alpha < \lambda$.

To show that $\mathbb{Q}^*(A, S)$ is κ^+ -Knaster, let $\langle p_\alpha \mid \alpha < \kappa^+ \rangle$ be an injective sequence of conditions in $\mathbb{Q}^*(A, S)$. Then there is an $X \in [\kappa^+]^{\kappa^+}$ and an $r \in [A]^{<\kappa}$ such that $s_{p_\alpha} = s_{p_{\bar{\alpha}}}$, $t_{p_\alpha} = t_{p_{\bar{\alpha}}}$, $r = a_{p_\alpha} \cap a_{p_{\bar{\alpha}}}$ and $c_x^{p_\alpha} = c_x^{p_{\bar{\alpha}}}$ for all $\alpha, \bar{\alpha} \in X$ with $\alpha \neq \bar{\alpha}$ and all $x \in r$. Given $\alpha, \bar{\alpha} \in X$ the tuple

$$\langle s_{p_\alpha}, t_{p_\alpha}, \langle c_x^{p_\alpha} \mid x \in a_{p_\alpha} \rangle \cup \langle c_x^{p_{\bar{\alpha}}} \mid x \in a_{p_{\bar{\alpha}}} \rangle \rangle$$

is a condition in $\mathbb{Q}^*(A, S)$ that extends both p_α and $p_{\bar{\alpha}}$.

The last claim of the lemma follows from a simple counting argument. \square

It follows from Lemma 3.3 that forcing with $\mathbb{Q}^*(A, S)$ preserves cofinalities, the continuum function and the stationarity of S .

Proposition 3.4. *If $x \in A$ and $\alpha < \kappa$, then the set*

$$D_{x,\alpha} = \{p \in \mathbb{Q}^*(A, S) \mid x \in a_p, c_x^p \setminus \alpha \neq \emptyset\}$$

is dense in $\mathbb{Q}^(A, S)$.*

Proof. Pick a condition p in $\mathbb{Q}^*(A, S)$. We may assume $x \in a_p$, because otherwise we work with the condition

$$p' = \langle s_p, t_p, \langle c_y^p \mid y \in a_p \rangle \cup \{(x, \emptyset)\} \rangle$$

Pick $\gamma > \gamma_p$ and define

- $s = s_p \cup \{(\beta, \emptyset) \mid \gamma_p \leq \beta \leq \gamma\}$.
- $t = t_p \cup \{(\beta, 0) \mid \gamma_p \leq \beta \leq \gamma\}$.
- $p_* = \langle s, t, \langle c_y^p \cup [\gamma_p, \gamma] \mid y \in a_p \rangle \rangle$.

Then p_* is a condition in $D_{x,\alpha}$ with $p_* \leq p$. □

Let \dot{s} and \dot{t} denote the canonical $\mathbb{Q}^*(A, S)$ -names such that

$$\dot{s}^G = \bigcup \{s_p \mid p \in G\}$$

and

$$\dot{t}^G = \bigcup \{t_p \mid p \in G\}$$

whenever G is $\mathbb{Q}^*(A, S)$ -generic over \mathbf{V} .

Theorem 3.5. *If G is $\mathbb{Q}^*(A, S)$ -generic over \mathbf{V} , then $\dot{s}^G : \kappa \rightarrow <^\kappa \kappa$, $\dot{t}^G : \kappa \rightarrow 2$ and A is equal to the set of all $x \in {}^\kappa \kappa$ with the property that*

$$\forall \alpha \in C \cap S [\dot{s}^G(\alpha) \subseteq x \rightarrow \dot{t}^G(\alpha) = 0] \quad (1)$$

holds for some club subset C of κ in $\mathbf{V}[G]$.

Proof. The first two statements follow directly from the above proposition. Pick $x \in A$ and define $C = \bigcup \{c_x^p \mid p \in G, x \in a_p\}$. Then the definition of $\mathbb{Q}^*(A, S)$ implies that C is a closed subset of κ that satisfies (1) and the above proposition shows that C is unbounded in κ .

Now work in the ground model \mathbf{V} , pick a $\mathbb{Q}^*(A, S)$ -name \dot{y} for an element of ${}^\kappa \kappa$ and a $\mathbb{Q}^*(A, S)$ -name \dot{C} for a club subset of κ and assume, towards a contradiction, that there is a condition p_0 in $\mathbb{Q}^*(A, S)$ with

$$p_0 \Vdash \dot{y} \notin \dot{A} \wedge \forall \alpha \in \dot{C} \cap S [\dot{s}(\alpha) \subseteq \dot{y} \rightarrow \dot{t}(\alpha) = 0]. \quad (2)$$

Let N be a countable elementary substructure of some large enough $H(\theta)$ containing $\mathbb{Q}^*(A, S)$, \dot{y} , \dot{C} and p_0 and such that $\gamma := N \cap \kappa \in S$. Let $\langle p_n \mid n < \omega \rangle$ be a descending $\langle N, \mathbb{Q}^*(A, S) \rangle$ -generic sequence of conditions extending p_0 . By the above proposition together with the genericity of $\langle p_n \mid n < \omega \rangle$,

- (i) $\sup_n \gamma_{p_n} = \gamma$, and

(ii) there is some $u : \gamma \rightarrow \kappa$ such that for every $n < \omega$ there is some $m \geq n$ such that p_m forces

- $\dot{y} \upharpoonright \gamma_{p_n} = u \upharpoonright \gamma_{p_n}$ and such that
- $x \upharpoonright \gamma_{p_n} \neq u \upharpoonright \gamma_{p_n}$ for all $x \in a_{p_n}$.

Now we define

- $s = \{(\gamma, u)\} \cup \bigcup \{s_{p_n} \mid n < \omega\}$.
- $t = \{(\gamma, 1)\} \cup \bigcup \{t_{p_n} \mid n < \omega\}$.
- $a = \bigcup \{a_{p_n} \mid n < \omega\}$.
- $c_x = \{\gamma\} \cup \bigcup \{c_x^{p_n} \mid n < \omega, x \in a_{p_n}\}$ for all $x \in a$.

Then the tuple $p = \langle s, t, \langle c_x \mid x \in a \rangle \rangle$ is a condition in $\mathbb{Q}^*(A, S)$, because $u \not\subseteq x$ for all $x \in a$. But $p \leq p_0$ and

$$p \Vdash \check{\gamma} \in \dot{C} \wedge \dot{s}(\check{\gamma}) \subseteq \dot{y} \wedge \dot{t}(\check{\gamma}) = 1,$$

contradicting (2). □

Remark 3.6. (i) The above theorem shows that the set A is definable over the structure $\langle H(\kappa^+)^{\mathbf{V}^G}, \in \rangle$ by a Σ_1 -formula with parameters S , \dot{s}^G and \dot{t}^G whenever G is $\mathbb{Q}^*(A, S)$ -generic over the ground model \mathbf{V} .

(ii) A small variation of the above proof shows that this coding has nice persistence properties - the set A is still defined by the formula (1) after further forcing with a σ -strategically closed partial order that preserves the regularity of κ . The next lemma however provides the crucial, for our present purposes, property that club coding (relative to S) satisfies in contrast to the classical almost disjoint coding.

Lemma 3.7. *If $A_0 \subseteq A$, then $\mathbb{Q}^*(A_0, S)$ is a complete subforcing of $\mathbb{Q}^*(A, S)$.*

Proof. It is immediate to observe that $\mathbb{Q}^*(A_0, S) \subseteq \mathbb{Q}^*(A, S)$, and moreover that both the extension relation and the incompatibility relation of $\mathbb{Q}^*(A, S)$ extend the respective relations of $\mathbb{Q}^*(A_0, S)$. Thus it suffices to show that every maximal antichain \mathcal{A} of $\mathbb{Q}^*(A_0, S)$ is predense in $\mathbb{Q}^*(A, S)$. Pick a condition p in $\mathbb{Q}^*(A, S)$ and define

$$\bar{p} = \langle s_p, t_p, \langle c_x^p \mid x \in a_p \cap A_0 \rangle \rangle.$$

Then \bar{p} is a condition in $\mathbb{Q}^*(A_0, S)$ and there are conditions q and r in $\mathbb{Q}^*(A_0, S)$ such that $q \in \mathcal{A}$ and r is a common extension of \bar{p} and q in $\mathbb{Q}^*(A_0, S)$. Define

$$p_* = \langle s_r, t_r, \langle c_x^r \mid x \in a_r \rangle \cup \langle c_x^p \mid x \in a_p \setminus A_0 \rangle \rangle.$$

Then p_* is a condition in $\mathbb{Q}^*(A, S)$ and it is a common extension of p and q in $\mathbb{Q}^*(A, S)$. □

Remark 3.8. It is easy to see that the almost disjoint coding forcing does not possess the property stated in the above lemma: Assume, towards a contradiction, that $\mathbb{Q}(A)$ is a complete subforcing of $\mathbb{Q}({}^\kappa\kappa)$ for every $A \subseteq {}^\kappa\kappa$ and let G be $\mathbb{Q}({}^\kappa\kappa)$ -generic over \mathbf{V} . For each $A \subseteq {}^\kappa\kappa$, the generic filter in $\mathbb{Q}(A)$ induced by G yields a function $t_A \in {}^\kappa 2$ coding A . It is easy to see that the resulting function $[A \mapsto t_A]$ is an injection of $\mathcal{P}({}^\kappa\kappa)^{\mathbf{V}}$ into $({}^\kappa 2)^{\mathbf{V}[G]}$ in $\mathbf{V}[G]$. Since forcing with $\mathbb{Q}(A)$ preserves cardinalities and the value of 2^κ , this yields a contradiction. This consideration does not apply if we work with $\mathbb{Q}^*(A, S)$ instead: Suppose $A_0 \subseteq A$, G is generic for $\mathbb{Q}^*(A, S)$, and G' is the restriction of G to $\mathbb{Q}^*(A_0, S)$. Then G' adds $t_{A_0} \in {}^\kappa 2$ coding A_0 in $\mathbf{V}[G']$. However, the same code t_{A_0} will code A in $\mathbf{V}[G]$. The reason is that in moving from $\mathbf{V}[G']$ to $\mathbf{V}[G]$ we are adding new club subsets of κ that ensure this to be the case.

4 More Preliminaries

Let $\langle \cdot, \cdot \rangle: \text{Ord} \times \text{Ord} \rightarrow \text{Ord}$ denote Gödel's pairing function.³ We also let $\prec \cdot, \cdot, \succ: \text{Ord}^3 \rightarrow \text{Ord}$ be $\prec \cdot, \prec \cdot, \cdot \succ$.

It will be convenient to define the following notion of rank of an ordinal with respect to a set of ordinals and the corresponding notion of perfect ordinal (see for example [2] or [3]).

Definition 4.1. Let X be a set of ordinals and let η, μ be ordinals. We define the relation $\text{rank}_X(\eta) \geq \mu$ by recursion as follows:

- $\text{rank}_X(\eta) > 0$ if and only if η is a limit point (but not necessarily an element) of X .
- If $\mu > 0$, then $\text{rank}_X(\eta) > \mu$ if and only if η is a limit of ordinals ξ such that $\text{rank}_X(\xi) \geq \mu$.

We say that an ordinal η is *perfect* if and only if $\text{rank}_\eta(\eta) = \eta$.

Note that the first nonzero perfect ordinal is $\epsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\}$. Note also that $\text{rank}_\delta(\delta) \leq \delta$ for every ordinal δ and that, given any uncountable cardinal λ , the set of perfect ordinals below λ forms a club subset of λ of order type λ . Let $(\eta_\xi)_{\xi \in \text{Ord}}$ be the strictly increasing enumeration of all nonzero perfect ordinals of cofinality ω .

The notions defined in the following two paragraphs appear in [2] and [3].

Given two sets of ordinals X and Y , let $X \cap^* Y$ be the collection of all $\delta \in X \cap Y$ such that δ is not a limit point of X .⁴ A sequence $\vec{C} = \langle C_\delta \mid \delta \in \text{dom}(\vec{C}) \rangle$ is a club-sequence if $\text{dom}(\vec{C})$ is a set of ordinals and C_δ is a club subset of δ for each $\delta \in \text{dom}(\vec{C})$. We will say that \vec{C} is *coherent* if there is a club-sequence $\vec{D} = \langle D_\delta \mid \delta \in \text{dom}(\vec{D}) \rangle$ such that

- $\vec{C} \subseteq \vec{D}$ and
- for every $\delta \in \text{dom}(\vec{D})$ and every limit point γ of D_δ , $\gamma \in \text{dom}(\vec{D})$ and $D_\gamma = D_\delta \cap \gamma$.

³That is, $\prec \alpha, \beta \succ$ is the order type of $\{\langle \gamma, \delta \rangle \in \text{Ord} \times \text{Ord} \mid \langle \gamma, \delta \rangle < \langle \alpha, \beta \rangle\}$, where $\langle \gamma, \delta \rangle < \langle \alpha, \beta \rangle$ if and only if either $\max\{\gamma, \delta\} < \max\{\alpha, \beta\}$, or $\max\{\gamma, \delta\} = \max\{\alpha, \beta\}$ and $\gamma < \alpha$, or $\max\{\gamma, \delta\} = \max\{\alpha, \beta\}$, $\gamma = \alpha$ and $\delta < \beta$ (see for example [10], p. 30).

⁴Note that \cap^* is not commutative. For example, $\{\omega\} \cap^* (\omega+1) = \{\omega\}$ but $(\omega+1) \cap^* \{\omega\} = \emptyset$.

If $\vec{C} = \langle C_\delta \mid \delta \in \text{dom}(\vec{C}) \rangle$ is a club-sequence, we denote $\bigcup_{\delta \in \text{dom}(\vec{C})} C_\delta$ by $\text{range}(\vec{C})$. Also, we will say that an ordinal τ is the *height of \vec{C}* , and will write $\text{ht}(\vec{C}) = \tau$, if $\text{ot}(C_\delta) = \tau$ for all $\delta \in \text{dom}(\vec{C})$.⁵ A club-sequence is called a ladder system if it has height ω . We will say that a club-sequence $\vec{C} = \langle C_\delta \mid \delta \in \text{dom}(\vec{C}) \rangle$ with stationary domain such that $\text{sup}(\text{dom}(\vec{C})) = \chi$ is *strongly type-guessing* if for every club subset $C \subseteq \chi$ there is a club $D \subseteq \chi$ such that $\text{ot}(C_\delta \cap^* C) = \text{ot}(C_\delta)$ for every $\delta \in \text{dom}(\vec{C}) \cap D$.

The following related form of club-guessing will also be used:⁶ A ladder system $\langle C_\delta \mid \delta \in S \rangle$, where S is a stationary subset of some κ , is *strongly guessing* if for every club $C \subseteq \kappa$ there is a club $D \subseteq \kappa$ such that $C_\delta \setminus C$ is bounded in δ for every $\delta \in D \cap S$.

The following notion of closure for partial orders will be useful: Let \mathbb{P} be a partial order, κ a cardinal, $\beta_{\mathbb{P}} : \mathbb{P} \rightarrow \text{Ord}$ a function, and S a set of ordinals. We will say that a partial order \mathbb{P} is *$<\kappa$ -closed relative to $\beta_{\mathbb{P}}$ outside S* if for every $\lambda < \kappa$ and every decreasing sequence $\langle p_i \mid i < \lambda \rangle$ of conditions in \mathbb{P} , if $\text{sup}(\beta_{\mathbb{P}} \text{``}\{p_i \mid i < \lambda\}\text{'}) \notin S$, then there is a condition in \mathbb{P} extending all p_i . If $S = \emptyset$, this simply says that \mathbb{P} is *$<\kappa$ -closed*.

It will be convenient to define the following notion of hereditary internal approachability. Let θ be an infinite cardinal. Given $x \in H(\theta)$ we define, by recursion on the cardinals less than θ , the notion of being a *hereditarily internally approachable* (HIA) elementary substructure of $\langle H(\theta), \in \rangle$ containing x as follows:

We say that $\langle N_i \mid i < \gamma \rangle$ is an *\in -chain* if $\langle N_j \mid j \leq i \rangle \in N_{i+1}$ for every i with $i + 1 < \gamma$. A structure $N \prec \langle H(\theta), \in \rangle$ such that $x \in N$ is HIA if $N = \bigcup_{i < \text{cof}(|N|)} N_i$ for a \subseteq -continuous \in -chain $\langle N_i \mid i < \text{cof}(|N|) \rangle$ of sets of size less than $|N|$ such that $\langle N_i, \in \rangle$ is an HIA elementary substructure of $\langle H(\theta), \in \rangle$ containing x , whenever N_i is infinite and i is a successor ordinal.

Lemma 4.2. *The set of HIA elementary substructures of $\langle H(\theta), \in \rangle$, that contain x and are of size μ , is a stationary subset of $[H(\theta)]^\mu$ whenever $x \in H(\theta)$ and $\mu \leq |H(\theta)|$ is an infinite cardinal.*

Proof. By induction on μ . Note that if $\mu = \omega$, every countable $N \prec \langle H(\theta), \in \rangle$ is HIA. Thus assume the lemma holds for all $\omega \leq \nu < \mu$, we need to verify it holds for μ . It suffices to show that whenever $f : [H(\theta)]^{<\omega} \rightarrow H(\theta)$, then we can find an HIA elementary substructure N of $\langle H(\theta), \in \rangle$ with x in its domain, which is of size μ and is closed under f . For this purpose, we may assume that whenever $N \subseteq H(\theta)$ is closed under f , then $\langle N, \in \rangle$ is an elementary substructure of $\langle H(\theta), \in \rangle$, and construct a \subseteq -continuous \in -chain $\langle N_i \mid i \leq \mu \rangle$, so that N_i is an HIA elementary substructure of $\langle H(\theta), \in \rangle$ whenever i is a successor ordinal, and so that each N_i is closed under f , as follows. Let N_0 be the closure under f of $\{x\}$. If i is a limit ordinal, let $N_i = \bigcup_{j < i} N_j$. Given N_i for some $i < \mu$, we use our inductive assumption to choose N_{i+1} of size $|N_i|$, so that $N_{i+1} \supseteq N_i$, $\langle N_j \mid j \leq i \rangle \in N_{i+1}$ and N_{i+1} is closed under f . In the end, $N = N_\mu$ is as desired. \square

Lemma 4.3. *Let κ be a cardinal and let \mathbb{P} be a partial order which is $<\kappa$ -closed. If $\theta > |\mathbb{P}|$ is a cardinal, Δ is a well-order of $H(\theta)$ and N is an HIA elementary*

⁵The height of a club-sequence may of course not be defined.

⁶This notion is rather standard; see for example [1].

substructure of $\langle H(\theta), \in, \Delta \rangle$ containing \mathbb{P} and of size at most κ with $p \in \mathbb{P} \cap N$, then there is an $\langle N, \mathbb{P} \rangle$ -generic sequence of conditions extending p .

Proof. We will verify the lemma by induction on $\mu = |N| \leq \kappa$. Since the lemma obviously holds if $\mu = \omega$, we assume that $\mu \geq \omega_1$ and that the lemma holds for $\nu < \mu$. We want to verify the lemma holds for μ . Let $\langle N_i \mid i < \text{cof}(|N|) \rangle$ be a \subseteq -continuous \in -chain with union N , such that $\langle N_i, \in \rangle$ is an HIA elementary substructure of $\langle H(\theta), \in, \Delta \rangle$ of size less than μ for every successor ordinal $i < \text{cof}(|N|)$. Using our inductive assumption, we can construct a decreasing $\langle N, \mathbb{P} \rangle$ -generic sequence $\langle p_i \mid i < \mu \rangle$ as follows. Let $p_0 = p \in N_1$ and for every $i < \text{cof}(|N|)$, $p_{i+1} \in N_{i+2}$ is the Δ -least lower bound of the Δ -least $\langle N_{i+1}, \mathbb{P} \rangle$ -generic sequence of conditions extending p_i , and $p_i \in N_{i+1}$ is the Δ -least lower bound of $\langle p_j \mid j < i \rangle$ for limit ordinals $i < \mu$ (we use here that $\langle N_j \mid j < i \rangle \in N_{i+1}$). The sequence $\langle p_i \mid i < \mu \rangle$ is now as desired. \square

Note that the above proof still goes through if, within the proof, we choose the p_i , $i \neq 0$ to be canonical rather than Δ -least lower bounds of the respective sequences, assuming that the forcing \mathbb{P} is such that those canonical lower bounds exist and can be constructed inside N_{i+1} . We will frequently make use of this fact.

We will also need the following technical lemma in the next section.

Lemma 4.4. *Assume κ is regular and uncountable, $S \subseteq \kappa \cap \text{cof } \omega$ is stationary, $(\text{cof } \omega \cap \kappa) \setminus S$ is stationary and θ is large enough and regular. For every countable $X \subseteq H(\theta)$, there is an \in -chain $\langle N_n \mid n < \omega \rangle$ of countable elementary substructures of $\langle H(\theta), \in \rangle$ such that $X \subseteq N_0$, $\gamma_n := \sup(N_n) \cap \kappa \notin S$ for any $n < \omega$ and $\sup_{n < \omega} \gamma_n \in S$.*

Proof. Let $\langle M_i \mid i < \kappa \rangle$ be a continuous \in -chain of elementary substructures of $H(\theta)$ of size less than κ such that $X \subseteq M_0$ and such that $\sup(M_i \cap \kappa) \in (\text{cof } \omega \cap \kappa) \setminus S$ whenever i is a successor ordinal. Let $\text{Lim}^2 := \lim(\lim(\kappa))$. $\{\sup(M_i) \cap \kappa \mid i \in \text{Lim}^2\}$ is a club subset of κ . Choose $i \in \text{Lim}^2$ least possible such that $\sup(M_i \cap \kappa) \in S$. Note that $\text{cof}(i) = \omega$. Pick a sequence $\langle i_n \mid n < \omega \rangle$ with supremum i , consisting only of limit ordinals of cofinality ω . This is possible for $i \in \text{Lim}^2$.

First assume that there is $n < \omega$ such that $\sup(M_{i_n} \cap \kappa) \in S$. By minimality of i , $i_n \notin \text{Lim}^2$. But this means that $i_n = k + \omega$ for some limit ordinal k . For $n < \omega$, let $N_n^* := M_{k+n+1}$.

Now assume $\sup(M_{i_n} \cap \kappa) \notin S$ for all $n < \omega$ and let $N_n^* := M_{i_n}$ for any $n < \omega$ in this case. In both cases, we have that $\sup(N_n^* \cap \kappa) \in (\text{cof } \omega \cap \kappa) \setminus S$ for any $n < \omega$. Now we inductively construct $\langle N_n \mid n < \omega \rangle$ as follows. Inside N_1^* , let N_0 be a countable elementary substructure of N_0^* with $\sup(N_0 \cap \kappa) = \sup(N_0^* \cap \kappa)$ and $X \subseteq N_0$. Given N_i for $i < \omega$, work inside of N_{i+2}^* to choose a countable elementary substructure N_{i+1} of N_{i+1}^* with $N_i \in N_{i+1}$ and $\sup(N_{i+1} \cap \kappa) = \sup(N_{i+1}^* \cap \kappa)$. Then $\sup_{n < \omega} (\sup(N_n) \cap \kappa) \in S$, as desired. \square

5 The single-step forcing

In this section we prove Theorem 1.1.

Proof of Theorem 1.1. Let us fix a regular uncountable cardinal κ such that $\kappa^{<\kappa} = \kappa$. \mathcal{Q} will consist of tuples of the form $\langle p, q \rangle$ where $p \in \vec{S}$ and $p \Vdash_{\vec{S}} \check{q} \in \dot{\mathbb{P}}$, for \vec{S} a notion of forcing in \mathbf{V} described below and $\dot{\mathbb{P}}$ an \vec{S} -name for a notion of forcing \mathbb{P} in $\mathbf{V}^{\vec{S}}$ described below, such that $\mathbf{1} \Vdash_{\vec{S}} \dot{\mathbb{P}} \subseteq \mathbf{V}$.⁷

For any ordinal β , $\beta^{<+\rangle}$ denotes the least ordinal greater than β that is closed under Gödel pairing. Let $C^{<+\rangle}$ denote the closed unbounded subset of κ consisting of 0 and all limit ordinals closed under Gödel pairing.

Conditions in \vec{S} will be pairs $\langle s, \sigma \rangle$ such that $s: \gamma \rightarrow 2$ for some ordinal $\gamma < \kappa$, $\{\delta < \gamma \mid s(\delta) = 1\} \subseteq C^{<+\rangle} \cap \text{cof}(\omega)$ and, letting $\vec{s} = \{\gamma \mid s(\gamma) = 1\}$, σ is a function $\sigma: \vec{s} \rightarrow {}^{<\kappa}\kappa$ (in a slight abuse of notation, we will sometimes identify s and \vec{s} later on). A condition $\langle s_1, \sigma_1 \rangle$ extends a condition $\langle s_0, \sigma_0 \rangle$ in \vec{S} if $s_0 \subseteq s_1$ and $\sigma_0 \subseteq \sigma_1$. Forcing with \vec{S} adds a stationary subset S of $\kappa \cap \text{cof}(\omega)$ such that $(\kappa \cap \text{cof}(\omega)) \setminus S$ is also stationary, and adds a generic enumeration \vec{s} of ${}^{<\kappa}\kappa$ with domain S and with the property that every element of ${}^{<\kappa}\kappa$ is enumerated stationarily often. Let \dot{S} and \dot{s} be the canonical \vec{S} -names for S and \vec{s} respectively.

Let $\lambda := 2^\kappa$. Let $\bar{\mathcal{W}}$ be a well-order of ${}^\kappa\kappa$ of order-type λ with smallest element $\bar{0}$. We want to use $\bar{\mathcal{W}}$ to construct a very specific well-order \mathcal{W} of ${}^\kappa\kappa$ of order-type $\lambda + 1$. If $x \in {}^{<\omega}\kappa$ and $y \in {}^\kappa\kappa$, we let $x \frown y$ denote the concatenation of x and y , i.e. if $x = \langle x_i \mid i < n \rangle$ we let $(x \frown y)(i) = x_i$ if $i < n$ and we let $(x \frown y)(n + \alpha) = y(\alpha)$ for $\alpha < \kappa$. \mathcal{W} will be made up of λ -many κ -blocks with $\bar{0}$ atop of them. Assuming that $x, y \in {}^\kappa\kappa$ are both not equal to $\bar{0}$, $x = \langle \alpha \rangle \frown \bar{x}$ and $y = \langle \beta \rangle \frown \bar{y}$, we set

$$x\mathcal{W}y \leftrightarrow [(\bar{x} = \bar{y} \wedge \alpha < \beta) \vee \bar{x}\bar{\mathcal{W}}\bar{y}].$$

We will need this well-order \mathcal{W} in our coding construction in order for every $\bar{x} \in {}^\kappa\kappa$ to be canonically connected to a κ -block of \mathcal{W} -consecutive elements. Having $\bar{0}$ as its largest element will just be notationally convenient.

Let $\bar{F}: \lambda \rightarrow H(\kappa^+)$ be a bookkeeping function for $H(\kappa^+)$ (i.e., for every $x \in H(\kappa^+)$, $\bar{F}^{-1}(x)$ is unbounded in λ) and let $F: {}^\kappa\kappa \setminus \{\bar{0}\} \rightarrow H(\kappa^+)$ be defined by $F(x) = \bar{F}(\text{ot}\{y \mid y\mathcal{W}x\})$.

Work in an \vec{S} -generic extension \mathbf{W} of \mathbf{V} until further notice and let G_0 denote the \vec{S} -generic filter. We want to construct by recursion along \mathcal{W} a collection of partial orders P_x for $x \in ({}^\kappa\kappa)^{\mathbf{W}}$ and set $\mathbb{P} = P_{\bar{0}}$. \mathbb{P} and the P_x will depend on S and \vec{s} and we write $\mathbb{P}(S, \vec{s})$ instead of \mathbb{P} when we want to emphasize this fact. Each P_x will have a canonical \vec{S} -name in \mathbf{V} , denoted by \dot{P}_x . Conditions in P_x will be of the form

$$p = \left\langle t, \vec{e}, \langle \langle \vec{C}^i, \vec{D}^i \rangle \mid i < \beta \rangle, \langle c_{\bar{x}} \mid \bar{x} \in a \rangle \right\rangle.$$

We will set $t^p = t$ and similarly for any other object appearing within p as above. Suppose now that p is a tuple as above such that

$$(1) \beta \in C^{<+\rangle},$$

⁷What we basically want to do here is to let \mathcal{Q} be the two-step iteration of $\vec{S} * \dot{\mathbb{P}}$. However for technical reasons, we choose it to be a dense subset of this two-step iteration. Since conditions in \vec{S} will be elements of $H(\kappa^+)$ and $\mathbf{1}_{\vec{S}}$ forces conditions in $\dot{\mathbb{P}}$ to be elements of $H(\kappa^+)^{\mathbf{V}}$, the above will in particular help us to obtain that $\mathcal{Q} \subseteq H(\kappa^+)$.

- (2) $t \in {}^{\beta+1}2$,
- (3) \vec{e} is a ladder system on $S \cap (\beta + 1)$,
- (4) for $i < \beta$, \vec{C}^i and \vec{D}^i are club-sequences with domains included in $\beta + 1$,
- (5) $a \in [({}^\kappa\kappa)^\mathbf{V} \setminus \{\vec{0}\}]^{<\kappa}$ and
- (6) for every $\bar{x} \in a$, $c_{\bar{x}}$ is a subset of $\beta + 1$.

Note that any such tuple is an element of \mathbf{V} for \vec{S} is $<\kappa$ -closed. We want to associate to p a certain set $C(p) \subseteq ({}^\kappa\kappa)^\mathbf{V}$ which canonically codes p .⁸

We code t , \vec{e} and $\langle\langle \vec{C}^i, \vec{D}^i \mid i < \beta \rangle\rangle$ by $b \in ({}^\kappa\kappa)^\mathbf{V}$ as follows. For $\gamma < \kappa$, let $b(2 \cdot \gamma) = 1$ iff $t(\gamma) = 1$, let $b(6 \cdot \gamma + 1) = 1$ iff $\gamma_0 \in E_{\gamma_1}$, $\gamma = \prec\gamma_0, \gamma_1\rangle$ and $\vec{e} = \langle E_\xi \mid \xi \in S \cap (\beta + 1) \rangle$, let $b(6 \cdot \gamma + 3) = 1$ iff $\gamma_0 \in C_{\gamma_1}^i$ and $\gamma = \prec\gamma_0, \gamma_1, i\rangle$,⁹ and let $b(6 \cdot \gamma + 5) = 1$ iff $\gamma_0 \in D_{\gamma_1}^i$ and $\gamma = \prec\gamma_0, \gamma_1, i\rangle$.

Now we want to define $C(p) \subseteq ({}^\kappa\kappa)^\mathbf{V} \setminus \{\vec{0}\}$ coding b and $\langle c_{\bar{x}} \mid \bar{x} \in a \rangle$. For $x \in {}^\kappa\kappa$, we let $x \in C(p)$ iff one of the following holds.

- There is $\alpha < \kappa$ such that $x = \langle 1 + \alpha \rangle \frown \vec{0}$ and $\alpha \in b$.
- There is $\alpha < \kappa$ and $\bar{x} \in a$ such that $x = \langle \alpha, 1 \rangle \frown \bar{x}$ and $\alpha \in c_{\bar{x}}$.

We code F by $F^* \subseteq ({}^\kappa\kappa)^\mathbf{V}$ as follows. Given $x, y \in {}^\kappa\kappa$, let $|x, y| \in {}^\kappa\kappa$ be defined by setting $|x, y|(\alpha) = \beta$ iff $\alpha = 2 \cdot \bar{\alpha}$ and $x(\bar{\alpha}) = \beta$ or $\alpha = 2 \cdot \bar{\alpha} + 1$ and $y(\bar{\alpha}) = \beta$. We set $|x, y|(\alpha) = 0$ whenever it is not given a value by the above. Set $z \in F^*$ iff there is (x, y) such that $x \in ({}^\kappa\kappa)^\mathbf{V}$, $y \subseteq \kappa$ codes $y^* \in H(\kappa^+)^\mathbf{V}$,¹⁰ $F(x) = y^*$ and $z = |x, y|$, where we identify y with its characteristic function in the latter. Let $\mathcal{W}^* \subseteq ({}^\kappa\kappa)^\mathbf{V}$ code \mathcal{W} by letting $z \in \mathcal{W}^*$ iff there is $(x, y) \in \mathcal{W}$ such that $z = |x, y|$.

Now we define $A^p \subseteq ({}^\kappa\kappa)^\mathbf{V} \setminus \{\vec{0}\}$ coding $C(p)$, F^* and \mathcal{W}^* by letting, for every $x \in {}^\kappa\kappa$, $x \in A^p$ iff one of the following holds.

- $x \in C(p)$.
- There is $\bar{x} \in {}^\kappa\kappa$ such that $x = \langle 0, 2 \rangle \frown \bar{x}$ and $\bar{x} \in F^*$.
- There is $\bar{x} \in {}^\kappa\kappa$ such that $x = \langle 0, 3 \rangle \frown \bar{x}$ and $\bar{x} \in \mathcal{W}^*$.

Let $s^* \in ({}^\kappa\kappa)^\mathbf{W}$ be a canonical code for \vec{s} , say if $\kappa > \alpha = \prec\beta, \gamma\rangle$ we set

$$s^*(\alpha) = \begin{cases} 0 & \text{if } \vec{s}(\beta)(\gamma) = 0 \\ 1 & \text{if } \vec{s}(\beta)(\gamma) = 1 \\ 2 & \text{if } \beta \notin \text{dom}(\vec{s}) \vee \gamma \geq \text{dom}(\vec{s}(\beta)) \end{cases}.$$

If $x \in {}^\kappa\kappa$, let x^- be defined by $x^-(\alpha) = x(1 + \alpha)$ for every $\alpha < \kappa$.¹¹ Let \mathcal{C} be the set of all $x \in ({}^\kappa\kappa)^\mathbf{V}$ such that either $x = \vec{0}$ or whenever $y\mathcal{W}x$ then both

⁸In fact we will not be able to read off from $C(p)$ whether $\bar{x} \in a$ in case $c_{\bar{x}} = \emptyset$. So one may rather say that $C(p)$ only codes partial information about p . This minor point will however be irrelevant.

⁹If \vec{C}^i is a club-sequence and $j \in \text{dom}(\vec{C}^i)$, we write C_j^i to abbreviate $\vec{C}^i(j)$.

¹⁰In the usual way of subsets of κ coding elements of $H(\kappa^+)$.

¹¹Thus when passing from x to x^- , we just *throw away* the first component of x .

$y^- \mathcal{W}x$ and for every $\alpha < \kappa$, $\langle \alpha \rangle \frown y \mathcal{W}x$.¹² Since $\text{cof}(\lambda) > \kappa$, $\mathcal{C} \cap \{y \mid y \mathcal{W} \vec{0}\}$ is a closed and unbounded (w.r.t. \mathcal{W}) subset of $\{y \mid y \mathcal{W} \vec{0}\}$. P_x will be defined iff $x \in \mathcal{C}$.

For $x \in (\kappa \kappa)^{\mathbf{V}}$ and $p = \langle t, \vec{e}, \langle \langle \vec{C}^i, \vec{D}^i \rangle \mid i < \beta \rangle, \langle c_{\bar{x}} \mid \bar{x} \in a \rangle \rangle$ as above, let

$$p \upharpoonright x = \langle t, \vec{e}, \langle \langle \vec{C}^i, \vec{D}^i \rangle \mid i < \beta \rangle, \langle c_{\bar{x}} \mid \bar{x} \in a \cap \{y \mid y \mathcal{W} x\} \rangle \rangle.$$

The next claim follows by the properties of \mathcal{C} .

Claim 5.1. *Let p be a tuple for which A^p is defined and let $x \in \mathcal{C}$. Then for every $y \mathcal{W}x$, $y \in A^p$ iff $y \in A^{p \upharpoonright x}$.*

Proof. Assume p is such that A^p is defined. Note that $A^{p \upharpoonright x} \subseteq A^p$ for any $x \in (\kappa \kappa)^{\mathbf{V}}$. Now assume $x \in \mathcal{C}$, $y \mathcal{W}x$ and $y \in A^p$. We want to show that $y \in A^{p \upharpoonright x}$. Clearly the only nontrivial case is when y is of the form $\langle \alpha, 1 \rangle \frown \bar{y}$, i.e. y codes the property that $\alpha \in c_{\bar{y}}$ for some $\alpha < \kappa$ and $\bar{y} \in (\kappa \kappa)^{\mathbf{V}} \setminus \{\vec{0}\}$. Using the fact that $x \in \mathcal{C}$, it follows that $\bar{y} \mathcal{W}x$. But this obviously implies that $y \in A^{p \upharpoonright x}$. \square

Given $x \in \mathcal{C}$ and assuming that P_y has been defined for all $y \mathcal{W}x$ with $y \in \mathcal{C}$, conditions in P_x are tuples of the form

$$p = \langle t, \vec{e}, \langle \langle \vec{C}^i, \vec{D}^i \rangle \mid i < \beta \rangle, \langle c_{\bar{x}} \mid \bar{x} \in a \rangle \rangle$$

satisfying the following properties (which imply properties (1)-(6) above).

- (i) $\beta \in C^{\prec+\succ}$.
- (ii) $t \in \beta^{+1} \mathbf{2}$
- (iii) $\vec{e} = \langle E_\delta \mid \delta \in S \cap (\beta + 1) \rangle$ is a ladder system.
- (iv) a is a subset of $W(\beta^{\prec+\succ}) \cap \{y \mid y \mathcal{W}x\}$ of size less than κ , where for $\xi < \kappa$,

$$W(\xi) = \left[\langle 1 \rangle \frown \vec{0}, \langle \xi \rangle \frown \vec{0} \right]^{\mathcal{W}} \cup \bigcup_{\bar{x} \in (\kappa \kappa)^{\mathbf{V}} \setminus \{\vec{0}\}} \left[\langle 0 \rangle \frown \bar{x}, \langle \xi \rangle \frown \bar{x} \right]^{\mathcal{W}}$$

and for any $x_0, x_1 \in (\kappa \kappa)^{\mathbf{V}}$, $[x_0, x_1]^{\mathcal{W}}$ denotes the interval $[x_0, x_1]$ w.r.t. \mathcal{W} , i.e. $[x_0, x_1]^{\mathcal{W}} = \{z \in \kappa \kappa \mid z = x_0 \vee x_0 \mathcal{W} z \mathcal{W} x_1\}$.

- (v) For each $\bar{x} \in a$, $c_{\bar{x}}$ is a closed subset of $\beta + 1$.
- (vi) $\forall \bar{x} \in a \cap A^p \forall \alpha \in c_{\bar{x}} \cap S [s_\alpha \subseteq \bar{x} \rightarrow t(\alpha) = 0]$
- (vii) For every $i < \beta$, \vec{C}^i and \vec{D}^i are club-sequences with domain included in $\beta + 1$, $\text{ht}(\vec{C}^i)$ is defined and is a perfect ordinal of countable cofinality, and \vec{C}^i is coherent as witnessed by \vec{D}^i . Moreover, for every $\xi < \beta$,

- (a) if $\xi = 2 \cdot \bar{\xi}$, then

$$\bar{\xi} \in S \leftrightarrow \exists i < \beta \text{ ht}(\vec{C}^i) = \eta_\xi.$$

¹²Note that in particular $\langle 1 \rangle \frown \vec{0}$, the \mathcal{W} -least element of $(\kappa \kappa)^{\mathbf{V}}$, is an element of \mathcal{C} .

(b) if $\xi = 4 \cdot \bar{\xi} + 1$ and $\bar{\xi} = \langle \xi_0, \xi_1 \rangle$, then

$$(\xi_0, \xi_1) \in s^* \leftrightarrow \exists i < \beta \text{ ht}(\vec{C}^i) = \eta_\xi.$$

(c) if $\xi = 4 \cdot \bar{\xi} + 3$ and $s_{\bar{\xi}} \neq s_\zeta$ for all $\zeta < \bar{\xi}$, then

$$s_{\bar{\xi}} \subseteq t \leftrightarrow \exists i < \beta \text{ ht}(\vec{C}^i) = \eta_\xi.$$

(viii) For every $i < \beta$,

(a) every successor point of every member of the range of \vec{C}^i has countable cofinality,

(b) $\text{dom}(\vec{D}^i) \cap (i + 1) = \emptyset$,

(c) $(\text{dom}(\vec{D}^i) \cup \text{range}(\vec{D}^i)) \cap S = \emptyset$,

(d) $\text{dom}(\vec{C}^i) \cap \text{range}(\vec{C}^j) = \emptyset$ for all $j < \beta$, and

(e) $\text{dom}(\vec{D}^i) \cap \text{dom}(\vec{D}^j) = \emptyset$ for all $j \neq i$.

(ix) Let $\bar{x} \in a$ be given and suppose there is a \mathcal{W} -least $z\mathcal{W}\bar{x}$ with $z \in \mathcal{C}$ such that $F(\bar{x})$ is a $\mathcal{Q} \cap (\vec{S} * \dot{P}_z)$ -name in \mathbf{V} for a club subset of κ , let $F(\bar{x})^{G_0}$ denote its partial evaluation by the \vec{S} -generic filter G_0 .¹³ Then $p \upharpoonright z$ is a condition in P_z and for every $\nu < \max(c_{\bar{x}})$, $p \upharpoonright z$ either forces $\nu \in F(\bar{x})^{G_0}$ or forces $\nu \notin F(\bar{x})^{G_0}$. Let $C_{\bar{x}}$ be the set of all $\nu < \max(c_{\bar{x}})$ such that $p \upharpoonright z \Vdash_{P_z} \nu \in F(\bar{x})^{G_0}$. Then

(a) $E_\delta \setminus C_{\bar{x}}$ is finite for every $\delta \in c_{\bar{x}} \cap S$, and

(b) $\text{ot}(C_\delta^i \cap^* C_{\bar{x}}) = \text{ht}(\vec{C}^i)$ for every $i < \beta$ and $\delta \in c_{\bar{x}} \cap \text{dom}(\vec{C}^i)$.

Given conditions $p_\epsilon = \langle t^\epsilon, \vec{e}^\epsilon, \langle \langle \vec{C}^{i,\epsilon}, \vec{D}^{i,\epsilon} \rangle \mid i < \beta^\epsilon \rangle, \langle c_{\bar{x}}^\epsilon \mid \bar{x} \in a^\epsilon \rangle \rangle$ for $\epsilon \in \{0, 1\}$, we order P_x by setting $p_1 \leq_x p_0$ iff

(i) $\beta^0 \leq \beta^1$, $t^0 \subseteq t^1$, $\vec{e}^0 \subseteq \vec{e}^1$, $a^0 \subseteq a^1$,

(ii) for all $\bar{x} \in a^0$, $c_{\bar{x}}^0 = c_{\bar{x}}^1 \cap (\beta^0 + 1)$, and

(iii) $\vec{C}^{i,0} = \vec{C}^{i,1} \upharpoonright (\beta^0 + 1)$ and $\vec{D}^{i,0} = \vec{D}^{i,1} \upharpoonright (\beta^0 + 1)$ for all $i < \beta^0$.

Note that $p_1 \leq_x p_0$ implies that $C(p_1) \supseteq C(p_0)$ and therefore $A^{p_1} \supseteq A^{p_0}$. Let \perp_x denote the incompatibility relation on P_x corresponding to \leq_x . We go down to \mathbf{V} for a moment to observe that our definitions yield the following.

Lemma 5.2. $\mathcal{Q} \subseteq H(\kappa^+)$. □

Back in \mathbf{W} , note that if $p_1 \leq p_0$ are conditions in \mathbb{P} , then $A^{p_1} \supseteq A^{p_0}$. The following is immediate by Claim 5.1 and noting (for the proof that (ix) holds for $p \upharpoonright z$) that if $y\mathcal{W}z\mathcal{W}x$ and $p \in P_x$, then $p \upharpoonright y = (p \upharpoonright z) \upharpoonright y$.

Claim 5.3. *If $x \in \mathcal{C}$, $p \in P_x$ and $z\mathcal{W}x$ with $z \in \mathcal{C}$, then $p \upharpoonright z \in P_z$. If p, q are both in P_x and $q \leq p$, then $q \upharpoonright z \leq p \upharpoonright z$.* □

¹³ $F(\bar{x})^{G_0}$ will be a P_z -name in \mathbf{W} for the same club subset of κ .

It is immediate (using Claim 5.1) that if $z\mathcal{W}x$ and $z, x \in \mathcal{C}$, then $P_z \subseteq P_x$. In fact, the following holds.

Claim 5.4. *If $z\mathcal{W}x$ and $z, x \in \mathcal{C}$, then P_z is a complete suborder of P_x .*

Proof. First note that if $p \perp_z q$, then $p \perp_x q$. To see this, assume $r \leq_x p, q$. Then $r \upharpoonright z \in P_z$ and $r \upharpoonright z \leq_z p, q$ by Claim 5.3. To see that P_z is a complete suborder of P_x , let B be a maximal antichain of P_z . Let $q \in P_x$. There is $b \in B$ which is compatible to $q \upharpoonright z$. Let $p \in P_z$ be stronger than both b and $q \upharpoonright z$. Let $c_{\bar{x}}^*$ be $c_{\bar{x}}^p$ if $\bar{x} \in a^p$ and let it be $c_{\bar{x}}^q$ if $\bar{x} \in a^q \setminus a^p$. Let

$$q^* = \langle t^p, \bar{e}^p, \langle \langle \vec{C}^{i,p}, \vec{D}^{i,p} \rangle \mid i < \beta^p \rangle, \langle c_{\bar{x}}^* \mid \bar{x} \in a^p \cup a^q \rangle \rangle.$$

To verify that B is maximal in P_x , it suffices to show that q^* is a condition in P_x extending both q and p . Given the former, the latter will be obvious considering the nature of the extension relation of P_x (which is end-extension). We will show, by induction on $t\mathcal{W}x$, that $q^* \upharpoonright t$ is a condition in P_t whenever $t \in \mathcal{C}$. For simplicity of notation, let us assume that $t = x$ and that $q^* \upharpoonright z$ is a condition in P_z for $z\mathcal{W}x$ whenever $z \in \mathcal{C}$. We want to show that q^* is a condition in P_x by showing that it satisfies conditions (i)-(ix) above. Conditions (i)-(v), (vii) and (viii) in the definition of P_x are immediate. For (vi), note that $A^{q^*} = A^q \cup A^p$. We thus have to show that

$$\forall \bar{x} \in (a^p \cup a^q) \cap (A^p \cup A^q) \forall \alpha \in c_{\bar{x}}^* \cap S [s_\alpha \subseteq \bar{x} \rightarrow t^p(\alpha) = 0].$$

If $\bar{x}\mathcal{W}z$, then $\bar{x} \in a^p \cap A^p$ and the above follows for \bar{x} from (vi) for p . Otherwise $\bar{x} \in a^q \cap A^q$ and the above follows for \bar{x} from (vi) for q .

We still need to verify (ix) - let $\bar{x} \in a^p \cup a^q$ be given. If $\bar{x} \in a^p$, then (ix) follows from (ix) for p as $c_{\bar{x}}^* = c_{\bar{x}}^p$, $q^* \upharpoonright z \in P_z$ by induction hypothesis, and $q^* \upharpoonright z \leq p$. So assume that $\bar{x} \in a^q \setminus a^p$. Then $\bar{x} \in a^q \setminus \{y \mid y\mathcal{W}z\}$. Suppose there is a \mathcal{W} -least $y\mathcal{W}\bar{x}$ with $y \in \mathcal{C}$ such that $F(\bar{x})^{G_0}$ is a P_y -name for a club subset of κ . As, by induction hypothesis, $q^* \upharpoonright y$ is a condition in P_y stronger than $q \upharpoonright y$, and as $c_{\bar{x}}^* = c_{\bar{x}}^q$, it follows that for every $\nu < \max c_{\bar{x}}^*$, $q^* \upharpoonright y$ either forces $\nu \in F(\bar{x})^{G_0}$ or forces $\nu \notin F(\bar{x})^{G_0}$. Let $C_{\bar{x}}$ be the set of all $\nu < \max(c_{\bar{x}}^*)$ such that $q^* \upharpoonright y \Vdash \nu \in F(\bar{x})^{G_0}$, which of course coincides with the set of $\nu < \max(c_{\bar{x}}^q)$ such that $q \upharpoonright y \Vdash \nu \in F(\bar{x})^{G_0}$. We have to show that

- (a) $E_\delta \setminus C_{\bar{x}}$ is finite for every $\delta \in c_{\bar{x}}^q \cap S$, and
- (b) $\text{ot}(C_\delta^{i,p} \cap^* C_{\bar{x}}) = \text{ht}(\vec{C}^{i,p})$ for every $i < \beta$ and $\delta \in c_{\bar{x}}^* \cap \text{dom}(\vec{C}^{i,p})$.

Condition (a) follows immediately from (ix) for q . For (b) fix some $i < \beta^p$ and $\delta \in c_{\bar{x}}^* \cap \text{dom}(\vec{C}^{i,p}) = c_{\bar{x}}^q \cap \text{dom}(\vec{C}^{i,p})$. It follows that $i < \delta \leq \beta^q$, as $\text{dom}(\vec{C}^{i,p}) \cap (i+1) = \emptyset$ by condition (viii). Therefore $C_\delta^{i,p} = C_\delta^{i,q}$ and thus $\text{ot}(C_\delta^{i,p} \cap^* C_{\bar{x}}) = \text{ot}(C_\delta^{i,q} \cap^* C_{\bar{x}}) = \text{ht}(\vec{C}^{i,q}) = \text{ht}(\vec{C}^{i,p})$. \square

Next we show that \mathbb{P} has the κ^+ -chain condition. In fact we show that \mathbb{P} is κ^+ -Knaster where, for a cardinal θ , a poset \mathbb{Q} is θ -Knaster if for every $\{q_\xi \mid \xi < \theta\} \subseteq \mathbb{Q}$ there is $I \subseteq \theta$ of size θ such that q_ξ and $q_{\xi'}$ are compatible conditions in \mathbb{Q} for all ξ, ξ' in I . We first need the following.

Claim 5.5. *If $x \in \mathcal{C}$ and $p \in P_x$, then $C(p) \subseteq W((\beta^p)^{\prec+\succ}) \cap \{z \mid z\mathcal{W}x\}$ and is of size less than κ .*

Proof. Assume $y \in C(p)$. We will only treat the case when y is of the form $y = \langle \alpha, 1 \rangle \frown \bar{y}$ for some $\alpha < \kappa$ and $\bar{y} \in (\kappa^\kappa)^\mathbb{V}$, i.e. y codes the fact that $\alpha \in c_{\bar{y}}^p$. But the latter implies that $\alpha \leq \beta^p$ and thus $y \in W(\beta^p + 2) \subseteq W((\beta^p)^{\prec+\succ})$, and it implies that $\bar{y}\mathcal{W}x$ and hence by the closure properties of elements of \mathcal{C} , this implies that $y\mathcal{W}x$. The case that y is of the form $y = \langle 1 + \alpha \rangle \frown \bar{0}$ is similar. That $C(p)$ is of size less than κ is obvious from its definition and the definition of conditions in P_x . \square

Lemma 5.6. *\mathbb{P} is κ^+ -Knaster.*

Proof. Let $\{p^\epsilon \mid \epsilon < \kappa^+\}$ be a set of conditions in \mathbb{P} . We want to show that there is $B \subseteq \kappa^+$ of size κ^+ such that p^ϵ and $p^{\epsilon'}$ are compatible whenever both ϵ and ϵ' are in B . Let

$$p^\epsilon = \langle t^\epsilon, \vec{e}^\epsilon, \langle \langle \vec{C}^{\epsilon,i}, \vec{D}^{\epsilon,i} \rangle \mid i < \beta^\epsilon \rangle, \langle c_{\bar{x}}^\epsilon \mid \bar{x} \in a^\epsilon \rangle \rangle.$$

By possibly strengthening the p^ϵ , we may assume that $a^\epsilon \supseteq C(p^\epsilon)$ for every $\epsilon < \kappa^+$, using Claim 5.5. This implies that if $\epsilon \neq \epsilon'$ then $A^{p^{\epsilon'}} \setminus A^{p^\epsilon} \subseteq a^{\epsilon'}$ and hence $(a^\epsilon \setminus a^{\epsilon'}) \cap (A^{p^{\epsilon'}} \setminus A^{p^\epsilon}) = \emptyset$. By a Δ -system argument using $2^{<\kappa} = \kappa$, we may assume that there are $\beta, t, \vec{e} = \langle E_\delta \mid \delta \in S \cap (\beta + 1) \rangle, a, \langle \langle \vec{C}^i, \vec{D}^i \rangle \mid i < \beta \rangle$ and $\langle c_{\bar{x}} \mid \bar{x} \in a \rangle$ such that for all distinct $\epsilon, \epsilon' < \kappa^+$,

- (i) $t^\epsilon = t, \vec{e}^\epsilon = \vec{e}, \langle \langle \vec{C}^{\epsilon,i}, \vec{D}^{\epsilon,i} \rangle \mid i < \beta^\epsilon \rangle = \langle \langle \vec{C}^i, \vec{D}^i \rangle \mid i < \beta \rangle,$
- (ii) $a^\epsilon \cap a^{\epsilon'} = a,$ and
- (iii) $c_{\bar{x}}^\epsilon = c_{\bar{x}}$ for all $\bar{x} \in a.$

We claim that any two such conditions p^ϵ and $p^{\epsilon'}$ are compatible, as

$$p^{\epsilon, \epsilon'} = \langle t, \vec{e}, \langle \langle \vec{C}^i, \vec{D}^i \rangle \mid i < \beta \rangle, \langle c_{\bar{x}}^\epsilon \mid \bar{x} \in \{a^\epsilon, a^{\epsilon'}\}, \bar{x} \in a^\epsilon \rangle \rangle$$

is a condition in \mathbb{P} stronger than both. It suffices to show, by induction along \mathcal{W} , that $p^{\epsilon, \epsilon'} \upharpoonright x$ is a condition in P_x whenever $x \in \mathcal{C}$. Thus assume that $x \in \mathcal{C}$ and inductively that $p^{\epsilon, \epsilon'} \upharpoonright z$ is a condition in P_z whenever $z\mathcal{W}x$ and $z \in \mathcal{C}$. We want to show that $p^{\epsilon, \epsilon'} \upharpoonright x$ is a condition in P_x . As in the proof of Claim 5.4, conditions (i)-(v), (vii) and (viii) are immediate. For (vi), by symmetry it suffices to show that

$$\forall \bar{x} \in a^\epsilon \cap (A^{p^\epsilon} \cup A^{p^{\epsilon'}}) \forall \alpha \in c_{\bar{x}}^\epsilon \cap S [s_\alpha \subseteq \bar{x} \rightarrow t(\alpha) = 0].$$

Now this follows from (vi) for p^ϵ in case $\bar{x} \in a^\epsilon \cap A^{p^\epsilon}$ or from (vi) for $p^{\epsilon'}$ if $\bar{x} \in a^{\epsilon'} \cap A^{p^{\epsilon'}}$ and thus we may assume that $\bar{x} \in (a^\epsilon \setminus a^{\epsilon'}) \cap (A^{p^{\epsilon'}} \setminus A^{p^\epsilon})$. But the latter set is empty by our above assumption.

We are left with proving that (ix) holds for $p^{\epsilon, \epsilon'} \upharpoonright x$. Given $\bar{x} \in a^\epsilon \cup a^{\epsilon'}$, we may assume (by symmetry) that $\bar{x} \in a^\epsilon$. Suppose there is a \mathcal{W} -least $z\mathcal{W}\bar{x}$ such that $z \in \mathcal{C}$ and $F(\bar{x})^{G_0}$ is a P_z -name for a club subset of κ . As, by induction, $p^{\epsilon, \epsilon'} \upharpoonright z$ is a condition in P_z stronger than $p^\epsilon \upharpoonright z$, it follows that for every $\nu < \max(c_{\bar{x}}^\epsilon)$, $p^{\epsilon, \epsilon'} \upharpoonright z$ either forces $\nu \in F(\bar{x})^{G_0}$ or forces $\nu \notin F(\bar{x})^{G_0}$. Let $C_{\bar{x}}$ be the set of all $\nu < \max(c_{\bar{x}}^\epsilon)$ such that $p^{\epsilon, \epsilon'} \upharpoonright z \Vdash \nu \in F(\bar{x})^{G_0}$. We have to show that

- (a) $E_\delta \setminus C_{\bar{x}}$ is finite for every $\delta \in c_{\bar{x}}^\varepsilon \cap S$, and
- (b) $\text{ot}(C_\delta^{i,p} \cap^* C_{\bar{x}}) = \text{ht}(\vec{C}^{i,p})$ for every $i < \beta$ and $\delta \in c_{\bar{x}}^\varepsilon \cap \text{dom}(\vec{C}^{i,p})$.

But this is immediate from (ix) for p^ε . \square

Let $\beta_{\mathbb{P}}$ be the function with domain \mathbb{P} mapping a condition p to β^p .

Lemma 5.7. \mathbb{P} is $<\kappa$ -closed relative to $\beta_{\mathbb{P}}$ outside S .

Proof. Given $\gamma < \kappa$ and a decreasing sequence of conditions $\langle p^k \mid k < \gamma \rangle$ in \mathbb{P} with

$$p^k = \langle t^k, \vec{e}^k, \langle \langle \vec{C}^{k,i}, \vec{D}^{k,i} \rangle \mid i < \beta^k \rangle, \langle c_{\bar{x}}^k \mid \bar{x} \in a^k \rangle \rangle,$$

let $\beta := \bigcup_{k < \gamma} \beta^k$, $t = \bigcup_{k < \gamma} t^k$, $\vec{e} = \bigcup_{k < \gamma} \vec{e}^k$, $\vec{C}^i = \bigcup_{k < \gamma} \vec{C}^{k,i}$ and $\vec{D}^i = \bigcup_{k < \gamma} \vec{D}^{k,i}$ for every $i < \beta$, $a = \bigcup_{k < \gamma} a^k$ and $c_{\bar{x}} = \bigcup \{c_{\bar{x}}^k \mid k < \gamma, \bar{x} \in a^k\}$ for every $\bar{x} \in a$. If there is $\bar{\gamma} < \gamma$ such that β^k is the same for all $k \geq \bar{\gamma}$, then $\langle t, \vec{e}, \langle \langle \vec{C}^i, \vec{D}^i \rangle \mid i < \beta \rangle, \langle c_{\bar{x}} \mid \bar{x} \in a \rangle \rangle$ is a condition stronger than each p^k . Otherwise, we may assume that $\beta \notin S$ and let p be defined by setting $\beta^p = \beta$, $t^p = t \cup (\beta, 0)$, $\vec{e}^p = \vec{e}$, $\vec{C}^{i,p} = \vec{C}^i$ and $\vec{D}^{i,p} = \vec{D}^i$ for every $i < \beta$, $a^p = a$ and $c_{\bar{x}}^p = c_{\bar{x}} \cup \{\text{sup}(c_{\bar{x}})\}$ for every $\bar{x} \in a$.

We claim that p is a condition in \mathbb{P} - if this holds true, p will obviously be stronger than each p^k . We will show by induction along \mathcal{W} that for every $x \in \mathcal{C}$, $p \upharpoonright x \in P_x$. Thus assume $x \in \mathcal{C}$ and for every $y \mathcal{W} x$ with $y \in \mathcal{C}$, $p \upharpoonright y \in P_y$. We want to check that conditions (i)-(ix) in the definition of P_x hold for $p \upharpoonright x$ and thus $p \upharpoonright x \in P_x$. Conditions (i), (ii), (v), (vii) and (viii) are immediate. Condition (iii) holds since $\beta \notin S$. Using that (iv) holds for the p^k and that $\beta^p > (\beta^k)^{<+\gamma}$ for every $k < \gamma$, we obtain

$$(*) \quad a^p \subseteq W(\beta^p)$$

and thus (iv) holds for p . For (vi), we have to check that

$$\forall \bar{x} \in a^p \cap A^p \forall \alpha \in c_{\bar{x}}^p \cap S [s_\alpha \subseteq \bar{x} \rightarrow t^p(\alpha) = 0.]$$

If $\bar{x} \in \bigcup_{k < \gamma} A^{p^k}$, this is immediate from (vi) for p^k for some sufficiently large $k < \gamma$ if $\alpha < \beta^p$ and, if $\alpha = \beta^p$, because we set $t^p(\beta^p) = 0$. If $\bar{x} \in A^p \setminus \bigcup_{k < \gamma} A^{p^k}$, it is easily checked using the definition of $C(p)$ that $f(\bar{x})$ is of the form $\kappa \cdot \delta + \xi$ for some $\delta < \lambda$ and $\xi \geq \beta^p$. But by (*) this means that $\bar{x} \notin a^p$ and therefore this case is vacuous.

It remains to show that (ix) holds for $p \upharpoonright x$. Let $\bar{x} \in a^p$ be given and suppose there is a \mathcal{W} -least $z \mathcal{W} \bar{x}$ with $z \in \mathcal{C}$ such that $F(\bar{x})^{G_0}$ is a P_z -name for a club subset of κ . As $p \upharpoonright z$ is a condition in P_z by induction hypothesis and $p \upharpoonright z \leq p^k \upharpoonright z$ for every $k < \gamma$, we have that for every $\nu < \max(c_{\bar{x}}^p)$, $p \upharpoonright z$ either forces $\nu \in F(\bar{x})^{G_0}$ or forces $\nu \notin F(\bar{x})^{G_0}$. Let $C_{\bar{x}}$ be the set of all $\nu < \max(c_{\bar{x}}^p)$ such that $p \upharpoonright z \Vdash \nu \in F(\bar{x})^{G_0}$. It remains to show that

- (a) $E_\delta \setminus C_{\bar{x}}$ is finite for every $\delta \in c_{\bar{x}} \cap S$, and
- (b) $\text{ot}(C_\delta^i \cap^* C_{\bar{x}}) = \text{ht}(\vec{C}^i)$ for every $i < \beta$ and $\delta \in c_{\bar{x}} \cap \text{dom}(\vec{C}^i)$.

Condition (a) holds since, as $\beta \notin S$, every $\delta \in c_{\bar{x}} \cap S$ is such that $\delta \in c_{\bar{x}}^k$ for some k . For condition (b), fix some $i < \beta^p$ and $\delta \in c_{\bar{x}} \cap \text{dom}(\vec{C}^i)$. Let $k < \gamma$ be such that $\bar{x} \in a^k$, $i < \beta^k$ and $\delta \in c_{\bar{x}}^k \cap \text{dom}(\vec{C}^{k,i})$. But then $\text{ot}(C_{\delta}^i \cap^* C_{\bar{x}}) = \text{ot}(C_{\delta}^{k,i} \cap^* C_{\bar{x}}) = \text{ht}(\vec{C}^i) = \text{ht}(\vec{C}^{i,p})$, where the middle equation holds as p^k is a condition in \mathbb{P} . \square

Note that the lower bound obtained in Lemma 5.7 is canonical and can easily be defined from the descending sequence of conditions (cf. the paragraph after Lemma 4.3).

Lemma 5.8. *Let $p = \langle t, \vec{e}, \langle \langle \vec{C}^i, \vec{D}^i \rangle \mid i < \beta \rangle, \langle c_{\bar{x}} \mid \bar{x} \in a \rangle \rangle$ be a condition in \mathbb{P} . Then $\forall \beta' < \kappa \exists p' \leq p \left[\beta^{p'} > \beta' \text{ and } \langle c_{\bar{x}}^{p'} \mid \bar{x} \in a^{p'} \rangle = \langle c_{\bar{x}} \mid \bar{x} \in a \rangle \right]$.*

Proof. Pick $\beta^* \in C^{\prec+\succ}$ such that $\beta^* > \beta'$, $\beta^* > \beta$, $\beta^* = \eta_{\beta^*}$ and such that for every $\gamma < \beta^*$, $\beta^* \setminus S$ contains a closed subset of order-type $\gamma + 1$. The latter is easily possible for S was added generically. We construct

$$p' = \langle t^*, \vec{e}^*, \langle \langle \vec{C}_*^i, \vec{D}_*^i \rangle \mid i < \beta^* \rangle, \langle c_{\bar{x}} \mid \bar{x} \in a \rangle \rangle \leq p$$

as follows. First we choose t^* of length $\beta^* + 1$ extending t and such that $t^* \upharpoonright [\beta + 1, \beta^*] = \vec{0}$. Let $\vec{e}^* = \langle E_{\xi}^* \mid \xi \in S \cap (\beta^* + 1) \rangle$ be any ladder system on $(\beta^* + 1) \cap S$ extending \vec{e} . Let $\langle \vec{C}_*^i, \vec{D}_*^i \rangle = \langle \vec{C}^i, \vec{D}^i \rangle$ if $i < \beta$ and for $i \in [\beta, \beta^*]$, let $\vec{C}_*^i = \{ \langle \sup(X_i), X_i \rangle \}$ and $\vec{D}_*^i = \{ \langle \gamma, X_i \cap \gamma \rangle \mid \gamma \text{ a limit point of } X_i \}$, where $X_i \subseteq \beta^* \setminus (\beta^{\prec+\succ} \cup (i+1))$ has order-type η_{ρ_i} , has all its non-accumulation points of cofinality ω , and is closed in $\sup(X_i)$, for $\langle \rho_i \mid i \in [\beta, \beta^*] \rangle$ as determined by S , s^* and t^* (up to permutation of the indices) via condition (vii) in the definition of \mathbb{P} . We also make sure that for $i \neq i'$, $(X_i \cup \{ \sup(X_i) \}) \cap (X_{i'} \cup \{ \sup(X_{i'}) \}) = \emptyset$ and $(X_i \cup \{ \sup(X_i) \}) \cap S = \emptyset$.¹⁴ We have to check that p' is a condition in \mathbb{P} . It is then obvious that p' is as desired. Conditions (i)-(v) in the definition of \mathbb{P} are immediate, (vii) and (viii) are ensured by our above choice of \vec{C}_*^i and \vec{D}_*^i for $i < \beta^*$, and (ix) is shown as usual.

Finally, condition (vi) in the definition of \mathbb{P} follows if we can show that $a \cap A^{p'} = a \cap A^p$, since (vi) holds for p . To show this is the case, assume $x \in a \cap A^{p'}$. Now if $x \in F^*$ or $x \in \mathcal{W}^*$ then trivially $x \in A^p$. Thus assume $x \in C(p')$. Assume first that there is $\alpha < \kappa$ such that $x = \langle 1 + \alpha \rangle \cap \vec{0}$. We distinguish several cases.

- If $\alpha = 2 \cdot \gamma$, $x \in A^{p'}$ codes the fact that $t^*(\gamma) = 1$. But having set $t^* \upharpoonright [\beta + 1, \beta^*] = \vec{0}$, this means that $\gamma \leq \beta$ and thus $x \in A^p$.
- If $\alpha = 6 \cdot \gamma + 1$ and $\gamma = \prec \gamma_0, \gamma_1 \succ$, $x \in A^{p'}$ codes the fact that $\gamma_0 \in E_{\gamma_1}^*$. If $\gamma_1 \leq \beta$, $x \in A^p$ as in the preceding case. $\gamma_1 > \beta$ implies that $\gamma_1 \geq \beta^{\prec+\succ}$ and thus $\alpha > \beta^{\prec+\succ}$. But then by condition (iv) for p , x could not have been an element of a .
- If $\alpha = 6 \cdot \gamma + 3$ and $\gamma = \prec \gamma_0, \gamma_1, i \succ$, $x \in A^{p'}$ codes the fact that $\gamma_0 \in C_*^{i, \gamma_1}$. If $i < \beta$, $x \in A^p$ as in the preceding cases. If $i \geq \beta$, by our choice of C_*^i it follows that γ_0 and γ_1 are both $\geq \beta^{\prec+\succ}$, which in turn implies that $\alpha > \beta^{\prec+\succ}$ and again by condition (iv) for p , x thus could not have been an element of a .

¹⁴This is easy to arrange by our requirements on β^* .

- The case that $\alpha = 6 \cdot \gamma + 5$ is handled just like the previous case.

Now assume that there is $\alpha < \kappa$ and $\bar{x} \in {}^\kappa \kappa$ such that $x = \langle \alpha, 1 \rangle \frown \bar{x}$. Then $x \in A^{p'}$ codes the fact that $\alpha \in c_{\bar{x}}$. But of course then $x \in A^p$. \square

Let us go down to \mathbf{V} for a moment. The following lemma follows from the proof of Lemma 5.7 and from Lemma 5.8.

Lemma 5.9. \mathcal{Q} has a dense subset which is $< \kappa$ -directed closed in \mathbf{V} .

Proof. We will only verify that \mathcal{Q} has a dense subset which is $< \kappa$ -closed in \mathbf{V} ; directed closure follows by essentially the same argument and will only be needed for large cardinal preservation arguments in Section 7. Let

$$\bar{\mathcal{Q}} = \{ \langle \langle s, \sigma \rangle, p \rangle \in \mathcal{Q} \mid \text{dom}(s) = \beta^p + 1 \}.$$

It is straightforward to see from the definition of \mathbb{P} that whenever $\langle \langle s, \sigma \rangle, p \rangle \in \mathcal{Q}$ and $\text{dom}(s) > \beta^p + 1$, then in fact $\langle \langle s \upharpoonright (\beta^p + 1), \sigma \upharpoonright (\beta^p + 1) \rangle, p \rangle \in \mathcal{Q}$. This together with Lemma 5.8 implies that $\bar{\mathcal{Q}}$ is a dense subset of \mathcal{Q} . Now assume $\langle p^k \mid k < \gamma \rangle$ is a decreasing sequence of conditions in $\bar{\mathcal{Q}}$ for some $\gamma < \kappa$ with

$$p^k = \left\langle \langle s^k, \sigma^k \rangle, \langle t^k, \bar{e}^k, \langle \langle \bar{C}^{k,i}, \bar{D}^{k,i} \rangle \mid i < \beta^k \rangle, \langle c_{\bar{x}}^k \mid \bar{x} \in a^k \rangle \right\rangle.$$

If $\langle \beta^{p^k} \mid k < \gamma \rangle$ is eventually constant we can obtain a lower bound of $\langle p^k \mid k < \gamma \rangle$ as in the first part of the proof of Lemma 5.7. Otherwise, let $\beta := \bigcup_{k < \gamma} \beta^k$, let $s := \{(\beta, 0)\} \cup \bigcup_{k < \gamma} s^k$ and let $\sigma := \bigcup_{k < \gamma} \sigma^k$. Define $t, \bar{e}, \langle \langle \bar{C}^i, \bar{D}^i \rangle \mid i < \beta \rangle, a$ and $\langle c_{\bar{x}} \mid \bar{x} \in a \rangle$ as in the proof of Lemma 5.7. Since $\langle s, \sigma \rangle$ forces (in \bar{S}) that $\beta \notin S$,

$$p = \left\langle \langle s, \sigma \rangle, \langle t, \bar{e}, \langle \langle \bar{C}^i, \bar{D}^i \rangle \mid i < \beta \rangle, \langle c_{\bar{x}} \mid \bar{x} \in a \rangle \right\rangle$$

is seen to be a condition in \mathcal{Q} (and thus in $\bar{\mathcal{Q}}$) as in the proof of Lemma 5.7. \square

Note again that the lower bound obtained above can canonically be defined from the given decreasing sequence of conditions.

From Lemma 5.9 we immediately obtain the following corollary, which will be used repeatedly.

Corollary 5.10. \mathcal{Q} is $< \kappa$ -distributive in \mathbf{V} and $\dot{\mathbb{P}}$ is $< \kappa$ -distributive in $\mathbf{V}^{\bar{S}}$.

Let us go back to \mathbf{W} now. The following is another corollary of Lemma 5.9.

Corollary 5.11. For every $p \in \mathbb{P}$, every collection \mathcal{X} of size $< \kappa$ of \mathbb{P} -names for unbounded subsets of κ and every $\beta < \kappa$ there is $p' \in \mathbb{P}$ stronger than p such that for every $\dot{X} \in \mathcal{X}$ there is some $\gamma > \beta$ such that p' forces $\gamma \in \dot{X}$. \square

Lemma 5.12. Let $p = \langle t, \bar{e}, \langle \langle \bar{C}^i, \bar{D}^i \rangle \mid i < \beta \rangle, \langle c_{\bar{x}} \mid \bar{x} \in a \rangle \rangle \in \mathbb{P}$.

$$(i) \quad \forall \bar{x} \in ({}^\kappa \kappa)^{\mathbf{V}} \setminus \{\bar{0}\} \exists p' \leq p \quad \bar{x} \in a^{p'}.$$

$$(ii) \quad \forall \bar{x} \in a \forall \nu < \kappa \exists p' \leq p \quad \left[\nu < \max(c_{\bar{x}}^{p'}) \text{ and } c_{\bar{x}}^{p'} \cap \nu = c_{\bar{x}} \cap \nu \right].$$

Proof. For (i), let $\bar{x} \in (\kappa^\kappa)^\mathbf{V} \setminus (a \cup \{\vec{0}\})$ be given. By Lemma 5.8, we may assume $\bar{x} \in W(\beta^{<+\gamma})$. Let c_z^* be equal to c_z for $z \in a$ and let $c_{\bar{x}}^* = \emptyset$. If we set

$$p' = \langle t, \vec{e}, \langle \langle \vec{C}^i, \vec{D}^i \rangle \mid i < \beta \rangle, \langle c_z^* \mid z \in a \cup \{\bar{x}\} \rangle \rangle,$$

then p' is easily seen to be a condition in \mathbb{P} for $A^{p'} = A^p$, and p' is as desired.

For (ii), let $\bar{x} \in a$ be given. Using Lemma 5.8, we may assume that $\beta > \nu$. Pick a countable elementary substructure N of some $H(\theta)$ containing \mathbb{P} , p , \bar{x} , F and G_0 with θ sufficiently large, and such that $\nu' := \sup(N \cap \kappa) \notin S$. We build a decreasing $\langle N, \mathbb{P} \rangle$ -generic sequence of conditions $\langle p_n \mid n \in \omega \rangle$ with $p_0 = p$. Note that $\langle \sup(c_{\bar{x}}^{p_n}) \mid n < \omega \rangle$ is either eventually constant or has supremum ν' by genericity, Lemma 5.8 and clause (ii) in the definition of the extension relation of \mathbb{P} . We build a condition q extending all p_n as in the proof of Lemma 5.7, except that we set $c_{\bar{x}}^q = \bigcup_{n \in \omega} c_{\bar{x}}^{p_n} \cup \{\nu'\}$, which is a closed subset of κ by the above. To argue that q is a condition, we only need to show that if $z \mathcal{W} \bar{x}$ is such that $z \in \mathcal{C}$ and z is \mathcal{W} -least such that $F(\bar{x})^{G_0}$ is a P_z -name for a club subset of κ and $\nu'' < \nu'$, then there is $n < \omega$ such that $p_n \upharpoonright z$ decides whether or not $\nu'' \in F(\bar{x})^{G_0}$. $\nu'' < \nu^*$ for some $\nu^* \in N \cap \kappa$ and, by Claim 5.4 together with Corollary 5.10, there is a dense set $D \in N$ of conditions in P_z deciding $F(\bar{x})^{G_0} \cap \nu^*$. It then follows from the genericity of $\langle p_n \mid n < \omega \rangle$ that there is some n such that $p_n \upharpoonright z$ decides for every $\xi \in \nu^*$ whether or not $\xi \in F(\bar{x})^{G_0}$. But of course ν'' is one such ξ . \square

Now let G be \mathcal{Q} -generic over \mathbf{V} , $G = G_0 * G_1$, where G_1 is $\mathbb{P}(S, \vec{s})$ -generic over $\mathbf{W} := \mathbf{V}[G_0]$. Work in $\mathbf{V}[G]$. Let $t^G = \bigcup_{p \in G_1} t^p$ and $\vec{E}^G = \bigcup_{p \in G_1} \vec{e}^p$. For each $i < \kappa$, let $\vec{C}^{i,G} = \bigcup \{ \vec{C}^{i,p} \mid p \in G_1, i < \beta^p \}$ and $\vec{D}^{i,G} = \bigcup \{ \vec{D}^{i,p} \mid p \in G_1, i < \beta^p \}$. By the definition of \mathcal{Q} and its extension relation and by Lemma 5.8, \vec{E}^G is a ladder system defined on all of S and each $\vec{C}^{i,G}$ is a coherent club-sequence (which is witnessed by $\vec{D}^{i,G}$) with nonempty domain disjoint from S and such that every successor point of every member of $\text{range}(\vec{C}^{i,G})$ has countable cofinality.

Write \vec{E}^G as $\vec{E}^G = \langle E_\delta \mid \delta \in S \rangle$ and let $c_{\bar{x}}^G = \bigcup \{ c_{\bar{x}}^p \mid p \in G_1, \bar{x} \in a^p \}$ for all $\bar{x} \in (\kappa^\kappa)^\mathbf{V} \setminus \{\vec{0}\}$. By Lemma 5.12, each $c_{\bar{x}}^G$ is a club subset of κ in $\mathbf{V}[G]$. Also, by condition (ix) (a) in the definition of \mathbb{P} , for every $\delta \in c_{\bar{x}}^G \cap S$, if there is a \mathcal{W} -least $z \mathcal{W} \bar{x}$ with $z \in \mathcal{C}$ such that $F(\bar{x})^{G_0}$ is a P_z -name \dot{C} for a club subset of κ and C is the G_1 -interpretation of \dot{C} , then $E_\delta^G \setminus C$ is finite. Let $A^G = \bigcup_{p \in G_1} A^p$. t^G will have a canonical \mathcal{Q} -name in \mathbf{V} which we will denote by \dot{t} . The partial evaluation of \dot{t} by G_0 will be denoted by \dot{t}^{G_0} and is a \mathbb{P} -name for t^G in \mathbf{W} . We will do the same for the other objects defined in $\mathbf{V}[G]$ above.

Lemma 5.13. *In $\mathbf{V}[G]$, S is stationary.*

Proof. We go back to working in $\mathbf{V}[G_0] = \mathbf{W}$. Let $p \in \mathbb{P}$ and let \dot{C} be a \mathbb{P} -name for a club subset of κ . We want to find an extension p^* of p and some $\gamma \in S$ such that $p^* \Vdash_{\mathbb{P}} \gamma \in \dot{C}$. For this, let $\langle N_n \mid n < \omega \rangle$ be an \in -chain of countable elementary substructures of some large enough $H(\theta)$ containing \mathbb{P} , \dot{C} , p , F and G_0 such that $\gamma_n := \sup(N_n \cap \kappa) \notin S$ for all n and $\gamma := \sup_n \gamma_n \in S$. This can be done using Lemma 4.4. Let $\langle p_n \mid n < \omega \rangle$ be a decreasing sequence of conditions in \mathbb{P} extending p such that for all n , $p_n \in N_{n+1}$ is a lower bound of a decreasing

$\langle N_n, \mathbb{P} \rangle$ -generic sequence of conditions in N_n extending p and extending p_{n-1} if $n > 0$. By Lemma 5.7, these lower bounds exist. For each n , p_{n+1} forces

- (i) $\gamma_n \in \dot{C}$ and
- (ii) $\gamma_n \in F(\bar{x})^{G_0}$ for all $\bar{x} \in a^{p_n}$ for which there is some $z \mathcal{W} \bar{x}$ such that $z \in \mathcal{C}$ and $F(\bar{x})^{G_0}$ is a P_z -name for a club subset of κ .

This is true by Corollary 5.11 since p_{n+1} is a lower bound of an $\langle N_n, \mathbb{P} \rangle$ -generic sequence. Let

- $\beta = \bigcup_{n < \omega} \beta^{p_n} = \gamma$,
- $t = (\bigcup_{n < \omega} t^{p_n}) \cup \{ \langle \gamma, 0 \rangle \}$,
- $\vec{e} = (\bigcup_{n < \omega} \vec{e}^{p_n}) \cup \{ \langle \gamma, \{ \gamma_n \mid n < \omega \} \rangle \}$,
- $\vec{C}^i = \bigcup \{ \vec{C}^{p_n, i} \mid n < \omega, i < \beta^{p_n} \}$ and $\vec{D}^i = \bigcup \{ \vec{D}^{p_n, i} \mid n < \omega, i < \beta^{p_n} \}$ for every $i < \beta$,
- $a = \bigcup_{n < \omega} a^{p_n}$, and
- $c_{\bar{x}} = \{ \gamma \} \cup \bigcup \{ c_{\bar{x}}^{p_n} \mid n < \omega, \bar{x} \in a^{p_n} \}$ for every $\bar{x} \in a$.

It is easy to check that $p^* = \langle t, \vec{e}, \langle \langle \vec{C}^i, \vec{D}^i \rangle \mid i < \beta \rangle, \langle c_{\bar{x}} \mid \bar{x} \in a \rangle \rangle$ is a condition extending p and forcing that $\gamma \in \dot{C}$ by condition (i) above. The proof of this is by induction on \mathcal{W} as usual, i.e., one proves by induction on x that $p^* \upharpoonright x$ is in P_x whenever $x \in \mathcal{C}$. The only nontrivial point is the verification of condition (ix) in the definition of P_x . But (ix) follows now thanks to (ii) above. \square

One could have verified the preceding lemma arguing in \mathbf{V} instead of \mathbf{W} and thus avoiding use of Lemma 4.4. We will perform this kind of argument in the proof of Claim 5.15 below. The following lemma is now easy.

Lemma 5.14. *In $\mathbf{V}[G]$, \vec{E}^G is a strongly guessing ladder system defined on the stationary set S .*

Proof. We have just seen that S is stationary. To see that \vec{E}^G is strongly guessing, let $\dot{C} \in \mathbf{W}$ be a nice \mathbb{P} -name for a club subset of κ . By the κ^+ -c.c. of \mathbb{P} together with $\text{cof}(\lambda) > \kappa$, we know that there is some $z \mathcal{W} \vec{0}$ in \mathcal{C} such that \dot{C} is in fact a P_z -name for a club subset of κ , and of course we also have $\dot{C} \in H(\kappa^+)$. Suppose also that z is \mathcal{W} -minimal with the above property. Since F is a book-keeping function, we can find $\bar{x} \neq z$ such that $z \mathcal{W} \bar{x}$ and $F(\bar{x})^{G_0} = \dot{C}$. Let $C = \dot{C}^{G_1}$. By the paragraph just before Lemma 5.13, for every δ in the intersection of the club $c_{\bar{x}}^G$ with S , $E_\delta^G \setminus C$ is finite, which finishes the proof. \square

Claim 5.15. *A^G is definable over $H(\kappa^+)^{\mathbf{V}[G]}$ using S , \vec{s} and t^G as parameters.*

Proof. We claim that in $\mathbf{V}[G]$,

$$A^G = \{ y \in \kappa \setminus \{ \vec{0} \} \mid \exists C \subseteq \kappa \text{ club } \forall \alpha \in C \cap S [\vec{s}(\alpha) \subseteq y \rightarrow t^G(\alpha) = 0] \}.$$

Now if $y \in A^G$, c_y^G is a club subset of κ and witnesses that y is an element of the set on the right hand side of the above equation. For the other direction, we need to combine the proofs of Theorem 3.5 and Lemma 5.13. Work in \mathbf{V} .

Similar to the proof of Theorem 3.5, pick a \mathcal{Q} -name \dot{y} for an element of ${}^\kappa\kappa \setminus \{\vec{0}\}$ and a \mathcal{Q} -name \dot{C} for a club subset of κ and assume, towards a contradiction, that there is $p \in \mathcal{Q}$ with

$$p \Vdash \dot{y} \notin \dot{A} \wedge \forall \alpha \in \dot{C} \cap \dot{S} [\dot{s}(\alpha) \subseteq \dot{y} \rightarrow \dot{t}(\alpha) = 0]. \quad (3)$$

Let $\langle N_n \mid n < \omega \rangle$ be an \in -chain of countable elementary substructures of some large enough $H(\theta)$ containing \mathcal{Q} , \dot{y} , \dot{C} , F and p . Let $N = \bigcup_{n < \omega} N_n$, let $\gamma_n = \sup(N_n \cap \kappa)$ for every $n < \omega$ and let $\gamma = \sup(N \cap \kappa) = \bigcup_{n < \omega} \gamma_n$. From Lemma 5.9 and its proof, we can obtain a sequence $\langle p_n \mid n < \omega \rangle$ of conditions in \mathcal{Q} below p such that for all n , $p_n \in N_{n+1}$ is a lower bound of a decreasing (N_n, \mathcal{Q}) -generic sequence of conditions in N_n extending p_{n-1} if $n > 0$, such that $s^{p_n}(\gamma_n) = 0$. As in the proof of Theorem 3.5, there is $u: \gamma \rightarrow \kappa$ such that for every $n < \omega$, p_{n+1} forces

- $\dot{y} \upharpoonright \gamma_n = u \upharpoonright \gamma_n$ and such that
- $x \upharpoonright \gamma_n \neq u \upharpoonright \gamma_n$ for all $x \in a_{p_n}$.

Moreover, as in the proof of Lemma 5.13, p_{n+1} forces

- $\gamma_n \in \dot{C}$ and
- $\gamma_n \in F(\bar{x})$ for all $\bar{x} \in a^{p_n}$ for which there is some $z \mathcal{W} \bar{x}$ such that $z \in \mathcal{C}$ and $F(\bar{x})$ is a $\mathcal{Q} \cap (\vec{S} * \dot{P}_z)$ -name for a club subset of κ .

Now we define

- $s = \{\langle \gamma, 1 \rangle\} \cup \bigcup_{n < \omega} s^n$.
- $\sigma = \{\langle \gamma, u \rangle\} \cup \bigcup_{n < \omega} \sigma^{p_n}$.
- $\beta = \bigcup_{n < \omega} \beta^{p_n} = \gamma$.
- $t = \{\langle \gamma, 1 \rangle\} \cup \bigcup_{n < \omega} t^{p_n}$.
- $\vec{e} = \{\langle \gamma, \{\gamma_n \mid n < \omega\} \rangle\} \cup \bigcup_{n < \omega} \vec{e}^{p_n}$.
- $\vec{C}^i = \bigcup_{n < \omega} \vec{C}^{p_n, i}$ and $\vec{D}^i = \bigcup_{n < \omega} \vec{D}^{p_n, i}$ for every $i < \beta$.
- $a = \bigcup_{n < \omega} a^{p_n}$.
- $c_x = \{\gamma\} \cup \bigcup_{n < \omega} c_x^{p_n}$ for all $x \in a$.

Then $q = \langle \langle s, \sigma \rangle, \langle t, \vec{e} \rangle, \langle \langle \vec{C}^i, \vec{D}^i \rangle \mid i < \beta \rangle, \langle c_x \mid \bar{x} \in a \rangle \rangle$ is a condition in \mathcal{Q} , because $u \not\subseteq x$ for all $x \in a$ and the other requirements on q can be verified as in the proof of Lemma 5.13. But $q \leq p$ and

$$q \Vdash \check{\gamma} \in \dot{C} \cap \dot{S} \wedge \dot{s}(\check{\gamma}) \subseteq \dot{y} \wedge \dot{t}(\check{\gamma}) = 1,$$

contradicting (3). □

Claim 5.16. G is definable over $H(\kappa^+)^{\mathbf{V}[G]}$ using S , \vec{s} and t^G as parameters.

Proof. By Claim 5.15, A^G is definable over $H(\kappa^+)^{\mathbf{V}[G]}$ using S , \vec{s} and t^G . As a first step, we show that using A^G as a predicate, we can define t^G , \vec{E}^G , $\vec{C}^{i,G}$ and $\vec{D}^{i,G}$ for every $i < \kappa$ and $c_{\vec{x}}^G$ for every $\vec{x} \in (\kappa^\kappa)^{\mathbf{V}} \setminus \{\vec{0}\}$ over $H(\kappa^+)^{\mathbf{V}[G]}$, for the following is ensured by forcing with \mathcal{Q} .

- $t^G = \{\gamma < \kappa \mid \langle 1 + 2 \cdot \gamma \rangle \frown \vec{0} \in A^G\}$.
- $\gamma_0 \in E_{\gamma_1}^G$ iff $\langle 1 + 6 \cdot \prec \gamma_0, \gamma_1 \succ + 1 \rangle \frown \vec{0} \in A^G$.
- $\gamma_0 \in C_{\gamma_1}^{G,i}$ iff $\langle 1 + 6 \cdot \prec \gamma_0, \gamma_1, i \succ + 3 \rangle \frown \vec{0} \in A^G$.
- $\gamma_0 \in D_{\gamma_1}^{G,i}$ iff $\langle 1 + 6 \cdot \prec \gamma_0, \gamma_1, i \succ + 5 \rangle \frown \vec{0} \in A^G$.
- $\alpha \in c_{\vec{x}}^G$ iff $\langle \alpha, 1 \rangle \frown \vec{x} \in A^G$.

Furthermore we can define F^* and \mathcal{W}^* , and thus F and \mathcal{W} over $H(\kappa^+)^{\mathbf{V}[G]}$, for

- $\vec{x} \in F^*$ iff $\langle 0, 2 \rangle \frown \vec{x} \in A^G$ and
- $\vec{x} \in \mathcal{W}^*$ iff $\langle 0, 3 \rangle \frown \vec{x} \in A^G$.

This allows us to define $H(\kappa^+)^{\mathbf{V}} = \text{dom}(\mathcal{W}) \cup \text{range}(\mathcal{W})$ over $H(\kappa^+)^{\mathbf{V}[G]}$. Thus by the definition of \mathcal{Q} , it is straightforward to see that \mathcal{Q} is definable over $H(\kappa^+)^{\mathbf{V}[G]}$ using the above parameters. Now assume that

$$p = \left\langle \langle s, \sigma \rangle, \langle t, \vec{e}, \langle \langle \vec{C}^i, \vec{D}^i \rangle \mid i < \beta \rangle, \langle c_{\vec{x}} \mid \vec{x} \in a \rangle \right\rangle$$

is a condition in \mathcal{Q} . $p \in G$ iff $s = S \upharpoonright \gamma$ and $\sigma = \vec{s} \upharpoonright \gamma$ for some $\gamma < \kappa$, $\beta < \kappa$, $t = t^G \upharpoonright (\beta + 1)$, $\vec{e} = \vec{E}^G \upharpoonright (\beta + 1)$, $\vec{C}^i = \vec{C}^{i,G}$ and $\vec{D}^i = \vec{D}^{i,G}$ for every $i < \beta$, $a \subseteq A^G \cap \mathcal{W}(\beta^{<+\kappa})$ is of size less than κ and $c_{\vec{x}} = c_{\vec{x}}^G \upharpoonright (\beta + 1)$ for every $\vec{x} \in a$. \square

Lemma 5.17. *In $\mathbf{V}[G]$ there is a well-order R of $H(\kappa^+)^{\mathbf{V}[G]}$ that is definable over $\langle H(\kappa^+), \in \rangle^{\mathbf{V}[G]}$ by a formula using S , \vec{s} and t^G as parameters.*

Proof. By the proof of Claim 5.16, \mathcal{W} , \mathcal{Q} and $H(\kappa^+)^{\mathbf{V}}$ are each definable over $H(\kappa^+)^{\mathbf{V}[G]}$ using parameters S , \vec{s} and t^G . But now we can obtain a well-order R of $H(\kappa^+)^{\mathbf{V}[G]}$ by setting xRy iff $\tilde{x}\mathcal{W}\tilde{y}$ where \tilde{x} is the \mathcal{W} -least characteristic function of a subset of κ in \mathbf{V} coding a collection \dot{x} of pairs of the form $\langle a, \nu \rangle$ with $a \in H(\kappa^+)^{\mathbf{V}}$ and $\nu < \kappa$ and such that $\{\nu < \kappa \mid (\exists a \in G)(\langle a, \nu \rangle \in \dot{x})\}$ is a subset of κ coding x (and analogously for \tilde{y} and y). By Claim 5.16, it follows that the relation R , which is clearly a well-order of $H(\kappa^+)^{\mathbf{V}[G]}$, is definable over $H(\kappa^+)^{\mathbf{V}[G]}$ using S , \vec{s} and t^G as parameters. \square

Let $X^G \subseteq \kappa$ be defined by setting $\xi \in X^G$ iff one of the following holds.

- (i) $\xi = 2 \cdot \bar{\xi}$ and $\bar{\xi} \in S$.
- (ii) $\xi = 4 \cdot \bar{\xi} + 1$, $\bar{\xi} = \prec \xi_0, \xi_1 \succ$ and $(\xi_0, \xi_1) \in s^*$.
- (iii) $\xi = 4 \cdot \bar{\xi} + 3$, $s_{\bar{\xi}} \neq s_{\zeta}$ for all $\zeta < \bar{\xi}$ and $s_{\bar{\xi}} \subseteq t^G$.

S , s^* and t^G (and therefore also \vec{s}) are obviously definable in $H(\kappa^+)^{\mathbf{V}[G]}$ from X^G . Also, by the definition of \mathcal{Q} together with Lemma 5.8 we have that for every $\xi < \kappa$, $\xi \in X^G$ if and only if there is some $i < \kappa$ such that $\text{ht}(\vec{C}^{i,G}) = \eta_\xi$.

Let $[S]$ be the class of S in $\mathcal{P}(\kappa)/\text{NS}_\kappa$, i.e., the collection of all $S' \subseteq \kappa$ such that the symmetric difference $S' \Delta S$ is non-stationary.

Lemma 5.18 is the final ingredient in the proof of Theorem 1.1. Its proof is essentially a copy of the proof of [3, Lemma 3.2].¹⁵ We reproduce that proof here for the reader's benefit (with the appropriate notational changes).

Lemma 5.18. *$[S]$ and X^G are lightface definable in $H(\kappa^+)^{\mathbf{V}[G]}$. In fact,*

- (i) *$[S]$ can be defined as the unique class K in $\mathcal{P}(\kappa)/\text{NS}_\kappa$ such that for every $S' \in K$ there is a strongly guessing ladder system defined on S' ,¹⁶ and*
- (ii) *X^G can be defined in $\mathbf{V}[G]$ as the set of all $\xi < \kappa$ such that there is a coherent strongly type-guessing club-sequence \vec{C} of height η_ξ , with $\text{dom}(\vec{C})$ a stationary subset of κ disjoint from S' for some $S' \in [S]$, and such that every successor point of every member of $\text{range}(\vec{C})$ has countable cofinality.*

Proof. We start by showing the following.

Claim 5.19. *In $\mathbf{V}[G]$, every $\vec{C}^{i,G}$ is a coherent strongly type-guessing club-sequence with $\text{dom}(\vec{C}^{i,G})$ a stationary subset of κ disjoint from S , and such that every successor point of every member of $\text{range}(\vec{C}^{i,G})$ has countable cofinality.*

Proof. We have already argued that $\vec{C}^{i,G}$ is a coherent club-sequence and it is immediate from the definition of \mathbb{P} that its domain is disjoint from S and that every successor point of every member of its range has countable cofinality. To see that the domain of $\vec{C}^{i,G}$ is, in $\mathbf{V}[G]$, a stationary subset of κ , let us move back to \mathbf{V} once again, let $\dot{C} \in \mathbf{V}$ be a \mathcal{Q} -name for a club subset of κ and let $r = \langle \langle s, \sigma \rangle, p \rangle \in \mathcal{Q}$ with $i < \beta^p$. It will suffice to see that there is a condition $\bar{r} = \langle \langle \bar{s}, \bar{\sigma} \rangle, \bar{p} \rangle \in \mathcal{Q}$ extending r and forcing $\text{dom}(\vec{C}^{i,\bar{p}}) \cap \dot{C} \neq \emptyset$.

Let $\eta = \text{ht}(\vec{C}^{i,p})$ and let us fix a \subseteq -continuous \in -chain $\langle N_\xi \mid \xi \leq \eta \rangle$ of HIA elementary substructures of $\langle H(\theta), \in, \Delta \rangle$ of size less than κ , for some large enough θ and some well-order Δ of $H(\theta)$, containing \mathcal{Q} , \dot{C} , F and p , and such that $\delta_\xi := N_\xi \cap \kappa \in \kappa$ for all $\xi < \eta$. We additionally make sure that $\text{cof}(\delta_{\xi+1}) = \omega$ for every $\xi < \eta$. We aim to build a decreasing sequence $\langle r_\xi \mid \xi \leq \eta \rangle$ of conditions extending r in such a way that for all ξ , letting $r_\xi = \langle \langle s_\xi, \sigma_\xi \rangle, p_\xi \rangle$,

- (i) $r_\xi \in N_{\xi+1}$ is an $\langle N_\xi, \mathcal{Q} \rangle$ -generic condition and $\beta^{p_\xi} = \delta_\xi$,
- (ii) if $\xi < \eta$ is a limit ordinal, then $\delta_\xi \in \text{dom}(\vec{D}^{i,p_\xi})$ and $D_{\delta_\xi}^{i,p_\xi} = \{\delta_{\xi'} \mid \xi' < \xi\}$,
- (iii) $\delta_\eta \in \text{dom}(\vec{C}^{i,p_\eta})$ and $C_{\delta_\eta}^{i,p_\eta} = D_{\delta_\eta}^{i,p_\eta} = \{\delta_\xi \mid \xi < \eta\}$.

¹⁵[3, Lemma 3.2] is stated in the context of the construction in that paper. In particular, κ is ω_1 in that lemma. However, the same proof works for general κ with almost no changes.

¹⁶Note that 'x belongs to the unique class $K \in \mathcal{P}(\kappa)/\text{NS}_\kappa$ such that ...' is indeed first order expressible in $\langle H(\kappa^+), \in \rangle$ despite the fact that an equivalence class in $\mathcal{P}(\kappa)/\text{NS}_\kappa$ is a proper class in $H(\kappa^+)$.

This is enough: Since r_η is $\langle N_\eta, \mathcal{Q} \rangle$ -generic, it forces $\delta_\eta \in \dot{C}$. Hence, r_η is a condition extending r and forces $\dot{C} \cap \text{dom}(\vec{C}^{i,p_\eta}) \neq \emptyset$. The construction of $\langle r_\xi \mid \xi \leq \eta \rangle$ is quite standard. Given ξ and assuming $r_{\xi'}$ has been built for all $\xi' < \xi$, we can find r_ξ in the following way.

Suppose ξ is 0 or a successor ordinal. Let $\langle r'_{\xi'} \mid \xi' < |N_\xi| \rangle$ be the Δ -least $\langle N_\xi, \mathcal{Q} \rangle$ -generic sequence of length $|N_\xi|$ of conditions extending r (if $\xi = 0$) or r_{ξ_0} (if $\xi = \xi_0 + 1$). This generic sequence exists thanks to Lemma 5.9 and Lemma 4.3. Let r_ξ be the canonical lower bound of $\langle r'_{\xi'} \mid \xi' < |N_\xi| \rangle$, as obtained in the proof of Lemma 5.9. Certainly this r_ξ is in $N_{\xi+1}$ and is an $\langle N_\xi, \mathcal{Q} \rangle$ -generic condition. Furthermore, by Lemma 5.8 we have that $\beta^{p_\xi} = \delta_\xi$.

If $\xi < \eta$ is a limit ordinal, we let r_ξ be the canonical lower bound of $\langle r_{\xi'} \mid \xi' < \xi \rangle$, as obtained in the proof of Lemma 5.9, however with $\delta_\xi \in \text{dom}(\vec{D}^{i,p_\xi})$, $\delta_\xi \notin \text{dom}(\vec{C}^{i,p_\xi})$ and $D_{\delta_\xi}^{i,p_\xi} = \{\delta_{\xi'} \mid \xi' < \xi\}$. Again, $\beta^{p_\xi} = \delta_\xi$. In this case, the verification that r_ξ is a condition in \mathcal{Q} and that it extends $r_{\xi'}$ for all $\xi' < \xi$ is exactly as in the proof of Lemma 5.9, using the fact that $\delta_\xi \notin \text{dom}(\vec{C}^{i',p_\xi})$ for all $i' < \delta_\xi$. $r_\xi \in N_{\xi+1}$ and it is $\langle N_\xi, \mathcal{Q} \rangle$ -generic because $N_\xi = \bigcup_{\xi' < \xi} N_{\xi'}$ and because each $r_{\xi'}$ is $\langle N_{\xi'}, \mathcal{Q} \rangle$ -generic.

If $\xi = \eta$, we let r_η be the canonical lower bound of $\langle r_\xi \mid \xi < \eta \rangle$, however with δ_η in both $\text{dom}(\vec{D}^{i,p_\eta})$ and $\text{dom}(\vec{C}^{i,p_\eta})$, and $C_{\delta_\eta}^{i,p_\eta} = D_{\delta_\eta}^{i,p_\eta} = \{\delta_\xi \mid \xi < \eta\}$.

Let us momentarily work in an \vec{S} -generic extension of \mathbf{V} for some \vec{S} -generic G_0 containing $\langle s_\eta, \sigma_\eta \rangle$. As usual we prove by induction along \mathcal{W} that $p_\eta \upharpoonright x$ is a P_x -condition extending all $p_\xi \upharpoonright x$ for $\xi < \eta$. We proceed as in the proof of Lemma 5.7. The only problem could come up in the verification of property (ix) for p_η . For this, suppose $\bar{x} \in a^{p_\eta \upharpoonright x}$ and suppose $z\mathcal{W}\bar{x}$ is \mathcal{W} -minimal with $z \in \mathcal{C}$ such that $F(\bar{x})^{G_0}$ is a P_z -name for a club subset of κ . By the construction of $\langle r_\xi \mid \xi \leq \eta \rangle$ as a generic sequence of conditions it is clear that $p_\eta \upharpoonright z$, which by induction hypothesis is a condition in P_z extending all $p_\xi \upharpoonright z$, decides whether or not ν is in $F(\bar{x})^{G_0}$ for every $\nu < \text{sup}(c_{\bar{x}}^{p_\eta})$. Hence we are left with checking that if $C_{\bar{x}}$ is the collection of all $\nu < \text{sup}(c_{\bar{x}}^{p_\eta})$ such that $p_\eta \upharpoonright z \Vdash_{P_z} \nu \in F(\bar{x})^{G_0}$, then $\text{ot}(C_{\delta_\eta}^{i',p_\eta} \cap^* C_{\bar{x}}) = \text{ht}(\vec{C}^{i',p_\eta})$ for every $i' < \beta^{p_\eta}$ and every $\delta \in c_{\bar{x}}^{p_\eta} \cap \text{dom}(\vec{C}^{i',p_\eta})$. The proof of this in the case when either there is some $\xi < \eta$ such that $\delta \in c_{\bar{x}}^{p_\xi}$ or $i' \neq i$ goes through easily from the way we have built $\langle r_\xi \mid \xi \leq \eta \rangle$.

The only nontrivial case is when $i' = i$ and $\delta = \delta_\eta$. We want to prove that $\text{ot}(C_{\delta_\eta}^{i,p_\eta} \cap^* C_{\bar{x}}) = \eta$. In this case we argue that, since $F(\bar{x})^{G_0}$ is a P_z -name for a club subset of κ and each r_ξ (for $\xi < \eta$) is (N_ξ, \mathcal{Q}) -generic, $p_\eta \upharpoonright z$ forces $\delta_\xi \in F(\bar{x})^{G_0}$ for all $\xi < \eta$ such that $z, \bar{x} \in a^{p_\xi} \subseteq N_\xi$. Hence, we have in fact that a final segment of $C_{\delta_\eta}^{i,p_\eta}$ is contained in $C_{\bar{x}}$. Now that we have that r_η is a condition in \mathcal{Q} , checking that it extends all r_ξ for $\xi < \eta$, is straightforward.

We still need to check that $\vec{C}^{i,G}$ is strongly type-guessing in $\mathbf{V}[G]$. For this we argue very much as in the proof of Lemma 5.14: Let us go back to \mathbf{W} . Let \dot{C} be a nice \mathbb{P} -name for a club subset of κ and let $p \in \mathbb{P}$. As in the proof of Lemma 5.14, we know that there is some $z\mathcal{W}\vec{0}$ such that $z \in \mathcal{C}$ and $\dot{C} \in H(\kappa^+)$ is a P_z -name for a club subset of κ . Now suppose z is \mathcal{W} -minimal with the above property and find $\bar{x}, z\mathcal{W}\bar{x}$, such that $F(\bar{x}) = \dot{C}$. Let $C = \dot{C}^{G_1}$.

By Lemma 5.12 (i) we may extend p to a condition p^* such that $\bar{x} \in a^{p^*}$. But by condition (ix) in the definition of \mathbb{P} we know that every $p' \in \mathbb{P}$ extending p^* is such that $p' \upharpoonright z$ decides, for every $\nu < \max(c_{\bar{x}}^{p'})$, whether or not ν is in

\dot{C} . Furthermore, for every such p' , letting $C_{\bar{x}}$ be the set of $\nu < \max(c_{\bar{x}}^{p'})$ such that $p' \upharpoonright z \Vdash_{P_z} \nu \in \dot{C}$, we know that every $\delta \in c_{\bar{x}}^{p'} \cap \text{dom}(\vec{C}^{i,p'})$ is such that $\text{ot}(C_{\delta}^{i,p'} \cap^* C_{\bar{x}}) = \eta$. That is, p' forces $\text{ot}(C_{\delta}^{i,\dot{G}} \cap^* \dot{C}) = \eta$ for every such δ . This shows that, in $\mathbf{V}[G]$, $\text{ot}(C_{\delta}^{i,G} \cap^* C) = \eta$ for all $\delta \in c_{\bar{x}}^G \cap \text{dom}(\vec{C}^{i,G})$. Hence, $c_{\bar{x}}^G$ is a witness for C to the fact that $\vec{C}^{i,G}$ is strongly type-guessing. \square

We have already seen that \vec{E}^G is a strongly guessing ladder system. It remains to see that there is no strongly guessing ladder system whose domain is a stationary subset of κ disjoint from S (this is shown in Claim 5.20 below in its case $\eta = \omega$) and that if $\xi < \kappa$ is such that $\xi \notin X^G$, then in $\mathbf{V}[G]$ there is no coherent strongly type-guessing club-sequence of height η_ξ whose domain is a stationary subset of κ disjoint from S and such that every successor point of every member of its range has countable cofinality (this is shown in Claim 5.20 below in its case $\eta > \omega$). This will finish the proof of Lemma 5.18.

Claim 5.20. *In $\mathbf{V}[G]$, let $\vec{C} = \langle C_\delta \mid \delta \in \text{dom}(\vec{C}) \rangle$ be a coherent club-sequence with $\text{dom}(\vec{C})$ a stationary subset of $\kappa \setminus S$. Let $\eta = \text{ht}(\vec{C})$. Suppose either*

- (i) $\eta = \omega$, or else
- (ii) η is a perfect ordinal of countable cofinality such that $\xi \notin X^G$ if $\eta_\xi = \eta$ and every successor point of every member of $\text{range}(\vec{C})$ has countable cofinality.

Then there is some $z \in (\kappa^\kappa)^\mathbf{V} \setminus \{\vec{0}\}$ such that in $\mathbf{V}[G]$,

$$\{\delta \in \text{dom}(\vec{C}) \mid \text{ot}(C_\delta \cap^* c_x^G) < \eta\}$$

is a stationary subset of κ for all $x \in (\kappa^\kappa)^\mathbf{V} \setminus \{\vec{0}\}$ such that $z \mathcal{W} x$.¹⁷

Proof. Let us work in \mathbf{W} . Using the κ^+ -chain condition of \mathbb{P} we may fix some $z \in \mathcal{C} \setminus \{\vec{0}\}$ such that $\vec{C} = \tau^{G_1}$, where $\tau \in H(\kappa^+)$ is a P_z -name for a coherent club-sequence of height η , and some $\bar{p} \in P_z \cap G_1$ forcing (in \mathbb{P}) that $\xi \notin \dot{X}^{G_0}$ if η is a perfect ordinal of countable cofinality and $\eta = \eta_\xi$ - where \dot{X}^{G_0} is a \mathbb{P} -name in \mathbf{W} for X^G - and that $\text{dom}(\tau)$ is a stationary subset of κ disjoint from S .

Let $x \in (\kappa^\kappa)^\mathbf{V} \setminus \{\vec{0}\}$ be such that $z \mathcal{W} x$ and let \dot{C} be a \mathbb{P} -name for a club subset of κ . Let p' be a condition extending \bar{p} in \mathbb{P} . By Lemma 5.12 (i) we may assume that $x \in a^{p'}$. It will suffice to find a condition $q \leq p'$ and some $\delta \in c_x^q$ such that $q \Vdash_{\mathbb{P}} \delta \in \dot{C} \cap \text{dom}(\tau)$ and such that $q \Vdash_{\mathbb{P}} \text{ot}(\tau_\delta \cap^* c_x^q) < \eta$ (where τ_δ is a name for $\tau(\delta)$).

Let G_* be P_x -generic over \mathbf{W} with $p' \upharpoonright x \in G_*$. Note that, since P_x is a complete suborder of \mathbb{P} , every generic filter G' for \mathbb{P}/G_* over $\mathbf{W}[G_*]$ - where \mathbb{P}/G_* is the suborder of \mathbb{P} consisting of those conditions q such that $q \upharpoonright x \in G_*$ - is such that $G' \cap P_x = G_*$ and is \mathbb{P} -generic over \mathbf{W} as a filter of \mathbb{P} , and that, conversely, every \mathbb{P} -generic filter G' over \mathbf{W} with $G' \cap P_x = G_*$ is \mathbb{P}/G_* -generic over $\mathbf{W}[G_*]$.

¹⁷The proof also shows, for \vec{C} as in the hypothesis, that if the set

$$\text{dom}(\vec{C}) \setminus \bigcup \{ \text{dom}(\vec{C}^{i,G}) \mid i < \kappa, \text{ht}(\vec{C}^{i,G}) < \eta \}$$

is in $\mathbf{V}[G]$ a stationary subset of κ , then in fact $\{\delta \in \text{dom}(\vec{C}) \mid \text{sup}(C_\delta \cap^* c_x^G) < \eta\}$ is stationary in $\mathbf{V}[G]$ for \mathcal{W} -cofinally many x in $(\kappa^\kappa)^\mathbf{V}$ below $\vec{0}$.

We will temporarily work in $\mathbf{W}[G_*]$. Let $\vec{C}^* = \tau^{G_*}$ and let

$$\vec{C}^{*i} = \bigcup \{ \vec{C}^{i,p} \mid p \in G_*, i < \beta^p \}$$

for all $i < \kappa$. Let θ be a sufficiently large regular cardinal and let Δ be a well-order of $H(\theta)^{\mathbf{W}[G_*]}$. Let $\langle N_\xi \mid \xi < \kappa \rangle$ be a \subseteq -continuous \in -chain of elementary substructures of $\langle H(\theta)^{\mathbf{W}[G_*]}, \in, \Delta \rangle$ of size less than κ containing everything relevant such that $\delta_\xi := N_\xi \cap \kappa \in \kappa^{18}$ for all $\xi < \kappa$ and let $D_0 = \{ \delta_\xi \mid \xi < \kappa \}$.

Subclaim 5.21. *There is a limit ordinal $\bar{\xi} < \kappa$ with $\delta_{\bar{\xi}} \in \text{dom}(\vec{C}^*)$, $\eta < \delta_{\bar{\xi}}$, with $(D_0 \cap \delta_{\bar{\xi}}) \setminus (C_{\delta_{\bar{\xi}}}^* \cup \bigcup_{i < \kappa} \text{dom}(\vec{C}^{*i}) \cup S)$ unbounded in $\delta_{\bar{\xi}}$, and such that $\text{ot}(C_{\delta_{\bar{\xi}}}^{*i} \cap^* D_0) = \text{ot}(C_{\delta_{\bar{\xi}}}^{*i})$ in case $i < \kappa$ is such that $\delta_{\bar{\xi}} \in \text{dom}(\vec{C}^{*i})$.¹⁹*

Proof. Note that, by Claim 5.19,

$$\mathcal{Y} := \{ \delta < \kappa \mid (\forall i) (\delta \in \text{dom}(\vec{C}^{*i}) \rightarrow \text{ot}(C_\delta^{*i} \cap^* D_0) = \text{ot}(C_\delta^{*i})) \}$$

is forced by \mathbb{P}/G_* to contain a club subset of κ . This is true because $\text{dom}(\vec{C}^{*i}) \cap (i+1) = \emptyset$ for all i - for every i there is, in $\mathbf{W}^{\mathbb{P}/G_*}$, a club

$$C_i \subseteq \{ \delta < \kappa \mid \delta \in \text{dom}(\vec{C}^{*i}) \rightarrow \text{ot}(C_\delta^{*i} \cap^* D_0) = \text{ot}(C_\delta^{*i}) \}.$$

Now the required club can be taken to be the diagonal intersection $\Delta_{i < \kappa} C_i$. $Z := D_0 \setminus (S \cup \bigcup_{i < \kappa} \text{dom}(\vec{C}^{*i}))$ is unbounded in κ (for example by an argument as in the proof of Lemma 5.7), and therefore $D_1 = \{ \delta \in D_0 \mid \text{rank}_Z(\delta) > \eta \}$ is a club subset of κ . Since $\text{dom}(\vec{C}^*)$ is forced by \mathbb{P}/G_* to be a stationary subset of κ , it must have stationary intersection with $\mathcal{Y} \cap D_1$. Pick $\bar{\xi}$ s.t. $\delta_{\bar{\xi}} > \eta$ is in this intersection. This is enough since then $(D_0 \cap \delta_{\bar{\xi}}) \setminus (C_{\delta_{\bar{\xi}}}^* \cup \bigcup_{i < \kappa} \text{dom}(\vec{C}^{*i}) \cup S)$ must be unbounded in $\delta_{\bar{\xi}}$ as $\text{rank}_Z(\delta_{\bar{\xi}}) > \eta$ and $\text{ot}(C_{\delta_{\bar{\xi}}}^*) = \eta$.²⁰ \square

Let $\bar{\xi}$ be as given by Subclaim 5.21. We will find, in \mathbf{W} , a condition q extending p' and forcing both $\delta_{\bar{\xi}} \in \dot{C} \cap \text{dom}(\tau)$ and $\text{ot}(\tau_{\delta_{\bar{\xi}}} \cap^* c_x^q) < \eta$.

The proof of the following subclaim is standard.

Subclaim 5.22. *For every dense set $\mathcal{D} \subseteq \mathbb{P}/G_*$, $q \in \mathbb{P}/G_*$, and every $u \in a^q$ with either $u = x$ or $x\mathcal{W}u$, there is a club $C \subseteq \kappa$ with the property that for every $\delta \in C$ and every $\delta' < \delta$ there is a condition $q' \in \mathcal{D}$ extending q with $\beta^{q'} < \delta$ and such that $c_u^{q'} \setminus c_u^q \subseteq (\delta', \delta)$.*

Proof. We may take this club to be $\{ M_j \cap \kappa \mid j < \kappa \}$ for a continuous \in -chain $\langle M_j \mid j < \kappa \rangle$ of elementary substructures of $H(\chi)$ (for some large enough χ) of size less than κ , containing \mathcal{D} and q , and such that $M_j \cap \kappa \in \kappa$ for all j . The subclaim then follows from an application of Clause (ii) of Lemma 5.12 within a relevant M_j . \square

¹⁸For this proof we do not need that the structures be HIA.

¹⁹There may or may not be such an i . If there is such an i , then of course it is unique.

²⁰Note that $Z \setminus Y$ is unbounded in $\text{sup}(Z)$ whenever Z and Y are sets of ordinals with $\text{rank}_Z(\text{sup}(Z)) > \text{ot}(Y)$.

In order to find the desired q extending p' we distinguish three cases.

Case 1: There is (a unique) i such that $\delta_{\bar{\xi}} \in \text{dom}(\vec{C}^{*i})$ and $\eta < \text{ht}(\vec{C}^{*i})$.

Let \mathcal{E} be the set of ordinals in $C_{\delta_{\bar{\xi}}}^{*}$ above $\min(C_{\delta_{\bar{\xi}}}^{*})$ which are not limit points of $C_{\delta_{\bar{\xi}}}^{*}$ and let $\langle t_k \mid k < \omega \rangle$ be an increasing sequence converging to the height of \vec{C}^{*i} . Since $i \in N_{\bar{\xi}}$ (because $i < \delta_{\bar{\xi}}$ using Clause (viii) in the definition of conditions), and therefore $N_{\bar{\xi}}$ can be assumed to contain $\langle t_k \mid k < \omega \rangle$, by disregarding an initial segment of $\langle N_{\bar{\xi}} \mid \xi < \kappa \rangle$ if necessary we may, and will, assume that $\langle t_k \mid k < \omega \rangle$ is in N_0 . Note that, since $\text{ht}(\vec{C}^{*i})$ is a perfect ordinal above η , for every k there are unboundedly many ordinals δ in \mathcal{E} such that $\text{ot}((C_{\delta_{\bar{\xi}}}^{*i} \cap^* D_0) \cap J_\delta) \geq t_k$, where J_δ is the interval $(\max(C_{\delta_{\bar{\xi}}}^{*} \cap \delta), \delta)$. Otherwise $\text{ht}(\vec{C}^{*i}) = \text{ot}(C_{\delta_{\bar{\xi}}}^{*i} \cap^* D_0)$ would be bounded by $t_k \cdot \eta$ for some k , which would contradict the fact that $\text{ht}(\vec{C}^{*i})$ is perfect and that t_k and η are less than $\text{ht}(\vec{C}^{*i})$. Since every ordinal in $C_{\delta_{\bar{\xi}}}^{*i} \cap^* D_0$ is of the form δ_ξ for some $\xi < \bar{\xi}$, it follows that we may find a strictly increasing sequence $(\xi_k)_{k < \omega}$ converging to $\bar{\xi}$ such that $\delta_{\xi_0} > i$ and such that

$$\text{ot}((C_{\delta_{\bar{\xi}}}^{*i} \cap^* D_0) \cap (\max(C_{\delta_{\bar{\xi}}}^{*} \cap \bar{\delta}_k), \bar{\delta}_k)) > t_{k+1}$$

for $\bar{\delta}_k := \min(\mathcal{E} \setminus \delta_{\xi_k})$ (for all k). It follows that there is a function h defined on \mathcal{E} such that $\max(C_{\delta_{\bar{\xi}}}^{*} \cap \delta) \leq h(\delta) < \delta$ for every $\delta \in \mathcal{E}$ and such that

$$\text{ot}((C_{\delta_{\bar{\xi}}}^{*i} \cap^* D_0) \cap \bigcup_{\delta' \in \mathcal{E} \cap \delta} (\max(C_{\delta_{\bar{\xi}}}^{*} \cap \delta'), h(\delta'))) \geq t_k$$

whenever $k < \omega$, $\delta \in \mathcal{E}$ and $\delta_{\xi_k} \leq \delta$. We may assume that h is defined inductively as follows. If $k < \omega$ is minimal such that $\delta \in \mathcal{E} \cap \delta_{\xi_k}$, let $h(\delta)$ be the least ϵ in $(\max(C_{\delta_{\bar{\xi}}}^{*} \cap \delta), \delta)$ such that

$$\text{ot} \left((C_{\delta_{\bar{\xi}}}^{*i} \cap^* D_0) \cap \left(\bigcup_{\delta' \in \mathcal{E} \cap \delta} (\max(C_{\delta_{\bar{\xi}}}^{*} \cap \delta'), h(\delta')) \cup (\max(C_{\delta_{\bar{\xi}}}^{*} \cap \delta), \epsilon) \right) \right) \geq \bar{t},$$

where \bar{t} is the maximal member t of the set $\{0\} \cup \{t_{k'} \mid k' \leq k\}$ for which there is some ϵ , $\max(C_{\delta_{\bar{\xi}}}^{*} \cap \delta) < \epsilon < \delta$, such that

$$\text{ot} \left((C_{\delta_{\bar{\xi}}}^{*i} \cap^* D_0) \cap \left(\bigcup_{\delta' \in \mathcal{E} \cap \delta} (\max(C_{\delta_{\bar{\xi}}}^{*} \cap \delta'), h(\delta')) \cup (\max(C_{\delta_{\bar{\xi}}}^{*} \cap \delta), \epsilon) \right) \right) \geq t.$$

Note that, since \vec{C}^{*i} and \vec{C}^{*} are coherent sequences, $h \upharpoonright (\mathcal{E} \cap \delta_\xi) \in N_{\xi+1}$ for every $\xi < \bar{\xi}$. The reason is that $N_{\xi+1}$ contains all initial segments of \vec{C}^{*i} and of \vec{C}^{*} of length less than $\delta_{\xi+1}$ and the sequences $\langle N_{\xi'} \mid \xi' \leq \xi \rangle$ and $\langle t_k \mid k < \omega \rangle$. Let

$$\Sigma = \{ \xi < \bar{\xi} \mid \delta_\xi \in (C_{\delta_{\bar{\xi}}}^{*i} \cap^* D_0) \cap \bigcup_{\delta \in \mathcal{E}} (\max(C_{\delta_{\bar{\xi}}}^{*} \cap \delta), h(\delta)) \}$$

and let $\bar{\Sigma}$ be the closure of Σ . Note that $\text{ot}(\Sigma) = \text{ht}(\vec{C}^{*i})$ and that Σ does not contain any of its accumulation points. In fact, if $\xi \in \Sigma$, then δ_ξ is a member

of $C_{\delta_{\bar{\xi}}}^{*i}$ which is not a limit point of $C_{\delta_{\bar{\xi}}}^{*i}$ (by the definition of \cap^*). Note also that $\max(\bar{\Sigma}) = \bar{\xi}$, that $\delta_{\xi} \notin S$ for any $\xi \in \bar{\Sigma}$ and that $\delta_{\xi} \notin \text{dom}(\vec{C}^{*j})$ for any $\xi \in \bar{\Sigma} \cap \delta_{\bar{\xi}}$ and $j < \kappa$, using Clause (viii) in the definition of conditions.

Now we can inductively build a decreasing sequence $\langle p_{\xi} \mid \xi \in \bar{\Sigma} \rangle$ of conditions in \mathbb{P}/G_* extending p' such that the following hold for each $\xi \in \bar{\Sigma}$.

- (i) $p_{\xi} \in N_{\xi+1}$. (to obtain this property, we will need all choices in the below arguments to be made in a canonical way; we will tacitly assume such choices in the following)
- (ii) If $\xi \in \bar{\Sigma} \setminus \Sigma$, then p_{ξ} is a lower bound of $\langle p_{\xi'} \mid \xi' \in \Sigma \cap \xi \rangle$.
- (iii) If $\xi \in \Sigma$, then p_{ξ} is a lower bound of a certain decreasing ω -sequence $\langle q_k^{\xi} \mid k < \omega \rangle$ of conditions in N_{ξ} (see below) and forces $\delta_{\xi} \in \dot{C}$.
- (iv) Given any two $\xi_0 < \xi_1$ in $\bar{\Sigma}$ and any $\bar{x} \in a^{p_{\xi_0}}$, if there is a minimal $\bar{z}\mathcal{W}\bar{x}$ in \mathcal{C} such that $F(\bar{x})^{G_0}$ is a $P_{\bar{z}}$ -name for a club subset of κ , then $p_{\xi_1} \upharpoonright \bar{z}$ forces $\delta_{\xi_1} \in F(\bar{x})^{G_0}$.
- (v) If $\xi \in \Sigma$ and $\delta \in \mathcal{E}$ is such that $\delta_{\xi} \in (\max(C_{\delta_{\bar{\xi}}}^{*} \cap \delta), h(\delta))$, then

$$\max(C_{\delta_{\bar{\xi}}}^{*} \cap \delta) < \min(c_x^{p_{\xi}} \setminus (c_x^{p'} \cup (\sup\{\delta_{\xi'} \mid \xi' \in \Sigma \cap \xi\} + 1))).$$

We want to show first, given any $\xi \in \Sigma$ and assuming $p_{\xi'}$ has been built for all $\xi' \in \bar{\Sigma} \cap \xi$, how to find p_{ξ} in $N_{\xi+1}$ so that (iii) and (v) hold about p_{ξ} , and if $\xi' = \max(\bar{\Sigma} \cap \xi)$, so that (iv) holds about the pair (ξ', ξ) . Moreover we want to show how to perform the construction in a uniformly definable way.

p_{ξ} can be built as a lower bound in $N_{\xi+1} \cap \mathbb{P}/G_*$ of a decreasing sequence $\langle q_k^{\xi} \mid k < \omega \rangle$ of \mathbb{P}/G_* -conditions in N_{ξ} extending $p_{\max(\bar{\Sigma} \cap \xi)}$ (if $\bar{\Sigma} \cap \xi \neq \emptyset$) or extending p' (if ξ is the first member of Σ) such that, for a suitable sequence $\langle \mathcal{D}_k \mid k < \omega \rangle$ of dense subsets of \mathbb{P}/G_* , all of them belonging to N_{ξ} ,

- (a) $q_k^{\xi} \in \mathcal{D}_k$ for all k ,
- (b) $\sup_{k' \geq k} \max(c_{u'}^{q_{k'}^{\xi}}) = \delta_{\xi}$ for every k and every $u \in a^{q_k^{\xi}}$, and
- (c) if $\delta \in \mathcal{E}$ is such that $\delta_{\xi} \in (\max(C_{\delta_{\bar{\xi}}}^{*} \cap \delta), h(\delta))$, then q_0^{ξ} puts some ordinal above $\max(C_{\delta_{\bar{\xi}}}^{*} \cap \delta)$, but no new ordinal below $\max(C_{\delta_{\bar{\xi}}}^{*} \cap \delta) + 1$, inside $c_x^{q_0^{\xi}}$.²¹

Since $\delta_{\xi} \notin S$ and $\delta_{\xi} \notin \vec{C}^{*j}$ for any $j < \kappa$, the sequence $\langle q_k^{\xi} \mid k < \omega \rangle$ has a lower bound r such that $r \upharpoonright x \in G_*$, by the arguments in the proof of Lemma 5.7. Conditions (a)-(c) can be met simultaneously once \mathcal{D}_k has been fixed since, by correctness, N_{ξ} contains a club as given by Subclaim 5.22 for $\mathcal{D} = \mathcal{D}_0$ and for q being either $p_{\max(\bar{\Sigma} \cap \xi)}$ or p' and since δ_{ξ} , being a non-accumulation point of $C_{\delta_{\bar{\xi}}}^{*i}$, has countable cofinality.

²¹This means that $c_x^{q_0^{\xi}} \setminus (c_x^r \cup (\max(C_{\delta_{\bar{\xi}}}^{*} \cap \delta) + 1)) \neq \emptyset$ and $c_x^{q_0^{\xi}} \cap (\max(C_{\delta_{\bar{\xi}}}^{*} \cap \delta) + 1) = c_x^r \cap (\max(C_{\delta_{\bar{\xi}}}^{*} \cap \delta) + 1)$, where $r = p_{\max(\bar{\Sigma} \cap \xi)}$ if $\bar{\Sigma} \cap \xi \neq \emptyset$ and $r = p'$ if $\xi = \min(\Sigma)$.

Let $\langle \epsilon_k \mid k < \omega \rangle \in N_{\xi+1}$ be an ω -sequence of ordinals in N_ξ converging to δ_ξ . We take each \mathcal{D}_k to be the set of conditions q' forcing some ordinal above ϵ_k to be in $F(\bar{x})^{G_0} \cap \dot{C} \cap c_x^{q'}$ whenever $\bar{x} \in a^q$ (for the right choice of q) and there exists $\bar{u} \in \mathcal{C}$ with $\bar{u}\mathcal{W}\bar{x}$ such that $F(\bar{x})^{G_0}$ is a $P_{\bar{u}}$ -name for a club subset of κ .

The sequence of conditions $\langle p_\xi \mid \xi \in \bar{\Sigma} \rangle$ can now be built in a canonical way. At limit stages $\xi < \bar{\xi}$ of the construction, we extend all conditions built up to that point by considering a canonical lower bound p_ξ of the sequence with $p_\xi \upharpoonright x \in G_*$. This lower bound exists by the proof of Lemma 5.7 because $\delta_\xi \notin S$ and $\delta_\xi \notin \text{dom}(\vec{C}^{*j})$ for any $j < \kappa$. The fact that $h \upharpoonright (\mathcal{E} \cap \delta_\xi) \in N_{\xi+1}$ for every ξ ensures that the choice of p_ξ takes place inside $N_{\xi+1}$.

Now if $\delta \in \mathcal{E} \cap c_x^{p_\xi}$, consider the least $\xi \in \bar{\Sigma} \setminus \{\xi\}$ such that $\delta \in \mathcal{E} \cap c_x^{p_\xi}$. Then if δ^* is the successor of δ in \mathcal{E} , ξ is also least in $\bar{\Sigma} \setminus \{\xi\}$ such that $\delta_\xi \in (\delta, \delta^*)$. But then by condition (v) in the construction of the p_ξ , it follows that $\delta \in \mathcal{E} \cap c_x^{p'_\xi}$. Considering that \mathcal{E} is the set of successor points of $C_{\delta_\xi}^*$, it follows that there is $\bar{\delta} < \delta_{\bar{\xi}}$ such that

$$(\star) \quad C_{\delta_{\bar{\xi}}}^* \cap^* c_x^{p_\xi} \subseteq \bar{\delta} \text{ for each } \xi.$$

Also, given any $x' \in (\kappa^\kappa)^{\mathbf{V}} \setminus \{\bar{0}\}$ such that $x\mathcal{W}x'$ and any $\xi_0 \in \Sigma$, if $x' \in a^{p_{\xi_0}}$, then $x' \in N_{\xi_0+1}$. Hence, if $\bar{z}\mathcal{W}x'$ is minimal such that $\bar{z} \in \mathcal{C}$ and $F(x')^{G_0}$ is a $P_{\bar{z}}$ -name for a club subset of κ , then each δ_ξ (for $\xi \in \Sigma$, $\xi > \xi_0$) is a member of $C_{\delta_{\bar{\xi}}}^{*i}$ which is not a limit point of $C_{\delta_{\bar{\xi}}}^{*i}$ and which is forced by p_ξ to be in $F(x')^{G_0}$. This allows us to obtain $p_{\bar{\xi}}$ ²² and extend it to a condition q such that $q \upharpoonright x \in G_*$ and such that q forces $\tau_{\delta_{\bar{\xi}}} = C_{\delta_{\bar{\xi}}}^*$. This finishes the proof in this case, since $q \Vdash_{\mathbb{P}} \delta_{\bar{\xi}} \in \dot{C}$ and since $q \Vdash_{\mathbb{P}} \tau_{\delta_{\bar{\xi}}} \cap^* c_x^q \subseteq \bar{\delta}$ (by (\star)).

Case 2: There is (a unique) i such that $\delta_{\bar{\xi}} \in \text{dom}(\vec{C}^{*i})$ and $\eta > \text{ht}(\vec{C}^{*i})$.

Let $\Sigma = \{\xi < \bar{\xi} \mid \delta_\xi \in C_{\delta_{\bar{\xi}}}^{*i}, \eta < \delta_\xi\}$. This time we build a decreasing sequence $\langle p_\xi \mid \xi \in \Sigma \rangle$ of conditions in \mathbb{P}/G_* extending p' and satisfying the following.

- (i) $p_\xi \in N_{\xi+1}$ for every ξ . (as in Case 1, this uses that we will tacitly make all below choices canonically)
- (ii) For every limit point ξ of Σ , p_ξ is a lower bound of $\langle p_{\xi'} \mid \xi' \in \Sigma \cap \xi \rangle$.
- (iii) Given any successor point ξ of Σ , p_ξ is a lower bound of a certain decreasing ω -sequence $\langle q_k^\xi \mid k < \omega \rangle$ of conditions in N_ξ and forces $\delta_\xi \in \dot{C}$.
- (iv) Given any $\xi_0 < \xi_1$ in Σ and any $x' \in a^{p_{\xi_0}}$, if $\bar{z}\mathcal{W}x'$ is \mathcal{W} -minimal in \mathcal{C} such that $F(x')^{G_0}$ is a $P_{\bar{z}}$ -name for a club subset of κ , then p_{ξ_1} forces $\delta_{\xi_1} \in F(x')^{G_0}$.
- (v) Given any successor point ξ of Σ ,

$$c_x^{p_\xi} \cap (\sup\{\delta_{\xi'} \mid \xi' \in \Sigma \cap \xi\}, \delta_\xi) \cap C_{\delta_{\bar{\xi}}}^* = \emptyset.$$

²²The crucial point here is that by the above, (b) in Condition (ix) in the definition of \mathbb{P} holds for $p_{\bar{\xi}}$. (a) in Condition (ix) and (vi) in the definition of \mathbb{P} hold trivially, for $\delta_{\bar{\xi}} \notin S$.

For any successor point ξ of Σ , assuming $p_{\xi'}$ for all $\xi' \in \Sigma \cap \xi$ has been defined, the choice of $p_\xi \in N_{\xi+1}$ can be made as in the previous case: p_ξ can be taken to be a lower bound of a decreasing sequence $\langle q_k^\xi \mid k < \omega \rangle$ of conditions in N_ξ meeting the members of a suitably chosen sequence $\langle \mathcal{D}_k \mid k < \omega \rangle$ of dense subsets of \mathbb{P}/G_* in N_ξ . This lower bound will exist exactly by the same reasons as in the previous case. This time we pick the conditions q_k^ξ in such a way that, for all k ,

- (a) $q_k^\xi \in \mathcal{D}_k$,
- (b) $\sup_{k' \geq k} \max(c_{u^{k'}}^{q_{k'}^\xi}) = \delta_\xi$ for every $u \in a^{q_k^\xi}$, and
- (c) q_k^ξ does not put any ordinal in $C_{\delta_\xi}^* \setminus (\sup\{\delta_{\xi'} \mid \xi' \in \Sigma \cap \xi\} + 1)$ inside $c_x^{q_k^\xi}$.

We again use that $\text{cof}(\delta_\xi) = \omega$. Conditions (a)-(c) can be met, once \mathcal{D}_k has been fixed, since $\text{ht}(\vec{C}^*) < \delta_\xi = N_\xi \cap \kappa$ and since N_ξ contains a club as given by Subclaim 5.22 for $\mathcal{D} = \mathcal{D}_k$ and for q being p_j or q_{k-1}^ξ if $k > 0$. The intersection of this club with δ_ξ has order type $\delta_\xi > \text{ht}(\vec{C}^*)$, so there are unboundedly many points δ in it such that $[\nu, \delta) \cap C_{\delta_\xi}^* = \emptyset$ for some $\nu < \delta$. The choice of $\langle \mathcal{D}_k \mid k < \omega \rangle$ is as in Case 1.

This is enough since then, by the same reasons as before, $\langle p_\xi \mid \xi \in \Sigma \rangle$ has a lower bound $p_{\bar{\xi}}$ in \mathbb{P}/G_* (using (iv) as in Case 1) such that $(c_x^{p_{\bar{\xi}}} \cap C_{\delta_{\bar{\xi}}}^*) \setminus (\delta_{\xi_0} + 1) \subseteq \{\delta_{\xi_j} \mid 0 < j < \text{ht}(\vec{C}^{*i})\}$ (by (v)), and forcing $\delta_{\bar{\xi}} \in \dot{C}$ (by (iii)). As in the previous case, we can extend $p_{\bar{\xi}}$ to a condition q , with $q \upharpoonright x \in G_*$, forcing $\tau_{\delta_{\bar{\xi}}} = C_{\delta_{\bar{\xi}}}^*$. This is enough, since then q forces $\text{ot}(\tau_{\delta_{\bar{\xi}}} \cap^* c_x^q) \leq \text{ht}(\vec{C}^{*i}) < \eta$.²³

Case 3: $\delta_{\bar{\xi}} \notin \bigcup_{i < \kappa} \text{dom}(\vec{C}^{*i})$.

The proof is now easier than in the previous two cases. Let $\langle \xi_k \mid k < \omega \rangle$ be a strictly increasing sequence converging to $\bar{\xi}$ and with $\{\delta_{\xi_k} \mid k < \omega\}$ disjoint from $C_{\delta_{\bar{\xi}}}^* \cup \bigcup_{i < \kappa} \text{dom}(\vec{C}^{*i})$.

We can now build by recursion a decreasing sequence $\langle p_k \mid k < \omega \rangle$ of conditions in \mathbb{P}/G_* extending p' such that, for each k ,

- (i) $p_k \in N_{\xi_{k+1}}$,
- (ii) p_k forces $\rho \in \dot{C}$ for some $\rho > \delta_{\xi_{k-1}}$, and
- (iii) if $k > 0$, then $\min(c_x^{p_k} \setminus (\delta_{\xi_{k-1}} + 1)) > \max(C_{\delta_{\bar{\xi}}}^* \cap \delta_{\xi_k})$.

Finally, since $\delta_{\bar{\xi}} \notin S \cup \bigcup_{i < \kappa} \text{dom}(\vec{C}^{*i})$, $\langle p_k \mid k < \omega \rangle$ has a lower bound \tilde{q} with $\tilde{q} \upharpoonright x \in G_*$ and forcing that $\delta_{\bar{\xi}}$ is in \dot{C} . Again, we can extend \tilde{q} to a condition q such that $q \upharpoonright x \in G_*$ and forcing $\tau_{\delta_{\bar{\xi}}} = C_{\delta_{\bar{\xi}}}^*$. It follows that q forces that $\tau_{\delta_{\bar{\xi}}} \cap^* c_x^q$ (and in fact $\tau_{\delta_{\bar{\xi}}} \cap c_x^q$) is bounded in $\delta_{\bar{\xi}}$.

The construction in this last case finishes the proof of Claim 5.20. □

This completes the proof of Lemma 5.18. □

²³In fact, q forces $\text{ot}((\tau_{\delta_{\bar{\xi}}} \setminus (\delta_{\xi_0} + 1)) \cap c_x^q) \leq \text{ht}(\vec{C}^{*i}) < \eta$.

It follows now, from Lemmas 5.17 and 5.18, that in $\mathbf{V}[G]$ there is a lightface definable well-order of $H(\kappa^+)^{\mathbf{V}[G]}$. This concludes the proof of Theorem 1.1. \square

6 The global iteration

In Section 5, given an uncountable cardinal κ with $\kappa^{<\kappa} = \kappa$, we obtained a partial order \mathcal{Q} which has a $<\kappa$ -directed closed dense subset, satisfies the κ^+ -cc, is a subset of $H(\kappa^+)$ and forces the existence of a lightface definable well-order of $H(\kappa^+)$. Note that the definition of \mathcal{Q} actually depended on the choice of some well-order $\bar{\mathcal{W}}$ of ${}^\kappa\kappa$ (we required that its order-type is 2^κ and that $\bar{0}$ is its least element) and an arbitrary bookkeeping function \bar{F} for $H(\kappa^+)$. Let $\mathcal{Q}_\kappa(\bar{\mathcal{W}}, \bar{F})$ denote the forcing \mathcal{Q} at κ relative to a particular choice of well-order $\bar{\mathcal{W}}$ and bookkeeping function \bar{F} . We may also assume that $\mathcal{Q}_\kappa(\bar{\mathcal{W}}, \bar{F})$ is $<\kappa$ -directed closed (by passing to its $<\kappa$ -directed closed dense subset).

If κ is an uncountable cardinal with $\kappa^{<\kappa} = \kappa$, then we let \mathcal{Q}_κ denote the two-step iteration which in the first step performs a lottery of all well-orders of ${}^\kappa\kappa$ with the above properties, and all bookkeeping functions for $H(\kappa^+)$ and thus chooses some particular $\bar{\mathcal{W}}_\kappa$ and \bar{F}_κ , and in the second step forces with $\mathcal{Q}_\kappa(\bar{\mathcal{W}}_\kappa, \bar{F}_\kappa)$. Otherwise, we let \mathcal{Q}_κ denote the trivial forcing.

Then \mathcal{Q}_κ is $<\kappa$ -directed closed and the set of conditions q in \mathcal{Q}_κ such that $\mathcal{Q}_\kappa(q)$ is κ^+ -cc is dense. In particular, forcing with \mathcal{Q}_κ preserves all cofinalities and \mathcal{Q}_κ -generic extensions have the κ^+ -cover property (every set of ordinals of cardinality at most κ in a \mathcal{Q}_κ -generic extension is covered by a set of cardinality at most κ that is contained in the ground model).

Proof of Theorem 1.2. Assume that the SCH holds. Let \mathbb{P} denote the class-iteration with Easton support which is trivial at stage κ unless κ is an uncountable cardinal satisfying $\kappa^{<\kappa} = \kappa$ in the ground model, in which case we force with the partial order \mathcal{Q}_κ of the \mathbb{P}_κ -generic extension at stage κ .

By induction, it follows that for every uncountable cardinal κ , the partial order \mathbb{P}_κ has cardinality at most $\kappa^{<\kappa}$ and the partial order $\mathbb{P}_{\kappa+1}$ has the $(\kappa^{<\kappa})^+$ -cover property. Moreover, if κ is an uncountable cardinal such that $\mathbb{P}_{\kappa+1}$ satisfies the κ^+ -cover property, then the corresponding tail forcing $\mathbb{P}^{\kappa+1}$ is $<(\kappa^{<\kappa})^+$ -directed closed in every $\mathbb{P}_{\kappa+1}$ -generic extension by [4, Proposition 7.12]. Since this implies that either \mathbb{P} is equivalent to a set-sized forcing or there are unboundedly many cardinals κ such that the corresponding tail forcing $\mathbb{P}^{\kappa+1}$ is $<(\kappa^{<\kappa})^+$ -closed in $\mathbb{P}_{\kappa+1}$ -generic extensions, we can conclude that forcing with \mathbb{P} preserves ZFC.

Assume, towards a contradiction, that forcing with \mathbb{P} changes the cofinality of a regular cardinal. Let ν be minimal such that \mathbb{P}_ν changes such a cofinality. By the definition of \mathbb{P} , our assumption implies that ν is a limit of cardinals λ with $\lambda^{<\lambda} = \lambda$. By the above remarks, this shows that forcing with \mathbb{P}_ν preserves cofinalities less than or equal to ν and greater than $\nu^{<\nu}$. This implies that $\nu^{<\nu} > \nu$ and we can conclude that ν is a singular cardinal with $\nu^{<\nu} = \nu^+$. By our assumptions, there is a $\lambda < \nu$ with $\lambda^{<\lambda} = \lambda$ such that \mathbb{P}_ν forces the cofinality of $\kappa = \nu^+$ to be smaller than λ . This yields a contradiction, because our assumption implies that κ is regular in every $\mathbb{P}_{\lambda+1}$ -generic extension and the above computations show that $\mathbb{P}^{\lambda+1}$ is $<\lambda$ -closed in all such extensions.

Fix an infinite cardinal κ . If $\kappa = \omega$, then forcing with \mathbb{P} preserves the value of 2^κ , because \mathbb{P} is σ -closed. From now on, assume that κ is uncountable and let ν denote the least upper bound of all cardinals $\lambda \leq \kappa$ with $\lambda^{<\lambda} = \lambda$. First, assume that $\nu^{<\nu} = \nu$. Then the partial orders $\mathbb{P}_{\kappa+1}$ and $\mathbb{P}_{\nu+1}$ are equivalent, \mathbb{P}_ν has cardinality at most κ and forcing with \mathcal{Q}_ν over any \mathbb{P}_ν -generic extension preserves the value of 2^κ . In particular, forcing with \mathbb{P} preserves the value of 2^κ in this case. Now, assume that $\nu^{<\nu} > \nu$ and $\kappa = \nu$. By the definition of ν , this implies that κ is a singular strong limit cardinal and there is a cardinal $\lambda < \kappa$ with $\text{cof}(\kappa) < \lambda = \lambda^{<\lambda}$. Since we know that \mathbb{P}_λ has cardinality less than κ in this case, forcing with $\mathbb{P}_{\lambda+1}$ preserves the value of $2^\kappa = \kappa^\lambda$ and forcing with $\mathbb{P}^{\lambda+1}$ over any $\mathbb{P}_{\lambda+1}$ -generic extension also preserves this value, we can conclude that forcing with \mathbb{P} preserves the value of 2^κ in this case. Finally, assume that $\nu^{<\nu} > \nu$ and $\nu < \kappa$. Then ν is a singular cardinal and the forcings $\mathbb{P}_{\kappa+1}$ and \mathbb{P}_ν are equivalent. By the above computations and our assumptions, \mathbb{P}_ν has cardinality $\nu^{<\nu} = \nu^+ \leq \kappa$ and this shows that forcing with \mathbb{P} also preserves the value of 2^κ in this case.

Let G be \mathbb{P} -generic over \mathbf{V} , κ be an uncountable cardinal with $\kappa^{<\kappa} = \kappa$ in $\mathbf{V}[G]$, G_κ be the filter on \mathbb{P}_κ induced by G and K be the filter on $\mathcal{Q}_\kappa^{\mathbf{V}[G_\kappa]}$ induced by G . By the above computations, $\kappa^{<\kappa} = \kappa$ holds both in \mathbf{V} and in $\mathbf{V}[G_\kappa]$ and there is a lightface definable well-order of $H(\kappa^+)$ in $\mathbf{V}[G_\kappa, K]$ by Theorem 1.1. Since the above computations show that $H(\kappa^+)^{\mathbf{V}[G]} = H(\kappa^+)^{\mathbf{V}[G_\kappa, K]}$, we can conclude that there is such a well-order in $\mathbf{V}[G]$. \square

7 Large Cardinal Preservation

When we force with the class-iteration \mathbb{P} constructed in the proof of Theorem 1.2 over a model of the SCH, various large cardinals can be preserved using (sometimes slight variations of) arguments to be found in the literature. We will state some of the resulting lemmas and mostly refer to the relevant articles for the proofs. Note that it is immediate that \mathbb{P} preserves the strong inaccessibility of all strongly inaccessible cardinals. Throughout this section, we assume that the SCH holds.

Lemma 7.1. *Assume that κ is λ -supercompact with $\kappa \leq \lambda$, $\lambda^{<\lambda} = \lambda$ and $2^\lambda = \lambda^+$. Then forcing with \mathbb{P} preserves the λ -supercompactness of κ .*

Proof-Sketch: The proof of this lemma is essentially as for [2, Theorem 4.1], noting that below any condition $p \in \mathbb{P}_\lambda$, there is $q \leq p$ such that q chooses \mathcal{W}_ν and \bar{F}_ν for every ν with $\theta \leq \nu \leq \lambda$ where we perform nontrivial forcing, with θ the largest inaccessible $\leq \lambda$. This implies that $\mathbb{P}(q)_\lambda$ has a dense subset of size λ and also $\mathcal{Q}_\lambda(q(\lambda))$ is sufficiently small (in a sense specified in (1) in the statement of [2, Theorem 4.1]) for the proof of [2, Theorem 4.1] to go through. Moreover one has to verify (in a straightforward way) that a suitable adaption of (2) from [2, Theorem 4.1] holds for $\mathcal{Q}_\lambda(\bar{\mathcal{W}}_\lambda, \bar{F}_\lambda)$.²⁴ \square

Under sufficient GCH hypothesis, this shows that \mathbb{P} preserves all supercompact cardinals as in [2]. Note however that one may well be in a situation where

²⁴Since we consider the present lemma to be rather weak (it requires instances of the GCH to hold while our interest here is to work in a non-GCH context), we do not want to go into any further details of its proof but rather concentrate on other notions of large cardinals in the following.

no λ as above exists. Thus we do not know whether it can be shown that \mathbb{P} preserves supercompact cardinals in general. Our below results however will show (under stronger large cardinal assumptions) that it is consistent for supercompact cardinals to exist after forcing with \mathbb{P} . The next two lemmas are based on large cardinal preservation results presented in [5].

Lemma 7.2. *Given a hyperstrong cardinal κ , there is a condition in \mathbb{P} forcing that the hyperstrength of κ is preserved. The same is true with hyperstrength replaced (in both the assumption and conclusion of the above statement) by n -superstrength for any n with $2 \leq n < \omega$.*

Proof-Sketch: Exactly as in the proof of [5, Theorem 9] (note that no GCH assumption is either made or needed there), working below a condition in \mathbb{P} that makes appropriate choices in its lottery components. \square

Lemma 7.3. *Given an ω -superstrong cardinal κ , there is a condition in \mathbb{P} forcing that the ω -superstrength of κ is preserved.*

Proof-Sketch: Essentially as in the proof of [5, Theorem 2]. Let $j: \mathbf{V} \rightarrow \mathbf{M}$ be the embedding witnessing that κ is ω -superstrong. Note that \mathbb{P} is trivial at the singular cardinal $j^\omega(\kappa)$ and therefore the tail of the iteration starting from $j^\omega(\kappa)$ is $(j^\omega(\kappa))^+$ -closed (and therefore one can use the argument of [5, Lemma 3] to generate the tail of the generic starting from $j^\omega(\kappa)$). However for the proof of [5, Lemma 4] to go through, one needs to choose (slightly different to the proof in [5], letting $G_{j^\omega(\kappa)}$ denote the $\mathbb{P}_{j^\omega(\kappa)}$ -generic filter as in [5]) $p \in G_{j^\omega(\kappa)}$ such that p reduces $f(\bar{a})$ below $j^{n+1}(\kappa)$ whenever \bar{a} belongs to $V_{j^n(\kappa)}$ and $f(\bar{a})$ is open dense on $\mathbb{P}_{j^\omega(\kappa)}$.²⁵ That such p exists is a standard reduction argument using that $V_{j^n(\kappa)}$ has size $j^n(\kappa)$, that $\mathbb{P}_{j^{n+1}(\kappa)}$ is $j^{n+1}(\kappa)$ -cc and that the iteration \mathbb{P} starting from $j^{n+1}(\kappa)$ is $j^{n+1}(\kappa)$ -closed.²⁶ \square

Note that starting with a proper class of large cardinals of any of the above kinds (i.e. hyperstrong or n -superstrong for $2 \leq n \leq \omega$) the arguments in [5] in fact show that the relevant large cardinal property is preserved for class-many of the given large cardinals by an easy density argument.²⁷

The above leaves out one type of large cardinal treated in [5] for which the corresponding arguments seem not to work in our present context, namely superstrong cardinals. As is the case with supercompacts, we do not know whether superstrong cardinals can be preserved by the forcing in general.

8 Other global Iterations

In Section 6, we provided a class sized iteration that introduces a lightface definable well-order of $H(\kappa^+)$ whenever this is possible by the methods developed in Section 5. We could however define *sparser* iterations that only introduce lightface definable well-orders of $H(\kappa^+)$ for certain κ . This can allow for better

²⁵This means that for every relevant \bar{a} , there is an open dense subset $f^*(\bar{a})$ of $\mathbb{P}_{j^{n+1}(\kappa)}$ such that whenever $q \leq p$ is such that $q \restriction j^{n+1}(\kappa) \in f^*(\bar{a})$, then $q \in f(\bar{a})$.

²⁶For example such a reduction argument is performed, in a somewhat more complicated context, in [6, Claim 23].

²⁷That is, if ϕ denotes the large cardinal property in question, for every $p \in \mathbb{P}$ and every ordinal δ , there is $\kappa > \delta$ with $\phi(\kappa)$ and $q \leq p$ such that $q \restriction \phi(\kappa)$.

results concerning the preservation of large cardinals. For example, one could use this to give an alternative proof of the main result of [6] (using the large cardinal preservation techniques of [6]) by using an iteration \mathbb{P} similar to that of Section 6, that only forces with \mathcal{Q}_κ at stage κ if κ is inaccessible. We will restate this result here.

Theorem 8.1 (Friedman-Holy-Lücke, [6]). *Assume that the SCH holds at singular fixed points of the \beth -function. There is a class sized notion of forcing \mathbb{P} such that the following hold.*

- (i) *Forcing with \mathbb{P} introduces a lightface definable well-order of $H(\kappa^+)$ for every inaccessible κ .*
- (ii) *\mathbb{P} is cofinality-preserving and preserves the continuum function.*
- (iii) *\mathbb{P} preserves the supercompactness of all supercompact cardinals.*
- (iv) *If κ is ω -superstrong then there is a condition in \mathbb{P} that forces κ to remain ω -superstrong.*

9 Open Questions

The results on large cardinal preservation leave the following open.

Question 9.1. *Assume the SCH holds. Is there a ZFC-preserving class forcing \mathbb{P} so that*

- *\mathbb{P} preserves cofinalities and the continuum function,*
- *\mathbb{P} introduces a lightface definable well-order of $H(\kappa^+)$ whenever $\kappa \geq \omega_1$ is such that $\kappa^{<\kappa} = \kappa$ and*
- *\mathbb{P} preserves the supercompactness of all supercompact cardinals?*

The results of this paper strongly suggest the following question. However to even partially answer it seems to require completely new techniques.

Question 9.2. *Is there an analogue of Theorem 1.1 in case $\kappa^{<\kappa} = \kappa$ fails?*

In [9], the second and third authors show that under strong assumptions on the ground model it is possible to introduce a lightface definable wellorder of certain $H(\kappa^+)$ of very low complexity, namely Σ_1 using only κ as parameter, by a very well-behaved forcing (more detail is provided in the last paragraph of Section 2 of the present paper). As is remarked in [9], results of Woodin show that such strong assumptions are indeed necessary in case $\kappa = \omega_1$, for the existence of infinitely many Woodin cardinals with a measurable cardinal above implies that $H(\omega_2)$ cannot have a wellordering which is Σ_1 -definable over $H(\omega_2)$, using only κ as parameter. We do not know whether an analogous result holds for larger κ as well. An answer to the following question, that is also posed in [9], would thus be very interesting.

Question 9.3. *Given a regular cardinal $\kappa > \omega_1$, does the existence of a wellorder of $H(\kappa^+)$ that is definable over $H(\kappa^+)$ by a Σ_1 -formula with parameter κ imply that there is no supercompact cardinal above κ ?*

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