Non-absoluteness of Hjorth's cardinal characterization

by

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Abstract. In [J. Math. Log. 2 (2002), 113–144], Hjorth proved that for every countable ordinal α, there exists a complete $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ_α that has models of all cardinalities less than or equal to $\aleph_\alpha$, but no models of cardinality $\aleph_{\alpha+1}$. Unfortunately, his solution does not yield a single $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ_α, but a set of $\mathcal{L}_{\omega_1,\omega}$-sentences, one of which is guaranteed to work. It was conjectured in [Notre Dame J. Formal Logic 55 (2014), 593–551] that it is independent of the axioms of ZFC which of these sentences has the desired property.

In the present paper, we prove that this conjecture is true. More specifically, we isolate a diagonalization principle for functions from ω_1 to ω_1 which is a consequence of the *Bounded Proper Forcing Axiom* (BPFA) and then we use this principle to prove that Hjorth's solution to characterizing $\aleph_2$ in models of BPFA is different than in models of CH. In addition, we show that large cardinals are not needed to obtain this independence result by proving that our diagonalization principle can be forced over models of CH.

1. Introduction. The present paper contributes to the study of the following model-theoretic concepts:

DEFINITION 1.1.

(1) An $\mathcal{L}_{\omega_1,\omega}$-sentence ψ *characterizes* an infinite cardinal κ if ψ has models in all infinite cardinalities less than or equal to κ, but no models in cardinality κ^+.

(2) A countable model characterizes some cardinal κ if the same is true for its Scott sentence.

We are interested in the following question: Given $\alpha < \omega_1$, is there a complete $\mathcal{L}_{\omega_1,\omega}$-sentence ψ_α that characterizes $\aleph_\alpha$? Although the problem is quite easy to solve if we allow the $\mathcal{L}_{\omega_1,\omega}$-sentence to be incomplete, it poses

[1]

a genuine challenge as stated. The question was answered in the affirmative
by Hjorth, who proved the following theorem.

THEOREM 1.2 ([5, Theorem 1.5]). *For every $\alpha < \omega_1$, there exists a complete $\mathcal{L}_{\omega_1,\omega}$-sentence that characterizes $\aleph_\alpha$.*

Unfortunately, Hjorth's solution is unsatisfactory. As observed in [2], for
every countable ordinal α, Hjorth produces not a single $\mathcal{L}_{\omega_1,\omega}$-sentence, but
a whole set S_α of $\mathcal{L}_{\omega_1,\omega}$-sentences ([1]). In [2], the authors notice that if α is a
finite ordinal, then the set S_α is finite. Otherwise, they state that the set S_α of
$\mathcal{L}_{\omega_1,\omega}$-sentences can be chosen to be countable. Below, we will argue that it is
possible to find a finite set S_α of $\mathcal{L}_{\omega_1,\omega}$-sentences with the desired properties
for every countable ordinal α. Hjorth's proof then shows that at least one of
the sentences in the set S_α characterizes the cardinal $\aleph_\alpha$, but it provides no
evidence which element of S_α has this property.

To see why this is the case, we briefly explain Hjorth's construction behind Theorem 1.2. First, assume that for some countable ordinal α, there
is a countable model $\mathcal{M}$ whose Scott sentence characterizes $\aleph_\alpha$. Working by
induction, Hjorth wants to create another countable model whose Scott sentence characterizes $\aleph_{\alpha+1}$. To achieve this, he first defines a countable model,
which he calls $(\mathcal{M}, \mathcal{N})$-*full*, using $\mathcal{M}$ and what we will call the *first Hjorth
construction* ([2]). Hjorth proves that the Scott sentence of this $(\mathcal{M}, \mathcal{N})$-full
model characterizes either $\aleph_\alpha$ or $\aleph_{\alpha+1}$. If the latter is the case, we are done.
Otherwise, Hjorth proceeds one more round to use the $(\mathcal{M}, \mathcal{N})$-full model
from the first step and what we will call the *second Hjorth construction*.
If the $(\mathcal{M}, \mathcal{N})$-full model characterizes some κ, then Hjorth's second construction characterizes κ^+. In particular, if the $(\mathcal{M}, \mathcal{N})$-full model characterizes $\aleph_\alpha$, then the second Hjorth construction produces a model that
characterizes $\aleph_{\alpha+1}$. Notice here that the failure of the $(\mathcal{M}, \mathcal{N})$-full model to
characterize $\aleph_{\alpha+1}$ is used to prove that the second Hjorth construction does
indeed characterize $\aleph_{\alpha+1}$. In either case, there exists some $\mathcal{L}_{\omega_1,\omega}$-sentence
that characterizes $\aleph_{\alpha+1}$, and the induction step is complete.

At limit stages, Hjorth takes disjoint unions of the previously constructed
models. For instance for $\alpha = \omega$, Hjorth considers the disjoint union of countable models $\mathcal{M}_n$, $n < \omega$, where each $\mathcal{M}_n$ characterizes $\aleph_n$. This union characterizes $\aleph_\omega$, but, as we mentioned, we do not know what the models $\mathcal{M}_n$
are.

Since at successor stages we have to choose between the first and the
second Hjorth construction and we repeat this process for every countable

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([1]) There is no mention of a set S_α in Hjorth's original proof. This notation was
introduced in [2].

([2]) In Hjorth's proof, the existence of the model $\mathcal{M}$ is an assumption. The existence
of $\mathcal{N}$ is something that comes out of the proof.

successor ordinal, the result is a binary tree of $\mathcal{L}_{\omega_1,\omega}$-sentences of height ω_1. The αth level of the tree gives us the set S_α. In particular, at least one of the sentences at the αth level of the tree characterizes $\aleph_\alpha$.

We now briefly observe that we can do slightly better for $\alpha = \omega$ and for each countable limit ordinal α in general. Consider the following countable models: $\mathcal{M}_0$ is a countable model which characterizes $\aleph_0$, and $\mathcal{M}_{n+1}$ is the second Hjorth construction which inductively uses $\mathcal{M}_n$ as input. If $\mathcal{M}_n$ characterizes some $\aleph_m$, then we mentioned that the first Hjorth construction characterizes either $\aleph_m$ or $\aleph_{m+1}$. It follows that the second Hjorth construction characterizes either $\aleph_{m+1}$ or $\aleph_{m+2}$. Therefore, we can prove inductively that for each $n < \omega$, the model $\mathcal{M}_{n+1}$ characterizes some $\aleph_k$ for $n < k \leq 2(n+1)$. Although we will prove that the value of k is independent of ZFC, the disjoint union of the $\mathcal{M}_n$'s always characterizes $\aleph_\omega$. In other words, we can isolate one $\mathcal{L}_{\omega_1,\omega}$-sentence that belongs to the set S_ω and provably characterizes $\aleph_\omega$. A similar argument applies to all countable limit ordinals α: There exists one $\mathcal{L}_{\omega_1,\omega}$-sentence that belongs to the set S_α and provably characterizes $\aleph_\alpha$. This greatly reduces the complexity of the tree whose levels are the S_α's, as we can assume that for limit α the set S_α is a singleton.

The problem for successor α remains, and in [10], it was conjectured that it is independent of the axioms of ZFC whether the Scott sentence of the $(\mathcal{M}, \mathcal{N})$-full model characterizes $\aleph_\alpha$ or $\aleph_{\alpha+1}$. This would imply that it is independent of ZFC whether the first or the second Hjorth construction characterizes $\aleph_{\alpha+1}$. Some evidence towards the validity of this conjecture was given by the following result.

THEOREM 1.3 ([10, Theorem 2.20]). *If $\mathcal{M}$ is a countable model that characterizes $\aleph_\alpha$ and $\aleph_\alpha^\omega = \aleph_\alpha$, then there is no $(\mathcal{M}, \mathcal{N})$-full structure of size $\aleph_{\alpha+1}$.*

In contrast, for $\alpha = 0$, Hjorth proves that the $(\mathcal{M}, \mathcal{N})$-full model characterizes $\aleph_1$. The proof works both under CH and its negation, but it uses results from descriptive set theory that cannot be used to prove the statement for cardinals bigger than $\aleph_1$.

The purpose of this paper is to prove the above conjecture by showing that the axioms of ZFC do not answer the given question for $\alpha = 1$. By Theorem 1.3, it is relatively consistent that for every countable model $\mathcal{M}$ that characterizes $\aleph_1$, there is no $(\mathcal{M}, \mathcal{N})$-full structure of size $\aleph_2$. In the following, we will prove that the negation of this statement is also relatively consistent. In Section 2, we discuss a result from [10] that shows that the existence of an $(\mathcal{M}, \mathcal{N})$-full structure of size $\aleph_{\alpha+1}$ is equivalent to the existence of a coloring of the two-element subsets of $\omega_{\alpha+1}$ with $\aleph_\alpha$-many colors that possesses certain almost disjointness and genericity properties. In Section 3,

we isolate a combinatorial principle ($⊿$) and prove that it is a consequence of the *Bounded Proper Forcing Axiom* (see [4]). In Section 4, we then show that the consistency of ZFC + ($⊿$) can be established from the consistency of ZFC alone ([3]) by showing that the principle ($⊿$) can be forced over models of CH. Finally, in Section 5, we prove that the principle ($⊿$) implies the existence of an ($\mathcal{M}, \mathcal{N}$)-full structure of size $\aleph_2$. We then end the paper by discussing possibilities to characterize cardinals in an *absolute* way and we propose two ways to formulate this concept in a mathematically sound way.

2. A reformulation. In the following, we recall the statement of a result in [10] which provides an equivalent condition to the existence of an ($\mathcal{M}, \mathcal{N}$)-full model of size $\aleph_{\alpha+1}$. Since it is easier to work with this equivalent condition, we will use it for the rest of this paper.

We start with some notation conventions. Given a set d, we let $[d]^2$ denote the set of all two-element subsets of d, and we let $[d]^{<\omega}$ denote the set of all finite subsets of d. Moreover, if c is a function whose domain is of the form $[d]^2$ for some set d, then we abbreviate $c(\{x, y\})$ by $c(x, y)$. In addition, given such a function c with domain $[d]^2$ and $x, y \in d$ with $x \neq y$, we define

$$\mathcal{A}^c_{x,y} = \big\{ z \in d \setminus \{x, y\} \mid c(x, z) = c(y, z) \big\}$$

to be the corresponding *set of agreements*. Finally, given sets $d \subseteq d'$, a function c with domain $[d]^2$ and a function c' with domain $[d']^2$ and $c' {\restriction} [d]^2 = c$, we say that c' *introduces no new agreement over* c if $\mathcal{A}^c_{x,y} = \mathcal{A}^{c'}_{x,y}$ holds for all $x, y \in d$.

LEMMA 2.1 ([10, Theorem 5.1]). *Assume that $\mathcal{M}$ is a countable model that characterizes $\aleph_\alpha$. Then the following statement are equivalent:*

(1) *There exists an ($\mathcal{M}, \mathcal{N}$)-full structure of cardinality $\aleph_{\alpha+1}$.*
(2) *There exists a function $c : [\omega_{\alpha+1}]^2 \to \omega_\alpha$ and a function $r : \omega_{\alpha+1} \to \omega_{\alpha+1}$ with the following properties:*

    (a) *(Finite agreement) For all $\beta < \gamma < \omega_{\alpha+1}$, the set $\mathcal{A}^c_{\beta,\gamma}$ is finite ([4]).*

    (b) *(Finite closure) For every $a \in [\omega_{\alpha+1}]^{<\omega}$, there is a $a \subseteq b \in [\omega_{\alpha+1}]^{<\omega}$ that is closed under $\mathcal{A}^c$, i.e. for all $\beta, \gamma \in b$ with $\beta \neq \gamma$, we have $\mathcal{A}^c_{\beta,\gamma} \subseteq b$.*

    (c) *(Finite extension) If d is a finite set and $e : [d]^2 \to \omega_\alpha$ is a function with $e {\restriction} [d \cap \omega_{\alpha+1}]^2 = c {\restriction} [d \cap \omega_{\alpha+1}]^2$ that introduces no new agreements over $c {\restriction} [d \cap \omega_{\alpha+1}]^2$, then there exists an injection $\iota : d \to \omega_{\alpha+1}$ with $\iota {\restriction} (d \cap \omega_{\alpha+1}) = \mathrm{id}_{d \cap \omega_{\alpha+1}}$ and $e(\beta, \gamma) = c(\iota(\beta), \iota(\gamma))$ for all $\beta, \gamma \in d$ with $\beta \neq \gamma$.*

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[3] Note that the results of [4] show that the consistency strength of BPFA lies strictly between the existence of an inaccessible cardinal and the existence of a Mahlo cardinal.

[4] Note that we allow the possibility that the set $\mathcal{A}^c_{\beta,\gamma}$ is empty.

(d) (Coloring) *The statement* (c) *holds true even if d is* colored, *i.e. if, in addition, there exists a function* $s : d \to \omega_{\alpha+1}$ *with* $s{\upharpoonright}(d \cap \omega_{\alpha+1}) = r{\upharpoonright}(d \cap \omega_{\alpha+1})$, *then there exists an injection* $\iota : d \to \omega_{\alpha+1}$ *with the above properties that also satisfies* $s(\beta) = r(\iota(\beta))$ *for all* $\beta \in d$.

REMARKS 2.2. (1) The requirement in (c) that "e introduces no new agreements over $c{\upharpoonright}[d \cap \omega_{\alpha+1}]^2$" was erroneously omitted in [10], but is needed for the equivalence.

(2) Since $(\mathcal{M}, \mathcal{N})$-full structures are defined as Fraïssé limits, they are sufficiently *generic*. The finite extension property (c) is an expression of this genericity.

(3) It follows from (c) that the function c is surjective. Given $\beta < \omega_\alpha$, consider the unique function $e : [\{\omega_{\alpha+1}, \omega_{\alpha+1} + 1\}]^2 \to \omega_\alpha$ with the property that $e(\omega_{\alpha+1}, \omega_{\alpha+1} + 1) = \beta$. Then the assumptions of (c) are satisfied and we find an injection $\iota : \{\omega_{\alpha+1}, \omega_{\alpha+1} + 1\} \to \omega_{\alpha+1}$ with the property that $c(\iota(\omega_{\alpha+1}), \iota(\omega_{\alpha+1} + 1)) = \beta$.

(4) The existence of the coloring function $r : \omega_{\alpha+1} \to \omega_{\alpha+1}$ is important for Hjorth's argument, but once one has constructed a function $c : [\omega_{\alpha+1}]^2 \to \omega_\alpha$ satisfying the statements (a)–(c) of the above theorem, it is easy to modify this function to also obtain a function $r : \omega_{\alpha+1} \to \omega_{\alpha+1}$ such that statement (d) holds too.

3. A diagonalization principle. We now introduce and study a diagonalization principle for families of functions from ω_1 to ω_1 that will be central for the independence results of this paper. In Section 5, we will use Lemma 2.1 to prove that this principle implies the existence of an $(\mathcal{M}, \mathcal{N})$-full structure of size $\aleph_2$.

DEFINITION 3.1.

(1) Given a set X, we say that a map $m : [X]^{<\omega} \to [X]^{<\omega}$ is *monotone* if $a \subseteq m(a)$ holds for every finite subset a of X.
(2) We let $(\varDelta)$ denote the statement that for every sequence $\langle f_\alpha \mid \alpha < \omega_1 \rangle$ of functions from ω_1 to ω_1, every finite subset F of ω_1 and every monotone function $m : [\omega_1]^{<\omega} \to [\omega_1]^{<\omega}$, there exists a function $g : \omega_1 \to \omega_1$ such that $F \cap \mathrm{ran}(g) = \emptyset$ and for every $a \in [\omega_1]^{<\omega}$, there exists $a \subseteq b \in [\omega_1]^{<\omega}$ with
$$\{\beta < \omega_1 \mid f_\alpha(\beta) = g(\beta)\} \subseteq m(b) \quad \text{for all } \alpha \in m(b).$$

A short argument shows that $(\varDelta)$ is not provable in ZFC:

PROPOSITION 3.2. *If* $(\varDelta)$ *holds, then* $2^{\aleph_0} > \aleph_1$.

Proof. If $2^{\aleph_0} = \aleph_1$, then there is a sequence $\langle f_\alpha : \omega_1 \to \omega \mid \alpha < \omega_1 \rangle$ of functions with the property that the set $\{f_\alpha{\upharpoonright}\omega \mid \alpha < \omega_1\}$ contains all

functions from ω to ω. In particular, for every function $g : \omega_1 \to \omega_1$, there exists $\alpha < \omega_1$ with $g(n) = f_\alpha(n)$ for all $n < \omega$. ∎

In addition, it is possible to use results of Baumgartner [3] to show that $(⊿)$ is also not a theorem of ZFC + ¬CH:

PROPOSITION 3.3. *If* CH *holds and* G *is* $\mathrm{Add}(\omega, \omega_2)$-*generic over* V, *then* $(⊿)$ *fails in* V[G].

Proof. Work in V[G] and assume, towards a contradiction, that $(⊿)$ holds. In this situation, we can construct a sequence $\langle f_\gamma : \omega_1 \to \omega_1 \mid \gamma < \omega_2 \rangle$ of functions such that for all $\delta < \gamma < \omega_2$, the set $\{\alpha < \omega_1 \mid f_\gamma(\alpha) = f_\delta(\alpha)\}$ is finite. By considering the graphs of these functions und using a bijection between $\omega_1 \times \omega_1$ and ω_1, we can now construct a sequence $\langle A_\gamma \mid \gamma < \omega_2 \rangle$ of unbounded subsets of ω_1 with the property that for all $\delta < \gamma < \omega_2$, the set $A_\gamma \cap A_\delta$ is finite. But this contradicts results in [3, Section 6] that show that no sequence of subsets with these properties exists in V[G]. ∎

In the remainder of this section, we use results of Larson to prove that the principle $(⊿)$ is a consequence of forcing axioms. These arguments rely on the following forcing notion defined in [7, Section 6]:

DEFINITION 3.4. We let $\mathbb{D}$ denote the partial order defined by the following clauses:

(1) A condition in $\mathbb{D}$ is a triple $p = \langle a_p, \mathscr{F}_p, \mathscr{X}_p \rangle$ such that the following statements hold:

   (a) a_p is a function from a finite subset d_p of ω_1 into ω_1.
   (b) $\mathscr{F}_p$ is a finite set of functions from ω_1 to ω_1.
   (c) $\mathscr{X}_p$ is a finite $\in$-chain of countable elementary submodels of $\mathrm{H}(\omega_2)$.
   (d) If $X \in \mathscr{X}_p$ and $\alpha \in d_p \cap X$, then $a_p(\alpha) \in X$.
   (e) If $X \in \mathscr{X}_p$, $\alpha \in d_p \setminus X$ and $f \in X$ is a function from ω_1 to ω_1, then $a_p(\alpha) \neq f(\alpha)$.

(2) Given conditions p and q in $\mathbb{D}$, we have $p \leq_{\mathbb{D}} q$ if and only if the following statements hold:

   (a) $d_q \subseteq d_p$, $a_q = a_p{\restriction}d_q$, $\mathscr{F}_q \subseteq \mathscr{F}_p$ and $\mathscr{X}_q \subseteq \mathscr{X}_p$.
   (b) If $\alpha \in d_p \setminus d_q$ and $f \in \mathscr{F}_q$, then $a_p(\alpha) \neq f(\alpha)$.

Given $\alpha < \omega_1$, we define D_α to be the set of all conditions p in $\mathbb{D}$ with $\alpha \in d_p$.

PROPOSITION 3.5. *If* q *is a condition in* $\mathbb{D}$ *and* $\alpha < \omega_1$, *then there is a condition* p *in* D_α *with* $p \leq_{\mathbb{D}} q$, $\mathscr{F}_p = \mathscr{F}_q$ *and* $\mathscr{X}_p = \mathscr{X}_q$.

Proof. First, assume that $\alpha \in X$ for some $X \in \mathscr{X}_q$. Let $Y \in \mathscr{X}_q$ be $\in$-minimal with this property. Then there exists $\beta \in Y \cap \omega_1$ with $f(\alpha) \neq \beta$ for

all $f \in \mathscr{F}_q$ and $g(\alpha) < \beta$ for all $X \in \mathscr{X}_q \cap Y$ and every function $g : \omega_1 \to \omega_1$ in X. Define
$$p = \langle a_q \cup \{\langle \alpha, \beta \rangle\}, \mathscr{F}_q, \mathscr{X}_q \rangle.$$
Then p is a condition in $\mathbb{D}$ that is an element of D_α. Moreover, this construction ensures that $p \leq_{\mathbb{D}} q$ holds.

Now, assume that $\alpha \notin X$ for all $X \in \mathscr{X}_q$. Pick $\beta < \omega_1$ with $f(\alpha) \neq \beta$ for all $f \in \mathscr{F}_q$ and $g(\alpha) < \beta$ for all $X \in \mathscr{X}_q$ and every function $g : \omega_1 \to \omega_1$ in X. If we define p as above, then we again obtain a condition in D_α below q. ∎

Next, for every function $f : \omega_1 \to \omega_1$, we let D_f denote the set of all conditions p in $\mathbb{D}$ with $f \in \mathscr{F}_p$.

PROPOSITION 3.6. *If q is a condition in $\mathbb{D}$ and $f : \omega_1 \to \omega_1$ is a function, then $\langle a_q, \mathscr{F}_q \cup \{f\}, \mathscr{X}_q \rangle$ is a condition in $\mathbb{D}$ below q that is an element of D_f.*

LEMMA 3.7 ([7, Theorem 6.2]). *The partial order $\mathbb{D}$ is proper.*

We are now ready to show that ($\vartriangleleft$) is a consequence of BPFA. Our proof relies on the following classical result of Bagaria that characterizes the validity of BPFA in terms of generic absoluteness.

THEOREM 3.8 ([1, Theorem 5]). *The following statements are equivalent:*

(1) BPFA *holds.*
(2) *If $\varphi(v)$ is a Σ_1-formula* ([5]), *z is an element of $\mathrm{H}(\omega_2)$, $\mathbb{P}$ is a proper forcing and p is a condition in $\mathbb{P}$ with $p \Vdash_{\mathbb{P}} \varphi(\check{z})$, then $\varphi(z)$ holds.*

THEOREM 3.9. BPFA *implies that ($\vartriangleleft$) holds.*

Proof. Assume that BPFA holds and fix a sequence
$$\vec{f} = \langle f_\alpha : \omega_1 \to \omega_1 \mid \alpha < \omega_1 \rangle$$
of functions, a finite subset F of ω_1 and a monotone function $m : [\omega_1]^{<\omega} \to [\omega_1]^{<\omega}$. Given $\alpha < \omega_1$, let $c_\alpha : \omega_1 \to \omega_1$ denote the constant function with value α. Moreover, define $p_F = \langle \emptyset, \{c_\alpha \mid \alpha \in F\}, \emptyset \rangle$. Then p_F is a condition in $\mathbb{D}$. Finally, given $d \in [\omega_1]^{<\omega}$, let E_d denote the set of all conditions p in $\mathbb{D}$ with the property that there exists $d \subseteq e \in [\omega_1]^{<\omega}$ with $d_p = m(e)$ and $f_\alpha \in \mathscr{F}_p$ for all $\alpha \in m(e)$.

CLAIM. *For every $d \in [\omega_1]^{<\omega}$, the set E_d is dense in $\mathbb{D}$.*

Proof of the Claim. Fix a condition r in $\mathbb{D}$ and set $e = d \cup d_r \in [\omega_1]^{<\omega}$. Since $d_r \subseteq e \subseteq m(e)$, we can now use Proposition 3.5 to find a condition q in $\mathbb{D}$ with $q \leq_{\mathbb{D}} r$ and $d_q = m(e)$. Finally, an application of Proposition 3.6

([5]) See [6, p. 5] for the definition of the *Levy hierarchy* of formulas. Note that, using a *universal* Σ_1-formula, it is possible to phrase this statement as a single sentence in the language of set theory.

yields a condition p in $\mathbb{D}$ with $p \leq_{\mathbb{D}} q$, $d_p = d_q = m(e)$ and $f_\alpha \in \mathscr{F}_p$ for all $\alpha \in m(e)$. We then have $p \leq_{\mathbb{D}} r$ and $p \in E_d$. ∎

Now, let G be $\mathbb{D}$-generic over the ground model V with $p_F \in G$. Work in V[G] and define $g = \bigcup \{a_p \mid p \in G\}$. Then Proposition 3.5 ensures that g is a function from ω_1 to ω_1.

CLAIM. $F \cap \mathrm{ran}(g) = \emptyset$.

Proof of the Claim. Fix $\alpha < \omega_1$ and $\beta \in F$. By Proposition 3.5, we can find $p \in G$ with $p \leq_{\mathbb{D}} p_F$ and $\alpha \in d_p$. Since $\alpha \notin d_{p_F} = \emptyset$ and $c_\beta \in \mathscr{F}_{p_F}$, the definition of $\mathbb{D}$ ensures that $g(\alpha) = a_p(\alpha) \neq c_\beta(\alpha) = \beta$. ∎

CLAIM. *For every $d \in [\omega_1]^{<\omega}$, there exists $d \subseteq e \in [\omega_1]^{<\omega}$ with*

$$\{\beta < \omega_1 \mid f_\alpha(\beta) = g(\beta)\} \subseteq m(e) \quad \text{for all } \alpha \in m(e).$$

Proof of the Claim. Using our first claim, we can find $q \in E_d \cap G$. Then there exists $d \subseteq e \in [\omega_1]^{<\omega}$ with $d_q = m(e)$. Fix $\alpha \in m(e)$ and $\beta \in \omega_1 \setminus m(e)$. By Proposition 3.5, there exists $p \in G$ with $p \leq_{\mathbb{D}} q$ and $\beta \in d_p$. Since $\beta \in d_p \setminus d_q$ and $f_\alpha \in \mathscr{F}_q$, the definition of $\mathbb{D}$ now implies that $g(\beta) = a_p(\beta) \neq f_\alpha(\beta)$. ∎

The above claims show that, in V[G], there exists a function $g : \omega_1 \to \omega_1$ with $F \cap \mathrm{ran}(g) = \emptyset$ and the property that for all $d \in [\omega_1]^{<\omega}$, there is $d \subseteq e \in [\omega_1]^{<\omega}$ such that $\{\beta < \omega_1 \mid f_\alpha(\beta) = g(\beta)\} \subseteq m(e)$ for all $\alpha \in m(e)$. Since this statement can be formulated by a Σ_1-formula with parameters $\vec{f}, F, m \in \mathrm{H}(\omega_2)^{\mathrm{V}}$, we can use [1, Theorem 5] to conclude that the given statement also holds in V. ∎

4. Forcing without large cardinals. In this section, we prove the following result that shows that no large cardinals are needed to establish the consistency of the principle ($\varDelta$).

THEOREM 4.1. *If* CH *holds, then there is a proper partial order $\mathbb{P}$ that satisfies the $\aleph_2$-chain condition such that $\mathbb{1}_{\mathbb{P}} \Vdash (\varDelta)$.*

Following the arguments in [11, Section 4], we now introduce a *matrix version* of Larson's forcing $\mathbb{D}$. We then use the results of [9, Section VIII.2] to show that the constructed partial order possesses the properties listed in Theorem 4.1.

DEFINITION 4.2. We let $\mathbb{E}$ denote the partial order defined by the following clauses:

(1) A condition in $\mathbb{E}$ is a triple $p = \langle a_p, \mathscr{F}_p, t_p \rangle$ such that the following statements hold:

    (a) a_p is a function from a finite subset d_p of ω_1 into ω_1.
    (b) $\mathscr{F}_p$ is a finite set of functions from ω_1 to ω_1.

(c) t_p is a function from a finite $\in$-chain $\mathscr{C}_p$ of countable transitive sets to the set of non-empty finite subsets of $H(\omega_2)$.

(d) If $M \in \mathscr{C}_p$ and $X \in t_p(M)$, then X is a countable elementary submodel of $H(\omega_2)$ and M is the transitive collapse of X.

(e) If $M, N \in \mathscr{C}_p$ with $M \in N$ and $X \in t_p(M)$, then there is $Y \in t_p(N)$ with $X \in Y$.

(f) If $M \in \mathscr{C}_p$, $X \in t_p(M)$ and $\alpha \in d_p \cap X$, then $a_p(\alpha) \in X$.

(g) If $M \in \mathscr{C}_p$, $X \in t_p(M)$, $\alpha \in d_p \setminus X$ and $f \in X$ is a function from ω_1 to ω_1, then $a_p(\alpha) \neq f(\alpha)$.

(2) Given conditions p and q in $\mathbb{E}$, we have $q \leq_{\mathbb{E}} p$ if and only if the following statements hold:

(a) $d_p \subseteq d_q$, $a_p = a_q \restriction d_p$, $\mathscr{F}_p \subseteq \mathscr{F}_q$, $\mathscr{C}_p \subseteq \mathscr{C}_q$ and $t_p(M) \subseteq t_q(M)$ for all $M \in \mathscr{C}_p$.

(b) If $\alpha \in d_q \setminus d_p$ and $f \in \mathscr{F}_p$, then $a_q(\alpha) \neq f(\alpha)$.

In order to prove that the partial order $\mathbb{E}$ is proper, we start by showing that Lemma 6.1 of [7] can directly be adapted to the matrix forcing $\mathbb{E}$, using the same proof.

LEMMA 4.3. *Let p be a condition in $\mathbb{E}$ and let D be a subset of $\mathbb{E}$ that is dense below p. Then there exists $\lambda < \omega_1$ with the property that for every finite set $\mathscr{F}$ of functions from ω_1 to ω_1, there exists $q \in D$ below p with $a_q \subseteq \lambda \times \lambda$ and $a_q(\alpha) \neq f(\alpha)$ for all $\alpha \in d_q \setminus d_p$ and $f \in \mathscr{F}$.*

Proof. Assume, towards a contradiction, that the above statement fails. Then there exists a sequence $\langle \mathscr{F}_\lambda \mid \lambda < \omega_1 \rangle$ of finite, non-empty sets of functions from ω_1 to ω_1 with the property that for every $\lambda < \omega_1$ and every $q \in D$ below p with $a_q \subseteq \lambda \times \lambda$, there exists $\alpha_q^\lambda \in d_q \setminus d_p$ and $f_q^\lambda \in \mathscr{F}_\lambda$ with $a_q(\alpha_q^\lambda) = f_q^\lambda(\alpha_q^\lambda)$. Without loss of generality, we may assume that there is $0 < n < \omega$ with $|\mathscr{F}_\lambda| = n$ for all $\lambda < \omega_1$. Given $\lambda < \omega_1$, pick functions $f_0^\lambda, \ldots, f_{n-1}^\lambda$ from ω_1 to ω_1 with $\mathscr{F}_\lambda = \{f_0^\lambda, \ldots, f_{n-1}^\lambda\}$.

Now, fix a uniform ultrafilter U on ω_1. For each $q \in D$ below p, we can now find $\alpha_q \in d_q \setminus d_p$ and $i_q < n$ with the property that

$$A_q = \{\lambda < \omega_1 \mid a_q \subseteq \lambda \times \lambda, \ \alpha_q^\lambda = \alpha_q, \ f_q^\lambda = f_{i_q}^\lambda\} \in U.$$

Then, for all $q, r \in D$ below p with $\alpha_q = \alpha_r$ and $i_q = i_r$, we can find $\lambda \in A_q \cap A_r$, and this allows us to conclude that

$$a_q(\alpha_q) = f_{i_q}^\lambda(\alpha_q) = a_r(\alpha_q).$$

This shows that there are functions $h_0, \ldots, h_{n-1} : \omega_1 \rightarrow \omega_1$ such that $h_{i_q}(\alpha_q) = a_q(\alpha_q)$ for all $q \in D$ below p. Define

$$q = \langle a_p, \mathscr{F}_p \cup \{h_i \mid i < n\}, t_p \rangle.$$

Then it is easy to see that q is a condition in $\mathbb{E}$ below p. Pick $r \in D$ below q. Then $\alpha_r \in d_r \setminus d_p = d_r \setminus d_q$ and $a_r(\alpha_r) = h_{i_r}(\alpha_r)$, contradicting the fact that $r \leq_{\mathbb{E}} q$. ∎

LEMMA 4.4. *Let θ be a sufficiently large regular cardinal and let Z be a countable elementary submodel of $\mathrm{H}(\theta)$. If p is a condition in $\mathbb{E}$ with $\mathrm{H}(\omega_2) \cap Z \in t_p(N)$ for some $N \in \mathscr{C}_p$, then p is a $(Z, \mathbb{E})$-generic condition.*

Proof. Pick a dense subset D of $\mathbb{E}$ that is contained in Z and a condition q in $\mathbb{E}$ below p. Set $\mathscr{C} = \mathscr{C}_q \cap N$. Then $\mathscr{C}$ is a finite $\in$-chain of countable transitive sets. Moreover, the definition of our forcing ensures that for every $M \in \mathscr{C}$, we can find $X \in t_q(M)$ and $Y \in t_q(N)$ with $X \in Y$. In particular, we know that every element of $\mathscr{C}$ is countable in N, and this shows that $\mathscr{C}$ is a subset of Z.

Now, let $|t_q(N)| = k > 0$ and pick sets $Z_0, \ldots, Z_{k-1}$ such that $Z_0 = \mathrm{H}(\omega_2) \cap Z$ and $t_q(N) = \{Z_0, \ldots, Z_{k-1}\}$. Given $i < k$, since N is the transitive collapse of Z_i, there exists a unique isomorphism $\pi_i : \langle Z_i, \in \rangle \to \langle Z_0, \in \rangle$. Next, for all $M \in \mathscr{C}$ and $X \in t_q(M)$, we let $I(X)$ denote the set of all $i < k$ with the property that $X \in Z_i$ and there exists a finite $\in$-chain C of elements of Z_i with the property that for all $M' \in \mathscr{C}$ with $M \in M'$, there exists $X' \in C$ with $X \in X' \in t_q(M')$. Note that, since $\mathscr{C}$ is finite, the definition of $\mathbb{E}$ then ensures that $I(X) \neq \emptyset$ holds for all $M \in \mathscr{C}$ and $X \in t_q(M)$. Define t to be the unique function with domain $\mathscr{C}$ and

$$t(M) = \{\pi_i(X) \mid X \in t_q(M), i \in I(X)\}$$

for all $M \in \mathscr{C}$. Finally, set

$$\bar{q} = \langle a_q \cap Z, \mathscr{F}_q \cap Z, t \rangle.$$

CLAIM. *$\bar{q}$ is a condition in $\mathbb{E}$ that is an element of Z.*

Proof of the Claim. First, fix $M \in \mathscr{C}$ and $X \in t(M)$. Then there are $X' \in t_q(M)$ and $i \in I(X')$ with $X = \pi_i(X')$. Since $X' \subseteq Z_i$ and both sets are elementary submodels of $\mathrm{H}(\omega_2)$, we know that X' is an elementary submodel of Z_i and therefore elementarity implies that X is an elementary submodel of both Z_0 and $\mathrm{H}(\omega_2)$. Moreover, since M is the transitive collapse of X', we can conclude that M is also the transitive collapse of X.

Now, fix $M_0, M_1 \in \mathscr{C}$ with $M_0 \in M_1$ and $X_0 \in t(M_0)$. Then there exists $X' \in t_q(M_0)$ and $i \in I(X')$ with $X = \pi_i(X')$. By the definition of $I(X')$, there exists a finite $\in$-chain C of elements of Z_i with the property that for all $M \in \mathscr{C}$ with $M_0 \in M$, there exists $X \in C$ with $X' \in X \in t_q(M)$. Pick $X'' \in C \cap t_q(M_1)$ and set $X_1 = \pi_i(X'')$. Then the $\in$-chain $\{X \in C \mid X'' \in X\}$ witnesses that $i \in I(X'')$ and hence X_1 is an element of $t(M_1)$ with $X_0 \in X_1$.

Next, pick $M \in \mathscr{C}$, $X \in t(M)$ and $\alpha \in d_q \cap X \cap Z = d_q \cap X$. Then there are $X' \in t_q(M)$ and $i \in I(X')$ with $X = \pi_i(X')$. Since $d_q \cap X = d_q \cap X'$,

we have $M \in \mathscr{C}_q$, $X' \in t_q(M)$ and $\alpha \in d_q \cap X'$. By the definition of $\mathbb{E}$, this implies that $a_q(\alpha) \in X'$, and since $X \cap \omega_1 = X' \cap \omega_1$, we can conclude that $a_q(\alpha) \in X$.

Finally, fix $M \in \mathscr{C}$, $X \in t(M)$, $\alpha \in (d_q \cap Z) \setminus X$ and a function f from ω_1 to ω_1 in X. Pick $X' \in t_q(M)$ and $i \in I(X')$ with $X = \pi_i(X')$. Then the fact that $X \cap \omega_1 = X' \cap \omega_1$ implies that $\alpha \in d_q \setminus X'$. In this situation, the definition of $\mathbb{E}$ and the fact that $\pi_i^{-1} {\restriction} (X \cap \omega_1) = \mathrm{id}_{X \cap \omega_1}$ imply that

$$a_q(\alpha) \neq (\pi_i^{-1}(f))(\alpha) = \pi_i^{-1}(f(\alpha)) = f(\alpha).$$

The above computations show that $\bar{q}$ is a condition in $\mathbb{E}$. Since all relevant sets are finite, the fact that $\mathscr{C}$ is a subset of Z allows us to conclude that $\bar{q}$ is an element of Z. ∎

An application of Lemma 4.3 in Z now yields an ordinal $\lambda \in Z \cap \omega_1$ with the property that for every finite set $\mathscr{F}$ of functions from ω_1 to ω_1, there exists $r \in D$ below $\bar{q}$ with $a_r \subseteq \lambda \times \lambda$ and $a_r(\alpha) \neq f(\alpha)$ for all $\alpha \in d_r \setminus d_{\bar{q}}$ and $f \in \mathscr{F}$. Hence, there exists $r \in D$ below $\bar{q}$ with $a_r \subseteq \lambda \times \lambda$ and $a_r(\alpha) \neq f(\alpha)$ for all $\alpha \in d_r \setminus d_{\bar{q}}$ and $f \in \mathscr{F}_q$. Since $a_r \subseteq \lambda \times \lambda \subseteq Z$, elementarity yields a condition $s \in D \cap Z$ with $a_r = a_s$ and $s \leq_{\mathbb{E}} \bar{q}$. Define $\mathscr{C}_* = \mathscr{C}_q \cup \mathscr{C}_s$ and let t_* denote the unique function with domain $\mathscr{C}_*$ such that $t_*(M) = t_q(M)$ for all $M \in \mathscr{C}_q \setminus \mathscr{C}_s$ and

$$t_*(M) = \{\pi_i^{-1}(X) \mid i < k,\ X \in t_s(M)\}$$

for all $M \in \mathscr{C}_s$. Finally, we set

$$u = \langle a_q \cup a_s, \mathscr{F}_q \cup \mathscr{F}_s, t_* \rangle.$$

CLAIM. *u is a condition in $\mathbb{E}$.*

Proof of the Claim. First, fix $\alpha \in d_q \cap d_s$. Since $d_s = d_r \subseteq \lambda \subseteq Z$, we know that $\alpha \in d_q \cap Z = d_{\bar{q}}$ and therefore the fact that $r \leq_{\mathbb{E}} \bar{q}$ allows us to conclude that

$$a_q(\alpha) = a_{\bar{q}}(\alpha) = a_r(\alpha) = a_s(\alpha).$$

In particular, we know that $a_q \cup a_s$ is a function.

Now, fix $M_0 \in \mathscr{C}_q \setminus \mathscr{C}_s$ and $M_1 \in \mathscr{C}_s \setminus \mathscr{C}_q$. Then $M_0 \notin \mathscr{C}_q \cap N = \mathscr{C}_{\bar{q}} \subseteq \mathscr{C}_s$ and hence $M_0 \notin N$. Since M_0 and N are both contained in the $\in$-chain $\mathscr{C}_q$, we now know that either $M_0 = N$ or $N \in M_0$. But $M_1 \in \mathscr{C}_s \subseteq Z$ implies that $M_1 \in N$, and therefore we know that $M_1 \in M_0$ in both cases. These computations show that $\mathscr{C}_*$ is an $\in$-chain.

Next, pick $M_0, M_1 \in \mathscr{C}_*$ with $M_0 \in M_1$ and $X_0 \in t_*(M_0)$. If $M_0, M_1 \in \mathscr{C}_q \setminus \mathscr{C}_s$, then $X_0 = t_*(M_0) = t_q(M_0)$ and there is $X_1 \in t_q(M_1) = t_*(M_1)$ with $X_0 \in X_1$. Now, assume that $M_1 \in \mathscr{C}_s$. Since $\mathscr{C}_s \subseteq Z$, we then have $M_1 \in N$ and, since $\mathscr{C}_q \cap N = \mathscr{C}_{\bar{q}} \subseteq \mathscr{C}_s$, we know that $M_0 \in \mathscr{C}_s$. We can now find $i < k$ and $X' \in t_s(M_0)$ with $X_0 = \pi_i^{-1}(X')$. Pick $X'' \in t_s(M_1)$ with $X' \in X''$ and set $X_1 = \pi_i^{-1}(X'')$. Then $X_0 \in X_1 \in t_*(M_1)$. Finally, assume that $M_0 \in \mathscr{C}_s$

and $M_1 \in \mathscr{C}_q \setminus \mathscr{C}_s$. As above, we know that either $M_1 = N$ or $N \in M_1$. In the first case, if $M_1 = N$ and $X_0 = \pi_i^{-1}(X)$ with $i < k$ and $X \in t_s(M)$, then $X_0 \in Z_i \in t_q(N) = t_*(M_1)$. In the other case, if $M_1 \in N$ and $X_0 = \pi_i^{-1}(X)$ with $i < k$ and $X \in t_s(M)$, then $Z_i \in t_q(N)$, there is $X_1 \in t_q(M_1) = t_*(M_1)$ with $Z_i \in X_1$, and elementarity implies that $X_0 \in X_1$. These computations show that, in all cases, there exists $X_1 \in t_*(M_1)$ with $X_0 \in X_1$.

Now, fix $M \in \mathscr{C}_*$, $X \in t_*(M)$ and $\alpha \in (d_q \cup d_s) \cap X$. First, assume that $M \in \mathscr{C}_q \setminus \mathscr{C}_s$ and $\alpha \in d_s \cap X$. Since $M, N \in \mathscr{C}_q$ and $M \notin \mathscr{C}_q \cap N = \mathscr{C}_{\bar{q}} \subseteq \mathscr{C}_s$, we know that either $M = N$ or $N \in N$, and both cases imply that $Z \cap \omega_1 \subseteq X$. In particular, we know that $(a_q \cup a_s)(\alpha) = a_s(\alpha) \in Z \cap \omega_1 \subseteq X$. Next, if $M \in \mathscr{C}_q \setminus \mathscr{C}_s$ and $\alpha \in d_q \cap X$, then $(a_q \cup a_s)(\alpha) = a_q(\alpha) \in X$. Finally, assume that $M \in \mathscr{C}_s$. Note that $d_q \cap X = d_{\bar{q}} \cap X \subseteq d_s \cap X$. In particular, we know that $\alpha \in d_s \cap X$. Pick $i < k$ and $X' \in t_s(M)$ with $X = \pi_i^{-1}(X')$. Then $\alpha \in d_s \cap X'$ and this implies that $a_s(\alpha) \in X'$. But this shows that $(a_q \cup a_s)(\alpha) = a_s(\alpha) \in X$. These computations show that $(a_q \cup a_s)(\alpha) \in X$.

Finally, assume that $M \in \mathscr{C}_*$, $X \in t_*(M)$, $\alpha \in (d_q \cup d_s) \setminus X$ and f is a function from ω_1 to ω_1 in X. First, assume that $M \in \mathscr{C}_q \setminus \mathscr{C}_s$. As above, we then know that either $M = N$ or $N \in M$, and in both cases we can conclude that $d_s \subseteq Z \cap \omega_1 \subseteq X$. This shows that $\alpha \in d_q \setminus X$ and hence $X \in t_q(M)$ implies that $(a_q \cup a_s)(\alpha) = a_q(\alpha) \neq f(\alpha)$. Next, assume that $M \in \mathscr{C}_s$ and $\alpha \in d_q \setminus (d_s \cup X)$. Since $d_q \cap Z = d_{\bar{q}} \subseteq d_s$, we now know that $\alpha \notin Z$ and therefore $f \in X \subseteq Z_0 \in t_q(N)$ implies that $(a_q \cup a_s)(\alpha) = a_q(\alpha) \neq f(\alpha)$. Finally, assume that $M \in \mathscr{C}_s$ and $\alpha \in d_s \setminus X$. Pick $i < k$ and $X' \in t_s(M)$ with $X = \pi_i^{-1}(X')$. Then $\alpha \in d_s \setminus X'$ and

$$(a_q \cup a_s)(\alpha) = a_s(\alpha) \neq \pi_i^{-1}(f)(\alpha) = f(\alpha).$$

These computations show that $(a_q \cup a_s)(\alpha) \neq f(\alpha)$ holds in all cases. ∎

CLAIM. $u \leq_{\mathbb{E}} q$.

Proof of the Claim. The definition of u directly implies that $d_q \subseteq d_u$, $a_q = a_u \restriction d_q$, $\mathscr{F}_q \subseteq \mathscr{F}_u$ and $\mathscr{C}_q \subseteq \mathscr{C}_u$. Now, assume that $M \in \mathscr{C}_q \cap \mathscr{C}_s$ and $X \in t_q(M)$. Then $M \in Z$ and therefore $M \in \mathscr{C}_q \cap N = \mathscr{C}_{\bar{q}}$. Fix $i \in I(X)$. Then $\pi_i(X) \in t_{\bar{q}}(M) \subseteq t_s(M)$ and therefore $X \in t_u(M)$. Since we also have $t_u \restriction (\mathscr{C}_q \setminus \mathscr{C}_s) = t_q \restriction (\mathscr{C}_q \setminus \mathscr{C}_s)$, we can conclude that $t_q(M) \subseteq t_u(M)$ for all $M \in \mathscr{C}_q$. Finally, fix $\alpha \in d_u \setminus d_q$ and $f \in \mathscr{F}_q$. Then $d_s \subseteq Z$ implies that $\alpha \in d_s \setminus d_q = d_r \setminus d_{\bar{q}}$ and therefore $a_u(\alpha) = a_s(\alpha) = a_r(\alpha) \neq f(\alpha)$. ∎

CLAIM. $u \leq_{\mathbb{E}} s$.

Proof of the Claim. The definition of u together with the fact that $\pi_0 = \mathrm{id}_{Z_0}$ directly implies that $d_s \subseteq d_u$, $a_s = a_u \restriction d_s$, $\mathscr{F}_s \subseteq \mathscr{F}_u$, $\mathscr{C}_s \subseteq \mathscr{C}_u$ and $t_s(M) \subseteq t_u(M)$ for all $M \in \mathscr{C}_s$. Fix $\alpha \in d_u \setminus d_s$ and $f \in \mathscr{F}_s$. Since $d_q \cap N \subseteq d_s$, we then know that $\alpha \in d_q \setminus Z_0$ and, since $N \in \mathscr{C}_q$, $Z_0 \in t_q(N)$ and $f \in Z_0$, we can conclude that $a_u(\alpha) = a_q(\alpha) \neq f(\alpha)$. ∎

Since s is an element of $D \cap Z$, the above claims show that p is a $(Z, \mathbb{E})$-generic condition. ∎

COROLLARY 4.5. *The partial order $\mathbb{E}$ is proper.*

Proof. Fix a sufficiently large regular cardinal θ, a countable elementary submodel X of $\mathrm{H}(\theta)$ and a condition p in $\mathbb{E}$ that is an element of X. Let M denote the transitive collapse of $\mathrm{H}(\omega_2) \cap X$ and define

$$q = \langle a_p, \mathscr{F}_p, t_p \cup \{\langle M, \{\mathrm{H}(\omega_2) \cap X\}\rangle\}\rangle.$$

Then it is easy to see that q is a condition in $\mathbb{E}$ below p. Moreover, Lemma 4.4 shows that q is $(X, \mathbb{E})$-generic. ∎

Following [9, Section VIII.2] and [11, Section 4], we give the following definition.

DEFINITION 4.6. A partial order $\mathbb{P}$ satisfies the $\aleph_2$-*isomorphism condition* ([6]) if for

- all sufficiently large regular cardinals θ,
- all well-orderings $\lhd$ of $\mathrm{H}(\theta)$,
- all ordinals $\alpha < \beta < \omega_2$,
- all countable elementary submodels Y and Z of $\langle \mathrm{H}(\theta), \in, \lhd \rangle$ with $\alpha \in Y$, $\beta \in Z$ and $\mathbb{P} \in Y \cap Z$, $Y \cap \omega_2 \subseteq \beta$ and $Y \cap \alpha = Z \cap \beta$,
- all conditions p in $\mathbb{P}$ that are contained in Y, and
- all isomorphisms $\pi : \langle Y, \in \rangle \to \langle Z, \in \rangle$ with $\pi(\alpha) = \beta$ and $\pi{\restriction}(Y \cap Z) = \mathrm{id}_{Y \cap Z}$,

there exists a $(Y, \mathbb{P})$-generic condition q below both p and $\pi(p)$ with the property that $\pi[G \cap Y] = G \cap Z$ holds whenever G is $\mathbb{P}$-generic over V with $q \in G$.

LEMMA 4.7. *The partial order $\mathbb{E}$ satisfies the $\aleph_2$-isomorphism condition.*

Proof. In the following, pick $\theta, \lhd, \alpha, \beta, Y, Z, p, \pi$ as in the definition of the $\aleph_2$-isomorphism condition. Then it is easy to see that $\pi(p)$ is again a condition in $\mathbb{E}$ with $d_p = d_{\pi(p)}$, $a_p = a_{\pi(p)}$ and $\mathscr{C}_p = \mathscr{C}_{\pi(p)}$. Let t denote the unique function with domain $\mathscr{C}_p$ and $t(M) = t_p(M) \cup t_{\pi(p)}(M)$ for all $M \in \mathscr{C}_p$. Then it is easy to see that the tuple

$$q = \langle a_p, \mathscr{F}_p \cup \mathscr{F}_{\pi(p)}, t \rangle$$

is a condition in $\mathbb{E}$ below both p and $\pi(p)$.

Now, let N denote the transitive collapse of $\mathrm{H}(\omega_2) \cap Y$ and define

$$r = \langle a_q, \mathscr{F}_q, t_q \cup \{\langle N, \{\mathrm{H}(\omega_2) \cap Y, \mathrm{H}(\omega_2) \cap Z\}\rangle\}\rangle.$$

Since our assumptions imply that $Y \cap \omega_1 = Z \cap \omega_1$ and $\pi{\restriction}(Y \cap \omega_1) = \mathrm{id}_{Z \cap \omega_1}$,

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([6]) In [9], this property is called $\aleph_2$-*properness isomorphism condition*. We follow the naming conventions of [11].

it follows that r is a condition in $\mathbb{E}$ below q. Moreover, Lemma 4.4 directly implies that r is both an $(Y, \mathbb{E})$- and a $(Z, \mathbb{E})$-generic condition.

In the following, let G be $\mathbb{E}$-generic over V with $r \in G$. Assume, towards a contradiction, that there is $s \in G \cap Y$ with $\pi(s) \notin G$. Fix a condition u in $\mathbb{E}$ below both r and s. Set $k = |t_u(N)| > 1$ and pick sets $W_0, \ldots, W_{k-1}$ with $W_0 = \mathrm{H}(\omega_2) \cap Y$, $W_1 = \mathrm{H}(\omega_2) \cap Z$ and $t_u(N) = \{W_0, \ldots, W_{k-1}\}$. For all $M \in \mathscr{C}_u \cap N$ and all $X \in t_u(M)$, we let $I(X)$ denote the set of all $i < k$ with the property that $X \in W_i$ and there exists a finite $\in$-chain C of elements of W_i with the property that for all $M \in M' \in C$, there exists $X \in X' \in C \cap t_u(M')$. Then $I(X) \neq \emptyset$ for all $M \in \mathscr{C}_u \cap N$ and all $X \in t_u(M)$. Next, given $i, j < k$, let $\pi_{i,j} : \langle W_i, \in \rangle \to \langle W_j, \in \rangle$ denote the unique isomorphism between these structures. We then have $\pi_{0,1} = \pi \restriction (W_0)$. We now define t_* to be the unique function with domain $\mathscr{C}_u$ such that $t_*(M) = t_u(M)$ holds for all $M \in \mathscr{C}_u \setminus N$ and

$$t_*(M) = \{\pi_{i,j}(X) \mid X \in t_u(M), i \in I(X), j < k\}$$

for all $M \in \mathscr{C}_u \cap N$. Set

$$v = \langle a_u, \mathscr{F}_{\pi(s)} \cup \mathscr{F}_u, t_* \rangle.$$

CLAIM. v is a condition in $\mathbb{E}$ below u.

Proof of the Claim. First, if $M \in \mathscr{C}_u \cap N$, $X \in t_u(M)$, $i \in I(X)$ and $j < k$, then X is an elementary submodel of W_i and this allows us to conclude that $\pi_{i,j}(X)$ is a countable elementary submodel of $\mathrm{H}(\omega_2)$ whose transitive collapse is equal to M.

Next, fix $M_0, M_1 \in \mathscr{C}_u$ with $M_0 \in M_1$ and $X_0 \in t_*(M_0)$. First, assume that $M_1 \in N$. Then $M_0 \in N$, and we can find $X \in t_u(M_0)$, $i \in I(X)$ and $j < k$ with $X_0 = \pi_{i,j}(X)$. Let C be the $\in$-chain of elements of W_i witnessing that $i \in I(X)$, and pick $X' \in C$ with $X \in X' \in C \cap t_u(M_1)$. Then $\{W \in C \mid X' \in W\}$ is an $\in$-chain witnessing that $i \in I(X')$, and hence we have $X_0 \in \pi_{i,j}(X') \in t_*(M_1)$. Next, if $M_0, M_1 \notin N$, then $X_0 \in t_u(M_0)$ and there exists $X_1 \in t_r(M_1) = t_*(M_1)$ with $X_0 \in X_1$. Finally, assume that $M_0 \in N$ and $M_1 \notin N$. Then there exists $j < k$ with $X_0 \in W_j \in t_u(N)$. Since M_1 and N are both contained in $\mathscr{C}_u$, we then know that either $M_1 = N$ or $N \in M_1$. If $N \in M_1$, then we can find $X_1 \in t_u(M_1)$ with $W_j \in X_1$ and we then also have $X_0 \in X_1$. We can therefore conclude that, in all cases, there exists $X_1 \in t_*(M_1)$ with $X_0 \in X_1$.

We now fix $M \in \mathscr{C}_u$, $X \in t_*(M)$ and $\alpha \in d_u \cap X$. If $M \notin N$, then $X \in t_u(M)$ and therefore $a_u(\alpha) \in X$. Now, assume that $M \in N$. Pick $X' \in t_u(M)$, $i \in I(X')$ and $j < k$ with $X = \pi_{i,j}(X')$. Then $\alpha \in d_u \cap X'$ and therefore $a_u(\alpha) \in X' \cap \omega_1 \subseteq X$.

Next, fix $M \in \mathscr{C}_u$, $X \in t_*(M)$, $\alpha \in d_u \setminus X$ and a function $f : \omega_1 \to \omega_1$ in X. If $M \notin N$, then $X \in t_u(M)$ and $a_u(\alpha) \neq f(\alpha)$. In the other case, if

$M \in N$ and $X = \pi_{i,j}(X')$ for some $X' \in t_u(M)$, $i \in I(X')$ and $j < k$, then $\alpha \in d_u \setminus X'$ and therefore $a_u(\alpha) \neq \pi_{i,j}^{-1}(f)(\alpha) = f(\alpha)$.

The above computations show that v is a condition in $\mathbb{E}$ with $a_u = a_v$, $\mathscr{F}_u \subseteq \mathscr{F}_v$ and $\mathscr{C}_u = \mathscr{C}_v$. Since our construction ensures that $t_u(M) \subseteq t_v(M)$ for all $M \in \mathscr{C}_u$, we can now conclude that $v \leq_{\mathbb{E}} u$. ∎

CLAIM. $v \leq_{\mathbb{E}} \pi(s)$.

Proof of the Claim. The fact that $v \leq_{\mathbb{E}} s$ directly implies that $d_{\pi(s)} = d_s \subseteq d_v$, $a_v \upharpoonright d_{\pi(s)} = a_s = a_{\pi(s)}$ and $\mathscr{C}_{\pi(s)} = \mathscr{C}_s \subseteq \mathscr{C}_v$. Moreover, the definition of v ensures that $\mathscr{F}_{\pi(s)} \subseteq \mathscr{F}_v$. In addition, the fact that $\pi_{0,1} = \pi \upharpoonright W_0$ and $\mathscr{C}_s \subseteq \mathscr{C}_u \cap N$ directly implies that

$$t_{\pi(s)}(M) = \pi[t_s(M)] \subseteq \pi[\{\pi_{0,1}(X) \mid X \in t_u(M),\ X \in W_0\}] \subseteq t_v(M)$$

for all $M \in \mathscr{C}_{\pi(s)}$. Finally, fix $\alpha \in d_v \setminus d_{\pi(s)}$ and $f \in \mathscr{F}_{\pi(s)}$. Then $\alpha \in d_v \setminus d_s$ and $\pi^{-1}(f) \in \mathscr{F}_s$. Since this allows us to conclude that

$$a_{\pi(s)}(\alpha) = a_s(\alpha) \neq \pi^{-1}(f)(\alpha) = f(\alpha),$$

the claim follows. ∎

A density argument now shows that there is a condition v in G that is stronger than $\pi(s)$, which contradicts our assumption. This shows that $\pi[G \cap Y] \subseteq G$.

Finally, assume, towards a contradiction, that there is $u \in G \cap Z$ with the property that $\pi^{-1}(u) \notin G$. Let D denote the set of all conditions in $\mathbb{E}$ that are either stronger than $\pi^{-1}(u)$ or incompatible with $\pi^{-1}(u)$. Then D is a dense subset of $\mathbb{E}$ that is contained in Y. Since $r \in G$, we can find $v \in D \cap G \cap Y$. In this situation, the above computations show that $\pi(v) \in G$, and elementarity implies that the element u and $\pi(v)$ are incompatible in $\mathbb{E}$, a contradiction. ∎

The statements of the following proposition can be proven in the same way as the corresponding results for the partial order $\mathbb{D}$ in Section 3. The details are left to the reader.

PROPOSITION 4.8. *Let G be $\mathbb{E}$-generic over* V *and set $g = \bigcup\{a_p \mid p \in G\}$.*

(1) *The set g is a function from ω_1 to ω_1.*

(2) *For every sequence $\langle f_\alpha \mid \alpha < \omega_1 \rangle$ of functions from ω_1 to ω_1 in* V*, every monotone map $m : [\omega_1]^{<\omega} \to [\omega_1]^{<\omega}$ in* V *and every $a \in [\omega_1]^{<\omega}$, there exists $a \subseteq b \in [\omega_1]^{<\omega}$ with*

$$\{\beta < \omega_1 \mid f_\alpha(\beta) = g(\beta)\} \subseteq m(b) \quad \text{for all } \alpha \in m(b).$$

(3) *Given a finite subset F of ω_1, the tuple*

$$q_F = \langle \emptyset, \{c_\beta \mid \beta \in F\}, \emptyset \rangle$$

is a condition in $\mathbb{E}$ in V *and, if $q_F \in G$, then $\text{ran}(g) \cap F = \emptyset$.* ∎

We are now ready to give the proof of Theorem 4.1.

Proof of Theorem 4.1. Assume that CH holds and let $\langle F_\gamma \mid \gamma < \omega_2 \rangle$ denote an enumeration of all finite subsets of ω_1 with the property that every such subset is enumerated unboundedly often in ω_2. Let

$$\left\langle \langle \vec{\mathbb{P}}_{<\gamma} \mid \gamma \leq \omega_2 \rangle, \langle \dot{\mathbb{P}}_\gamma \mid \gamma < \omega_2 \rangle \right\rangle$$

denote a forcing iteration with countable support with the property that for all $\gamma < \omega_2$, if G is $\vec{\mathbb{P}}_{<\gamma}$-generic over V, then $\dot{\mathbb{P}}_\gamma^G$ is equal to the suborder of $\mathbb{E}^{V[G]}$ consisting of all conditions below q_{F_γ}. Then Corollary 4.5 and the iteration theorem for proper forcings ensure that the partial order $\vec{\mathbb{P}}_{<\omega_2}$ is proper. In addition, Lemma 4.7 allows us to apply [9, Section VIII, Lemma 2.4] to show that $\vec{\mathbb{P}}_{<\omega_2}$ satisfies the $\aleph_2$-chain condition. In combination, these arguments show that forcing with $\vec{\mathbb{P}}_{<\omega_2}$ preserves all cardinals. Moreover, we know that every subset of ω_1 in a $\vec{\mathbb{P}}_{<\omega_2}$-generic extension is contained in a proper intermediate extension of the iteration.

Now, let G be $\vec{\mathbb{P}}_{<\omega_2}$-generic over V and, in V[G], fix a sequence $\vec{f} = \langle f_\alpha \mid \alpha < \omega_1 \rangle$ of functions from ω_1 to ω_1, a finite subset F of ω_1 and some monotone map $m : [\omega_1]^{<\omega} \to [\omega_1]^{<\omega}$. By the definition of our iteration and the above remarks, there exists $\gamma < \omega_2$ with the property that, if $\bar{G}$ denotes the filter on $\vec{\mathbb{P}}_{<\gamma}$ induced by G, then $F_\gamma = F$ and $\vec{f}, m \in V[\bar{G}]$. Let G_γ be the filter on $\dot{\mathbb{P}}_\gamma$ induced by G, and set $g = \bigcup \{a_p \mid p \in G_\gamma\}$. Then Proposition 4.8 shows that g is a function from ω_1 to ω_1 with the property that $F \cap \operatorname{ran}(g) = \emptyset$ and, for all $a \in [\omega_1]^{<\omega}$, there exists $a \subseteq b \in [\omega_1]^{<\omega}$ with $\{\beta < \omega_1 \mid f_\alpha(\beta) = g(\beta)\} \subseteq m(b)$ for all $\alpha \in m(b)$. This shows that $(\vartriangle)$ holds in V[G]. ∎

5. The coloring. We now use the principle $(\vartriangle)$ to construct an $(\mathcal{M}, \mathcal{N})$-full structure of cardinality $\aleph_2$. This implication is an immediate consequence of the next result.

THEOREM 5.1. *Assume that $(\vartriangle)$ holds. Then there exists*

- *a map $c : [\omega_2]^2 \to \omega_1$,*
- *a monotone map $m : [\omega_2]^{<\omega} \to [\omega_2]^{<\omega}$, and*
- *a map $r : \omega_2 \to \omega_2$*

such that the following statements hold:

(1) *If $a \in [\omega_2]^{<\omega}$ and $\alpha, \beta \in m(a)$ with $\alpha \neq \beta$, then $\mathcal{A}_{\alpha,\beta}^c \subseteq m(a)$.*
(2) *Given*

  - *a finite subset d of $\omega_2 + \omega$,*
  - *a function $e : [d]^2 \to \omega_1$, and*
  - *a function $s : d \to \omega_2$*

such that

- $c{\restriction}[d \cap \omega_2]^2 = e{\restriction}[d \cap \omega_2]^2$,
- $r{\restriction}d = s$, *and*
- $\mathcal{A}^e_{\alpha,\beta} \subseteq \omega_2$ *for all* $\alpha, \beta \in d \cap \omega_2$ *with* $\alpha \neq \beta$,

there exists an injection $\iota : d \to \omega_2$ *with*

- $\iota{\restriction}(d \cap \omega_2) = \mathrm{id}_{d \cap \omega_2}$,
- $c(\iota(\alpha), \iota(\beta)) = e(\alpha, \beta)$ *for all* $\alpha, \beta \in d$ *with* $\alpha \neq \beta$, *and*
- $r(\iota(\alpha)) = s(\alpha)$ *for all* $\alpha \in d$.

Before we present the proof of the above theorem, we briefly show how it can be applied to prove the desired independence result.

COROLLARY 5.2. *Assume that* $(⊿)$ *holds and let* $\mathcal{M}$ *be a countable model that characterizes* $\aleph_1$. *Then there exists an* $(\mathcal{M}, \mathcal{N})$-*full structure of cardinality* $\aleph_2$.

Proof. Let c, m and r be the functions given by Theorem 5.1. Then the function m directly witnesses that the function c possesses the properties (a) and (b) listed in Lemma 2.1. Now, fix a finite set d, a function $e : [d]^2 \to \omega_\alpha$ with $e{\restriction}[d \cap \omega_2]^2 = c{\restriction}[d \cap \omega_2]^2$ that introduces no new agreements over $c{\restriction}[d \cap \omega_2]^2$, and a function $s : d \to \omega_2$ with $s{\restriction}(d \cap \omega_2) = r{\restriction}(d \cap \omega_2)$. Without loss of generality, we may assume that d is a subset of $\omega_2 + \omega$. If $\alpha, \beta \in d \cap \omega_2$ with $\alpha \neq \beta$, then the fact that e introduces no new agreements over $c{\restriction}[d \cap \omega_2]^2$ implies that $\mathcal{A}^e_{\alpha,\beta} = \mathcal{A}^{c{\restriction}[d \cap \omega_2]^2}_{\alpha,\beta} \subseteq \omega_2$. This allows us to use conclusion (2) of Theorem 5.1 to find an injection $\iota : d \to \omega_2$ with $\iota{\restriction}(d \cap \omega_2) = \mathrm{id}_{d \cap \omega_2}$, $c(\iota(\alpha), \iota(\beta)) = e(\alpha, \beta)$ for all $\alpha, \beta \in d$ with $\alpha \neq \beta$, and $r(\iota(\alpha)) = s(\alpha)$ for all $\alpha \in d$. These computations allow us to conclude that the functions c and r possess the properties (c) and (d) listed in Lemma 2.1. We can therefore apply Lemma 2.1 to find an $(\mathcal{M}, \mathcal{N})$-full structure of cardinality $\aleph_2$. $\blacksquare$

Proof of Theorem 5.1. In the following, we let $\prec\cdot, \cdot\succ : \mathrm{On} \times \mathrm{On} \to \mathrm{On}$ denote the *Gödel pairing function*. In addition, let $p_0, p_1 : \omega_1 \to \omega_1$ denote the corresponding *projections* on ω_1, i.e. the unique pair of functions on ω_1 with $\alpha = \prec p_0(\alpha), p_1(\alpha)\succ$ for all $\alpha < \omega_1$. For each $0 < \alpha < \omega_2$, fix a surjection $s_\alpha : \omega_1 \to \alpha$. In addition, pick an enumeration $\langle \langle e_\xi, s_\xi \rangle \mid \xi < \omega_2 \rangle$ of all pairs $\langle e, s \rangle$ of functions with $e : [d]^2 \to \omega_1$ and $s : d \to \omega_2$ for some finite subset d of $\omega_2 + \omega$ such that the enumeration has the property that for all $\zeta < \omega_2$, the set $\{\xi < \omega_2 \mid e_\xi = e_\zeta\}$ is unbounded in ω_2.

Given $0 < \alpha < \omega_2$, a finite subset F of ω_1, a map $c_0 : [\alpha]^2 \to \omega_1$ and a monotone map $m_0 : [\alpha]^{<\omega} \to [\alpha]^{<\omega}$, we call a pair (c, m) an F-*good extension* of (c_0, m_0) if there exists a function $g : \alpha \to \omega_1$ such that the following statements hold:

- $F \cap \mathrm{ran}(g) = \emptyset$.

- $c : [\alpha + 1]^2 \to \omega_1$ is a map with $c{\upharpoonright}[\alpha]^2 = c_0$ and

$$c(\alpha, \beta) = \prec g(\beta), \min(s_\alpha^{-1}\{\beta\}) \succ$$

  for all $\beta < \alpha$.
- $m : [\alpha + 1]^{<\omega} \to [\alpha + 1]^{<\omega}$ is a map with $m{\upharpoonright}[\alpha]^{<\omega} = m_0$ and the property that for all $a \in [\alpha]^{<\omega}$, there exists $a \subseteq b \in [\alpha]^{<\omega}$ satisfying

$$m(a \cup \{\alpha\}) = m_0(b) \cup \{\alpha\}$$

  and

$$\{\gamma \in \alpha \setminus \{\beta\} \mid p_0(c_0(\beta, \gamma)) = g(\gamma)\} \subseteq m_0(b)$$

  for all $\beta \in m_0(b)$.

Note that ($\varangle$) implies that an F-good extension of (c_0, m_0) exists.

In the following, we construct

- a strictly increasing sequence $\langle \alpha_\xi \mid \xi < \omega_1 \rangle$ of ordinals less than ω_2 with $\alpha_0 = 0$,
- a sequence $\langle c_\xi : [\alpha_\xi]^2 \to \omega_1 \mid \xi < \omega_2 \rangle$ of maps with $c_\xi{\upharpoonright}[\alpha_\zeta]^2 = c_\zeta$ for all $\zeta \leq \xi < \omega_2$,
- a sequence $\langle m_\xi : [\alpha_\xi]^{<\omega} \to [\alpha_\xi]^{<\omega} \mid \xi < \omega_2 \rangle$ of monotone maps satisfying $m_\xi{\upharpoonright}[\alpha_\zeta]^{<\omega} = m_\zeta$ for all $\zeta \leq \xi < \omega_2$, and
- a sequence $\langle r_\xi : \alpha_\xi \to \omega_2 \mid \xi < \omega_2 \rangle$ of functions with $r_\xi{\upharpoonright}\alpha_\zeta = r_\zeta$ for all $\zeta \leq \xi < \omega_2$.

Fix $0 < \xi < \omega_2$ and assume that α_ζ, c_ζ and m_ζ with the above properties are defined for all $\zeta < \xi$. We define

- $\alpha_* = \sup_{\zeta < \xi} \alpha_\zeta$,
- $c_* = \bigcup_{\zeta < \xi} c_\zeta : [\alpha_*]^2 \to \omega_1$,
- $m_* = \bigcup_{\zeta < \xi} m_\zeta : [\alpha_*]^{<\omega} \to [\alpha_*]^{<\omega}$,
- $r_* : \alpha_* \to \omega_2$, and
- $d = \mathrm{dom}(s_\xi) \in [\omega_2 + \omega]^{<\omega}$.

If either $d \subseteq \alpha_*$, or $d \cap \omega_2 \not\subseteq \alpha_*$, or $c_*{\upharpoonright}[d \cap \omega_2]^2 \neq e_\xi{\upharpoonright}[d \cap \omega_2]^2$, or $r_*{\upharpoonright}(d \cap \omega_2) \neq s_\xi{\upharpoonright}(d \cap \omega_2)$, or there exist $\beta, \gamma \in d \cap \omega_2$ with $\beta \neq \gamma$ and $\mathcal{A}_{\beta,\gamma}^{e_\xi} \not\subseteq \omega_2$, then we say that ξ has *Type 0*, we set $\alpha_\xi = \alpha_* + 1$, we define $r_\xi = r_* \cup \{\langle \alpha_*, 0 \rangle\}$, and we pick c_ξ and m_ξ such that the pair (c_ξ, m_ξ) is a $\emptyset$-good extension of (c_*, m_*). In the following, assume that $d \not\subseteq \alpha_*$, $d \cap \omega_2 \subseteq \alpha_*$, $c_*{\upharpoonright}[d \cap \omega_2]^2 = e_\xi{\upharpoonright}[d \cap \omega_2]^2$, $r_*{\upharpoonright}(d \cap \omega_2) = s_\xi{\upharpoonright}(d \cap \omega_2)$ and $\mathcal{A}_{\beta,\gamma}^{e_\xi} \subseteq \omega_2$ for all $\beta, \gamma \in d \cap \omega_2$ with $\beta \neq \gamma$. We then say that ξ has *Type 1*. Let $|d \setminus \omega_2| = n > 0$, set $\alpha_\xi = \alpha_* + n$ and pick an injection $\iota_\xi : d \to \alpha_\xi$ with $\iota_\xi{\upharpoonright}(d \cap \omega_2) = \mathrm{id}_{d \cap \omega_2}$ and $\iota_\xi[d \setminus \omega_2] = [\alpha_*, \alpha_\xi)$. In addition, set $F = \mathrm{ran}(p_0 \circ e_\xi)$. Pick a map $c^* : [\alpha_\xi]^2 \to \omega_1$ and a monotone map $m^* : [\alpha_\xi]^{<\omega} \to [\alpha_\xi]^{<\omega}$ such that $c^*{\upharpoonright}[\alpha_*]^2 = c_*$, $m^*{\upharpoonright}[\alpha_*]^{<\omega} = m_*$ and for all $i < n$, the pair $(c^*{\upharpoonright}[\alpha_* + i + 1]^2, m^*{\upharpoonright}[\alpha_* + i + 1]^{<\omega})$ is an F-good extension of $(c^*{\upharpoonright}[\alpha_* + i]^2, m^*{\upharpoonright}[\alpha_* + i]^{<\omega})$.

Define $c_\xi : [\alpha_\xi]^2 \to \omega_1$ to be the unique map with $c_\xi(\iota_\xi(\beta), \iota_\xi(\gamma)) = e_\xi(\beta, \gamma)$ for all $\beta, \gamma \in d$ with $\beta \neq \gamma$ and $c_\xi(\beta, \gamma) = c^*(\beta, \gamma)$ for all $\beta < \gamma < \alpha_\xi$ with $\{\beta, \gamma\} \nsubseteq \mathrm{ran}(\iota_\xi)$. Our assumptions on e_ξ then ensure that $c_\xi {\restriction} [\alpha_*]^2 = c_*$. In addition, let $m_\xi : [\alpha_\xi]^{<\omega} \to [\alpha_\xi]^{<\omega}$ denote the unique function with $m_\xi {\restriction} [\alpha_*]^{<\omega} = m_*$ and $m_\xi(a) = m^*(a \cup \mathrm{ran}(\iota_\xi))$ for all $a \in [\alpha_\xi]^{<\omega}$ with $a \cap [\alpha_*, \alpha_\xi) \neq \emptyset$. Finally, define r_ξ to be the unique function with domain α_ξ satisfying $r_\xi {\restriction} \alpha_* = r_*$ and $r_\xi(\iota_\xi(\alpha)) = s_\xi(\alpha)$ for all $\alpha \in d$.

We can now define

- $c = \bigcup_{\xi < \omega_2} c_\xi : [\omega_2]^2 \to \omega_1$,
- $m = \bigcup_{\xi < \omega_2} m_\xi : [\omega_2]^{<\omega} \to [\omega_2]^{<\omega}$, and
- $r = \bigcup_{\xi < \omega_2} r_\xi : \omega_2 \to \omega_2$.

Our construction then ensures that m is a monotone map.

CLAIM. *If $0 < \xi < \omega_2$, $a \in [\alpha_\xi]^{<\omega}$ and $\alpha, \beta \in m(a)$ with $\alpha \neq \beta$, then* $\mathcal{A}^{c_\xi}_{\alpha, \beta} \subseteq m(a)$.

Proof of the Claim. Assume that $0 < \xi < \omega_2$ has the property that $\mathcal{A}^{c_\zeta}_{\alpha, \beta} \subseteq m(a)$ for all $0 < \zeta < \xi$, every $a \in [\alpha_\zeta]^{<\omega}$ and all $\alpha, \beta \in m(a)$ with $\alpha \neq \beta$. Set $\alpha_* = \sup_{\zeta < \xi} \alpha_\zeta$ and $c_* = c {\restriction} [\alpha_*]^2$. Then our assumptions imply that $\mathcal{A}^{c_*}_{\beta, \gamma} \subseteq m(a)$ holds for every $a \in [\alpha_*]^{<\omega}$ and all $\beta, \gamma \in m(a)$ with $\beta \neq \gamma$.

First, assume that ξ has Type 0. Then $\alpha_\xi = \alpha_* + 1$. Fix $a \in [\alpha_\xi]^{<\omega}$ and let $g : \alpha_* \to \omega_1$ denote the function used in the construction of c_ξ. If $\gamma < \beta < \alpha_*$, then

$$s_{\alpha_*}(p_1(c_\xi(\alpha_*, \beta))) = \beta \neq \gamma = s_{\alpha_*}(p_1(c_\xi(\alpha_*, \gamma)))$$

and therefore $c_\xi(\alpha_*, \beta) \neq c_\xi(\alpha_*, \gamma)$. In particular, if $a \subseteq \alpha_*$ and $\beta, \gamma \in m(a)$ with $\beta \neq \gamma$, then $\mathcal{A}^{c_\xi}_{\beta, \gamma} = \mathcal{A}^{c_*}_{\beta, \gamma} \subseteq m(a)$. In the following, assume that $\alpha_* \in a$. Then there exists $a \cap \alpha_* \subseteq b \in [\alpha_*]^{<\omega}$ with $m(a) = m(b) \cup \{\alpha_*\}$ and

$$\{\gamma \in \alpha_* \setminus \{\beta\} \mid p_0(c_*(\beta, \gamma)) = g(\gamma)\} \subseteq m(b)$$

for all $\beta \in m(b)$. Now, if $\beta, \gamma \in m(a) \cap \alpha_* = m(b)$ with $\beta \neq \gamma$, then the above computations show that $\mathcal{A}^{c_\xi}_{\beta, \gamma} = \mathcal{A}^{c_*}_{\beta, \gamma} \subseteq m(b) \subseteq m(a)$. Moreover, if we have $\beta \in m(a) \cap \alpha_* = m(b)$ and $\gamma \in \alpha_* \setminus \{\beta\}$ satisfying $c_\xi(\alpha_*, \gamma) = c_\xi(\beta, \gamma)$, then $p_0(c_*(\beta, \gamma)) = g(\gamma)$ and therefore $\gamma \in m(b) \subseteq m(a)$. This shows that $\mathcal{A}^{c_\xi}_{\alpha_*, \beta} \subseteq m(a)$ for all $\beta \in m(a)$ with $\alpha_* \neq \beta$.

Now, assume that ξ has Type 1. Let d denote the unique finite subset of $\omega_2 + \omega$ with the property that the domain of e_ξ is equal to the set $[d]^2$. In addition, let $c^* : [\alpha_\xi]^2 \to \omega_1$ and $m^* : [\alpha_\xi]^{<\omega} \to [\alpha_\xi]^{<\omega}$ denote the functions used in the construction of c_ξ. Then the above computations show that $\mathcal{A}^{c^*}_{\beta, \gamma} \subseteq \alpha_*$ for all $\gamma < \beta < \alpha_*$ and $\mathcal{A}^{c^*}_{\beta, \gamma} \subseteq m^*(a)$ for every $a \in [\alpha_\xi]^{<\omega}$ and all $\beta, \gamma \in m^*(a)$ with $\beta \neq \gamma$.

SUBCLAIM. *If $\gamma < \beta < \alpha_*$, then $\mathcal{A}^{c_\xi}_{\beta, \gamma} \subseteq \alpha_*$.*

Proof of the Subclaim. Assume, towards a contradiction, that $c_\xi(\beta, \delta) = c_\xi(\gamma, \delta)$ holds for some $\alpha_* \leq \delta < \alpha_\xi$. Then $\delta \in \text{ran}(\iota_\xi)$ and we know that $\{\beta, \gamma\} \nsubseteq \text{ran}(\iota_\xi)$, because otherwise we would have $\beta, \gamma \in d \cap \omega_2$ and $\mathcal{A}_{\beta, \gamma}^{e_\xi} \nsubseteq \omega_2$. Now, if $\gamma \notin \text{ran}(\iota_\xi)$, then the fact that

$$p_0(c_\xi(\beta, \delta)) = p_0(c_\xi(\gamma, \delta)) = p_0(c^*(\gamma, \delta)) \notin \text{ran}(p_0 \circ e_\xi)$$

implies that $\beta \notin \text{ran}(\iota_\xi)$. The same argument shows that $\beta \notin \text{ran}(\iota_\xi)$ implies that $\gamma \notin \text{ran}(\iota_\xi)$. Hence, we can conclude that β and γ are both not contained in $\text{ran}(\iota_\xi)$. But then our assumption implies that $c^*(\beta, \delta) = c^*(\gamma, \delta)$ and, by the above remarks, this shows that $\delta < \alpha_*$, a contradiction. ∎

Fix $a \in [\alpha_\xi]^{<\omega}$. If $a \subseteq \alpha_*$, then our subclaim shows that $\mathcal{A}_{\beta, \gamma}^{c_\xi} = \mathcal{A}_{\beta, \gamma}^{c_*} \subseteq m(a)$ for all $\beta, \gamma \in m(a)$ with $\beta \neq \gamma$. In the following, assume that $a \cap [\alpha_*, \alpha_\xi) \neq \emptyset$. Then $m(a) = m^*(a \cup \text{ran}(\iota_\xi))$. Pick $\beta, \gamma \in m(a)$ with $\beta \neq \gamma$ and $\delta \in \mathcal{A}_{\beta, \gamma}^{c_\xi} \setminus \text{ran}(\iota_\xi)$. Then the definition of c_ξ ensures that $c^*(\beta, \delta) = c^*(\gamma, \delta)$ and hence we know that $\delta \in \mathcal{A}_{\beta, \gamma}^{c^*} \subseteq m^*(a \cup \text{ran}(\iota_\xi)) = m(a)$. This shows that

$$\mathcal{A}_{\beta, \gamma}^{c_\xi} \subseteq \text{ran}(\iota_\xi) \cup \mathcal{A}_{\beta, \gamma}^{c^*} \subseteq m^*(a \cup \text{ran}(\iota_\xi)) = m(a)$$

for all $\beta, \gamma \in m(a)$ with $\beta \neq \gamma$. ∎

CLAIM. *If $\zeta < \omega_2$ and $\beta < \alpha < \alpha_\zeta$, then $\mathcal{A}_{\alpha, \beta}^c \subseteq \alpha_\zeta$.*

Proof of the Claim. Fix $\alpha_\zeta \leq \gamma < \omega_2$. Let $\xi < \omega_2$ be minimal with $\gamma < \alpha_\xi$ and let d be the unique finite subset of $\omega_2 + \omega$ such that the domain of e_ξ is equal to $[d]^2$.

First, assume that ξ has Type 0. Then $\alpha_\xi = \gamma + 1$ and the above constructions ensure that

$$s_\gamma(p_1(c(\alpha, \gamma))) = \alpha \neq \beta = s_\gamma(p_1(c(\beta, \gamma))).$$

This allows us to conclude that $c(\alpha, \gamma) \neq c(\beta, \gamma)$ in this case.

Next, assume that ξ has Type 1. Set $\alpha_* = \sup_{\eta < \xi} \alpha_\eta$ and let $c^* : [\alpha_\xi]^2 \to \omega_1$ denote the function used in the construction of c_ξ. Then $\alpha < \alpha_\zeta \leq \alpha_* \leq \gamma < \alpha_\xi$ and hence $\gamma \in \text{ran}(\iota_\xi)$. Moreover, the above computations show that $c^*(\alpha, \gamma) \neq c^*(\beta, \gamma)$. Set $F = \text{ran}(p_0 \circ e_\xi)$ and $n = |d \setminus \omega_2|$. Now, if $\alpha, \beta \in \text{ran}(\iota_\xi)$, then $\alpha, \beta \in d \cap \alpha_*$ and, since $\mathcal{A}_{\alpha, \beta}^{e_\xi} \subseteq \omega_2$, our construction ensures that $c(\alpha, \gamma) \neq c(\beta, \gamma)$. Next, if $\alpha \in \text{ran}(\iota_\xi)$ and $\beta \notin \text{ran}(\iota_\xi)$, then we have $c(\alpha, \gamma) \in \text{ran}(e_\xi)$, $p_0(c(\beta, \gamma)) = p_0(c^*(\beta, \gamma)) \notin \text{ran}(p_0 \circ e_\xi)$, and therefore we know that $c(\alpha, \gamma) \neq c(\beta, \gamma)$. The same argument shows that, if $\alpha \notin \text{ran}(\iota_\xi)$ and $\beta \in \text{ran}(\iota_\xi)$, then $c(\alpha, \gamma) \neq c(\beta, \gamma)$. Finally, if $\alpha, \beta \notin \text{ran}(\iota_\xi)$, then $c(\alpha, \gamma) = c^*(\alpha, \gamma) \neq c^*(\beta, \gamma) = c(\beta, \gamma)$. ∎

CLAIM. *If $a \in [\omega_2]^{<\omega}$ and $\alpha, \beta \in m(a)$ with $\alpha \neq \beta$, then $\mathcal{A}_{\alpha, \beta}^c \subseteq m(a)$.*

Proof. Pick $\xi < \omega_2$ with $m(a) \subseteq \alpha_\xi$. Then the previous claim shows that $\mathcal{A}_{\alpha, \beta}^c \subseteq \alpha_\xi$ and we can use our first claim to conclude that $\mathcal{A}_{\alpha, \beta}^c = \mathcal{A}_{\alpha, \beta}^{\alpha_\xi} \subseteq m(a)$. ∎

CLAIM. *Given a finite subset d of $\omega_2 + \omega$, a function $e : [d]^2 \to \omega_1$ and a function $s : d \to \omega_2$ such that $e{\restriction}[d \cap \omega_2]^2 = c{\restriction}[d \cap \omega_2]^2$, $r{\restriction}(d \cap \omega_2) = s{\restriction}(d \cap \omega_2)$ and $\mathcal{A}^e_{\alpha,\beta} \subseteq \omega_2$ for all $\alpha, \beta \in d \cap \omega_2$ with $\alpha \neq \beta$, there exists an injection $\iota : d \to \omega_2$ with $\iota{\restriction}(d \cap \omega_2) = \mathrm{id}_{d \cap \omega_2}$, $r(\iota(\alpha)) = s(\alpha)$ for all $\alpha \in d$ and $c(\iota(\alpha), \iota(\beta)) = e(\alpha, \beta)$ for all $\alpha, \beta \in d$ with $\alpha \neq \beta$.*

Proof of the Claim. Without loss of generality, we may assume that $d \setminus \omega_2 \neq \emptyset$. The above choices ensure that we can find $\xi < \omega_2$ with the property that $e = e_\xi$, $s = s_\xi$ and $d \cap \omega_2 \subseteq \alpha_* = \sup_{\zeta < \xi} \alpha_\zeta$. We define $c_* = c{\restriction}[\alpha_*]^2 = \bigcup_{\zeta < \xi} c_\zeta$ and $r_* = r{\restriction}\alpha_* = \bigcup_{\zeta < \xi} r_\zeta$. We then know that $d \nsubseteq \alpha_*$, $d \cap \omega_2 \subseteq \alpha_*$, $c_*{\restriction}[d \cap \omega_2]^2 = e_\xi{\restriction}[d \cap \omega_2]^2$, $r_*{\restriction}(d \cap \omega_2) = s_\xi{\restriction}(d \cap \omega_2)$ and $\mathcal{A}^{e_\xi}_{\alpha,\beta} \subseteq \omega_2$ for all $\alpha, \beta \in d \cap \omega_2$ with $\alpha \neq \beta$. In particular, this shows that ξ has Type 1 and $\iota_\xi : d \to \alpha_\xi$ is an injection with $\iota_\xi{\restriction}(d \cap \omega_2) = \mathrm{id}_{d \cap \omega_2}$,

$$r(\iota_\xi(\alpha)) = r_\xi(\iota(\alpha)) = s_\xi(\alpha)$$

for all $\alpha \in d$ and

$$c(\iota_\xi(\beta), \iota_\xi(\gamma)) = c_\xi(\iota_\xi(\alpha), \iota_\xi(\beta)) = e_\xi(\alpha, \beta)$$

for all $\alpha, \beta \in d$ with $\alpha \neq \beta$. ∎

This completes the proof of the theorem. ∎

6. Concluding remarks and restating the problem. We summarize the current situation of the problem motivating the results of this paper: Hjorth proved that there exists some countable model $\mathcal{M}$ that belongs to the constructible universe L and which characterizes $\aleph_1$ in all transitive models of ZFC. Using the Scott sentence of $\mathcal{M}$, he constructed two complete sentences, call them σ_1 and σ_2, using what we called the first and the second Hjorth construction. Moreover, in all transitive models of ZFC, exactly one of these sentences characterizes $\aleph_2$. If CH holds, then σ_1 characterizes $\aleph_1$ and σ_2 characterizes $\aleph_2$. If ($\vartriangleleft$) holds (and CH necessarily fails by Proposition 3.2), then σ_1 characterizes $\aleph_2$ and σ_2 characterizes $\aleph_3$.

Therefore, Hjorth's solution to the problem of characterizing $\aleph_2$ is dependent on the underlying model of set theory. One may ask whether the same holds true for $\aleph_3$ and, in general, for successor $\aleph_\alpha$ with $2 < \alpha < \omega_1$. For $\alpha < \omega$, this is easily seen to be true, because Hjorth's characterization of $\aleph_3$ uses inductively the characterization of $\aleph_2$, etc. For $\alpha > \omega$ our construction does not yield an answer. One would have to extend our results for functions from ω_1 to ω_1 into results for functions from $\omega_{\omega+1}$ to $\omega_{\omega+1}$. However, we think the main question here is how to characterize $\aleph_\alpha$, $\alpha < \omega_1$, *in an absolute way*. To make things precise:

QUESTION 6.1. *Does there exist a formula $\Phi(v_0, v_1)$ in the language of set theory such that ZFC proves the following statements hold for all ordinals α?*

(1) In L, there exists a unique code (^{7}) c for a complete $\mathcal{L}_{\alpha^+,\omega}$-sentence ψ_α such that $\Phi(\alpha, c)$ holds.
(2) If α is countable and ψ_α is as above, then ψ_α characterizes $\aleph_\alpha$.

As we mentioned, this is true for limit ordinals α. In [2], the authors provide a characterization of all $\aleph_n$, for n finite, that is absolute in the way described above. For successor ordinals $\alpha > \omega$ the question remains open.

Another canonical way to formulate the existence of absolute characterizations is given by *Shoenfield absoluteness* (see [6, Theorem 13.15]) and the fact that Σ_3^1-statements are upwards absolute between transitive models of set theory with the same ordinals.

QUESTION 6.2. Is there a Σ_3^1-formula $\Phi(v_0, v_1)$ in the language of second-order arithmetic with the property that the axioms of ZFC prove that the following statements hold?

(1) For every real a, there is a unique real b such that $\Phi(a, b)$ holds.
(2) If α is a countable ordinal, c is a code for a complete $\mathcal{L}_{\omega_1,\omega}$-sentence that characterizes $\aleph_\alpha$ and d is a real with the property that $\Phi(c, d)$ holds, then d is a code for a complete $\mathcal{L}_{\omega_1,\omega}$-sentence that characterizes $\aleph_{\alpha+1}$.

Note that since it is possible to force CH over a model of ($\vartriangle$) without adding new real numbers, the results of this paper show that the property of an $\mathcal{L}_{\omega_1,\omega}$-sentence to characterize $\aleph_2$ is not absolute between models of set theory with the same real numbers.

In the light of results of Woodin [12] that show the existence of a proper class of Woodin cardinals implies that the theory of $L(\mathbb{R})$ with real parameters is generically absolute, it also seems natural to consider the following question:

QUESTION 6.3. Is there a formula $\Phi(v_0, v_1)$ in the language of set theory with the property that the theory

$$\text{ZFC} + \text{``there exists a proper class of Woodin cardinals''}$$

proves the following statements hold?

(1) For every real a, there is a unique real b such that $\Phi(a, b)$ holds in $L(\mathbb{R})$.
(2) If α is a countable ordinal, c is a code for a complete $\mathcal{L}_{\omega_1,\omega}$-sentence that characterizes $\aleph_\alpha$ and d is a real with the property that $\Phi(c, d)$ holds in $L(\mathbb{R})$, then d is a code for a complete $\mathcal{L}_{\omega_1,\omega}$-sentence that characterizes $\aleph_{\alpha+1}$.

We end this paper by considering the question whether versions of the combinatorial principle ($\vartriangle$) can hold at cardinals larger than ω_1. Note that many of the techniques used in the consistency proofs of Sections 3 and 4

(^{7}) Using some canonical Gödelization of $\mathcal{L}_{\kappa,\omega}$-formulas.

have no obvious analogs at higher cardinals. The following question considers two interesting test cases for such generalizations.

QUESTION 6.4. Are the following statements consistent with the axioms of ZFC?

(1) For every sequence $\langle f_\alpha : \omega_2 \to \omega_2 \mid \alpha < \omega_2 \rangle$ of functions, there is a function $g : \omega_2 \to \omega_2$ such that the set $\{\xi < \omega_2 \mid f_\alpha(\xi) = g(\xi)\}$ is finite for every $\alpha < \omega_2$.

(2) For every sequence $\langle f_\alpha : \omega_\omega \to \omega_\omega \mid \alpha < \omega_\omega \rangle$ of functions, there is a function $g : \omega_\omega \to \omega_\omega$ such that the set $\{\xi < \omega_\omega \mid f_\alpha(\xi) = g(\xi)\}$ is finite for every $\alpha < \omega_\omega$.

Note that the above question for ω_2 should be closely connected with recent work of Moore and Todorčević [8] and Neeman on *Baumgartner's isomorphism problem for $\aleph_2$-dense suborders of the reals*, because, by adding all constant functions from ω_2 to ω_2 to a given family of functions, the statement can be used to show that for every family $\mathcal{F}$ of function from ω_2 to ω_2 of cardinality $\aleph_2$, there exists a finite-to-one function $g : \omega_2 \to \omega_2$ with the property that the set $\{\xi < \omega_2 \mid f(\xi) = g(\xi)\}$ is finite for all $f \in \mathcal{F}$, and therefore it implies the principle $\mathrm{ED}(\omega_2, \omega_2, \omega)$, introduced in [8, Section 3].

References

[1] J. Bagaria, *Bounded forcing axioms as principles of generic absoluteness*, Arch. Math. Logic 39 (2000), 393–401.

[2] J. T. Baldwin, M. Koerwien, and M. C. Laskowski, *Disjoint amalgamation in locally finite AEC*, J. Symbolic Logic 82 (2017), 98–119.

[3] J. E. Baumgartner, *Almost-disjoint sets, the dense set problem and the partition calculus*, Ann. Math. Logic 9 (1976), 401–439.

[4] M. Goldstern and S. Shelah, *The bounded proper forcing axiom*, J. Symbolic Logic 60 (1995), 58–73.

[5] G. Hjorth, *Knight's model, its automorphism group, and characterizing the uncountable cardinals*, J. Math. Logic 2 (2002), 113–144.

[6] A. Kanamori, *The Higher Infinite. Large Cardinals in Set Theory From Their Beginnings*, 2nd ed., Springer Monogr. in Math., Springer, Berlin, 2003.

[7] P. B. Larson, *The nonstationary ideal in the $\mathbb{P}_{\max}$ extension*, J. Symbolic Logic 72 (2007), 138–158.

[8] J. T. Moore and S. Todorčević, *Baumgartner's isomorphism problem for $\aleph_2$-dense suborders of $\mathbb{R}$*, Arch. Math. Logic 56 (2017), 1105–1114.

[9] S. Shelah, *Proper Forcing*, Lecture Notes in Math. 940, Springer, Berlin, 1982.
[10] I. Souldatos, *Notes on cardinals that are characterizable by a complete (Scott) sentence*, Notre Dame J. Formal Logic 55 (2014), 533–551.
[11] S. Todorčević, *Directed sets and cofinal types*, Trans. Amer. Math. Soc. 290 (1985), 711–723.
[12] W. H. Woodin, *The Axiom of Determinacy, Forcing Axioms, and the Nonstationary Ideal*, revised ed., De Gruyter Ser. Logic Appl. 1, De Gruyter, Berlin, 2010.

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