EXTERNAL AUTOMORPHISMS OF ULTRAPRODUCTS OF FINITE MODELS

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Abstract. Let $\mathcal{L}$ be a finite first-order language and $\langle \mathcal{M}_n \mid n < \omega \rangle$ be a sequence of finite $\mathcal{L}$-models containing models of arbitrarily large finite cardinality. If the intersection of less than continuum-many dense open subsets of Cantor Space ${}^{\omega}2$ is non-empty, then there is a non-principal ultrafilter $\mathcal{U}$ over ω such that the corresponding ultraproduct $\prod_{\mathcal{U}} \mathcal{M}_n$ has an automorphism that is not induced by an element of $\prod_{n<\omega} \mathrm{Aut}(\mathcal{M}_n)$.

1. Introduction

Let $\mathcal{L}$ be a finite first-order language and $\langle \mathcal{M}_n \mid n < \omega \rangle$ be a sequence of finite $\mathcal{L}$-models. Given a non-principal ultrafilter $\mathcal{U}$ over ω, we let $\prod_{\mathcal{U}} \mathcal{M}_n$ denote the corresponding ultraproduct and $[\vec{x}]_{\mathcal{U}}$ denote the $\equiv_{\mathcal{U}}$-equivalence class of an element $\vec{x}$ in $\prod_{n<\omega} \mathcal{M}_n$. An automorphism π of $\prod_{\mathcal{U}} \mathcal{M}_n$ is *internal* if there is a sequence $\langle \pi_n \in \mathrm{Aut}(\mathcal{M}_n) \mid n < \omega \rangle$ such that $\pi([\vec{x}]_{\mathcal{U}}) = [\vec{y}]_{\mathcal{U}}$ holds for all $\vec{x}, \vec{y} \in \prod_{n<\omega} \mathcal{M}_n$ with $\vec{y}(n) = \pi_n(\vec{x}(n))$ for all $n < \omega$. We call an automorphism π of $\prod_{\mathcal{U}} \mathcal{M}_n$ *external* if it is not internal.

If the continuum hypothesis (CH) holds and $\prod_{\mathcal{U}} \mathcal{M}_n$ is infinite, then it is a saturated model of cardinality $\aleph_1$ (see [1, 6.1.1] and [4]) and the group of automorphisms $\mathrm{Aut}(\prod_{\mathcal{U}} \mathcal{M}_n)$ has cardinality $2^{\aleph_1} > \aleph_1$ (see [1, 5.3.15]). In particular, $\prod_{\mathcal{U}} \mathcal{M}_n$ has many external automorphisms in this situation.

If (CH) fails, then things can look totally different, as the following result (due to the second author) on ultraproducts of the finite fields $\mathbb{F}_p$ of prime order p shows.

Theorem 1.1 ([5]). *It is consistent with the axioms of* ZFC *that there is a non-principal ultrafilter $\mathcal{F}$ over the set $\mathbb{P}$ of primes such that the field $\prod_{\mathcal{F}} \mathbb{F}_p$ has no non-trivial automorphisms.*

Using this result, Simon Thomas and the first author proved the following result on ultraproducts of finite symmetric groups. Recall that for $n \neq 6$ every automorphism of the finite symmetric group $\mathrm{Sym}(n)$ is inner. In particular, every internal automorphism of the group $\prod_{\mathcal{U}} \mathrm{Sym}(n)$ is inner.

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Theorem 1.2 ([3]). *It is consistent with the axioms of* ZFC *that there is a non-principal ultrafilter $\mathcal{U}$ over ω such that all automorphisms of the group $\prod_{\mathcal{U}} \mathrm{Sym}(n)$ are internal.*

The work of this note is motivated by the following questions.

Questions 1. (i) *Is it consistent with the axioms of* ZFC *that for every non-principal ultrafilter $\mathcal{F}$ over the set $\mathbb{P}$ of primes, the field $\prod_{\mathcal{F}} \mathbb{F}_p$ has no non-trivial automorphisms?*

(ii) *Is it consistent with the axioms of* ZFC *that for every non-principal ultrafilter $\mathcal{U}$ over ω, the group $\prod_{\mathcal{U}} \mathrm{Sym}(n)$ has only inner automorphisms?*

The result of this note shows that a mild set-theoretic assumption implies the existence of ultraproducts of finite symmetric groups with external automorphisms.

Theorem 1.3. *Assume that the intersection of less than continuum-many dense open subsets of Cantor Space ${}^{\omega}2$ is non-empty. If $\mathcal{L}$ is a finite first-order language and $\langle \mathcal{M}_n \mid n < \omega \rangle$ is a sequence of finite $\mathcal{L}$-models containing models of arbitrarily large finite cardinality, then there is a non-principal ultrafilter $\mathcal{U}$ over ω such that $\prod_{\mathcal{U}} \mathcal{M}_n$ has an external automorphism.*

The above assumption is known as *Martin's Axiom for Cohen-Forcing* and it is implied both by the continuum hypothesis and Martin's Axiom. Moreover, if $\kappa \geq 2^{\aleph_0}$ is a cardinal of uncountable cofinality, then this assumption holds in every generic extension of the ground model obtained by adding κ-many Cohen reals. This observation is interesting, because it is possible to construct a model of set theory witnessing the conclusions of Theorem 1.1 by adding $\aleph_3$-many Cohen reals to Gödel's constructible universe L (see [5, page 65]).

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2. Extending Partial Elementary Maps

From now on, we let $\mathcal{L}$ denote a finite first-order language and $\langle \mathcal{M}_n \mid n < \omega \rangle$ denote a sequence of finite $\mathcal{L}$-models. We may also assume that $|\mathcal{M}_n| < |\mathcal{M}_{n+1}|$ holds for all $n < \omega$. For the remainder of this note, we fix a sufficiently large regular cardinal θ such that the set $H_\theta = \{x \mid |\mathrm{tc}(x)| < \theta\}$ contains all relevant objects.

Our aim is to inductively construct a non-principal ultrafilter $\mathcal{U}$ over ω and a function $f : \prod_{n<\omega} \mathcal{M}_n \to \prod_{n<\omega} \mathcal{M}_n$ such that there is an external automorphism $\bar{f}$ of $\prod_{\mathcal{U}} \mathcal{M}_n$ with $\bar{f}([\vec{x}]_{\mathcal{U}}) = [\vec{y}]_{\mathcal{U}}$ for all $\vec{x}, \vec{y} \in \prod_{n<\omega} \mathcal{M}_n$ with $f(\vec{x}) = \vec{y}$. We will approximate both $\mathcal{U}$ and f by their intersections with certain elementary submodels of H_θ of cardinality less than $2^{\aleph_0}$. The following definition captures this idea.

Definition 2.1. We call a triple $\langle \mathcal{U}_0, N_0, f_0 \rangle$ *adequate* if it satisfies the following properties.

(1) $\mathcal{U}_0$ is a non-principal ultrafilter over ω.
(2) N_0 is an elementary submodel of H_θ of cardinality less than $2^{\aleph_0}$ containing $f_0, \mathcal{U}_0$ and the sequence $\langle \mathcal{M}_n \mid n < \omega \rangle$.
(3) $f_0 : \prod_{n<\omega} \mathcal{M}_n \xrightarrow{part} \prod_{n<\omega} \mathcal{M}_n$ is a partial function and there is a partial elementary map $\bar{f}_0 : \prod_{\mathcal{U}_0} \mathcal{M}_n \xrightarrow{part} \prod_{\mathcal{U}_0} \mathcal{M}_n$ with the following properties.
   (a) $\mathrm{dom}\left(\bar{f}_0\right) \subseteq N_0 \cap \prod_{\mathcal{U}_0} \mathcal{M}_n$.

(b) $\mathrm{dom}\,(f_0) = \{\vec{x} \in \prod_{n<\omega} \mathcal{M}_n \mid [\vec{x}]_{\mathcal{U}_0} \in \mathrm{dom}\,(\bar{f}_0)\}$.
(c) If $\vec{x} \in \mathrm{dom}\,(f_0)$ with $f_0(\vec{x}) = \vec{y}$, then $\bar{f}_0([\vec{x}]_{\mathcal{U}_0}) = [\vec{y}]_{\mathcal{U}_0}$.

Given adequate triples $\langle \mathcal{U}_0, N_0, f_0 \rangle$ and $\langle \mathcal{U}_1, N_1, f_1 \rangle$, we define

$$\langle \mathcal{U}_0, N_0, f_0 \rangle \preceq \langle \mathcal{U}_1, N_1, f_1 \rangle$$

to hold if $N_0 \cap \mathcal{U}_0 \subseteq \mathcal{U}_1$, $N_0 \subseteq N_1$ and $f_0 \cap N_0 \subseteq f_1$.

The idea behind this definition is that the submodel N_0 fixes the subsets of $\mathcal{U}_0$ and f_0 that will be contained in the final objects $\mathcal{U}$ and f (in the sense that $N_0 \cap \mathcal{U}_0 \subseteq \mathcal{U}$ and $f_0 \cap N_0 \subseteq f$ will hold). Note that f_0 might be the empty function in the above definition. In particular, there always exists an adequate triple. Moreover, if $\langle \mathcal{U}_0, N_0, f_0 \rangle$ is an adequate triple, then the partial map $\bar{f}_0$ is uniquely determined. Finally, it is easy to see that the above ordering of adequate triples is transitive.

Given $\mathcal{U}_0$ and N_0 as in Definition 2.1, we define $\mathcal{M}_{\mathcal{U}_0,N_0}$ to be the unique $\mathcal{L}$-substructure of $\prod_{\mathcal{U}_0} \mathcal{M}_n$ with domain $N_0 \cap \prod_{\mathcal{U}_0} \mathcal{M}_n$. We list some basic properties.

Lemma 2.2. (1) *Let $\mathcal{U}$ and $\mathcal{U}_0$ be non-principal ultrafilters over ω and N_0 be an elementary submodel of H_θ with $\mathcal{U}_0 \in N_0$ and $N_0 \cap \mathcal{U}_0 \subseteq \mathcal{U}$. Then the map*

$$i : \mathcal{M}_{\mathcal{U}_0,N_0} \to \prod_{\mathcal{U}} \mathcal{M}_n \ ; \ [\vec{x}]_{\mathcal{U}_0} \mapsto [\vec{x}]_{\mathcal{U}}$$

is an elementary embedding.

(2) *Given an adequate triple $\langle \mathcal{U}_0, N_0, f_0 \rangle$, we have $\bar{f}_0 : \mathcal{M}_{\mathcal{U}_0,N_0} \xrightarrow{part} \mathcal{M}_{\mathcal{U}_0,N_0}$ and this is a partial elementary map.*

(3) *If $\langle \mathcal{U}_0, N_0, f_0 \rangle \preceq \langle \mathcal{U}_1, N_1, f_1 \rangle$ holds for two adequate triples, then the induced map $i : \mathcal{M}_{\mathcal{U}_0,N_0} \to \mathcal{M}_{\mathcal{U}_1,N_1}$ is an elementary embedding with*

$$i \circ \bar{f}_0 = (\bar{f}_1 \circ i) \restriction \mathrm{dom}\,(\bar{f}_0).$$

Proof. Let $\varphi \equiv \varphi(v_0, \dots, \varphi_{k-1})$ be an $\mathcal{L}$-formula and $\vec{x}_0, \dots, \vec{x}_{k-1} \in N_0 \cap \prod_{n<\omega} \mathcal{M}_n$. By elementarity, we have $\{n < \omega \mid \mathcal{M}_n \models \varphi(\vec{x}_0(n), \dots, \vec{x}_{k-1}(n))\} \in N_0$ and the assumption $N_0 \cap \mathcal{U}_0 \subseteq \mathcal{U}$ allows us to run an easy induction on the complexity of the formula φ to show

$$\begin{aligned} & \mathcal{M}_{\mathcal{U}_0,N_0} \models \varphi([\vec{x}_0]_{\mathcal{U}_0}, \dots, [\vec{x}_{k-1}]_{\mathcal{U}_0}) \\ \Longleftrightarrow & \prod_{\mathcal{U}_0} \mathcal{M}_n \models \varphi([\vec{x}_0]_{\mathcal{U}_0}, \dots, [\vec{x}_{k-1}]_{\mathcal{U}_0}) \\ \Longleftrightarrow & \{n < \omega \mid \mathcal{M}_n \models \varphi(\vec{x}_0(n), \dots, \vec{x}_{k-1}(n))\} \in N_0 \cap \mathcal{U}_0 \subseteq \mathcal{U} \\ \Longleftrightarrow & \prod_{\mathcal{U}} \mathcal{M}_n \models \varphi([\vec{x}_0]_{\mathcal{U}}, \dots, [\vec{x}_{k-1}]_{\mathcal{U}}) \end{aligned}$$

The other statements follow directly from the above computations and the definitions of $\mathcal{M}_{\mathcal{U},N}$ and $\bar{f}$. □

Proposition 2.3. *Suppose $\mathcal{U}_0$, X, f^* and p satisfy the following statements.*

(1) *$\mathcal{U}_0$ be a non-principal ultrafilter over ω and $X \subseteq H_\theta$ with $|X| < 2^{\aleph_0}$.*
(2) *$p : \prod_{n<\omega} \mathcal{M}_n \xrightarrow{part} \prod_{n<\omega} \mathcal{M}_n$ is a partial function.*
(3) *$f^* : \prod_{\mathcal{U}_0} \mathcal{M}_n \xrightarrow{part} \prod_{\mathcal{U}_0} \mathcal{M}_n$ is a partial elementary map with $|\mathrm{dom}\,(f^*)| < 2^{\aleph_0}$, $\{[\vec{x}]_{\mathcal{U}_0} \mid \vec{x} \in \mathrm{dom}\,(p)\} \subseteq \mathrm{dom}\,(f^*)$ and $f^*([\vec{x}]_{\mathcal{U}_0}) = [\vec{y}]_{\mathcal{U}_0}$ for all $\vec{x} \in \mathrm{dom}\,(p)$ with $p(\vec{x}) = \vec{y}$.*

Then there is an adequate triple $\langle \mathcal{U}_0, N_0, f_0\rangle$ such that $p \subseteq f_0$, $f^* = \bar{f}_0$, $X \subseteq N_0$ and $|N_0| = |X| + |\mathrm{dom}\,(f^*)| + \aleph_0$.

Proof. We can find a partial function $f_0 : \prod_{n<\omega} \mathcal{M}_n \xrightarrow{part} \prod_{n<\omega} \mathcal{M}_n$ with $p \subseteq f_0$, $\mathrm{dom}\,(f_0) = \{\vec{x} \mid [\vec{x}]_{\mathcal{U}_0} \in \mathrm{dom}\,(f^*)\}$ and $f^*([\vec{x}]_{\mathcal{U}_0}) = [\vec{y}]_{\mathcal{U}_0}$ for all $\vec{x} \in \mathrm{dom}\,(f_0)$ with $f_0(\vec{x}) = \vec{y}$. Let $D \subseteq \prod_{n<\omega} \mathcal{M}_n$ be a complete set of representatives for elements in $\mathrm{dom}\,(f^*)$. Then clearly $|D| = |\mathrm{dom}\,(f^*)| < 2^{\aleph_0}$. Finally, pick an elementary submodel N_0 of H_θ with $D \cup X \cup \{f_0, \mathcal{U}_0, \langle \mathcal{M}_n \mid n < \omega\rangle\} \subseteq N_0$ and $|N_0| = |D| + |X| + \aleph_0$. □

Lemma 2.4. *Let $\langle \mathcal{U}_0, N_0, f_0\rangle$ be an adequate triple and $x \in H_\theta$. There is an adequate triple $\langle \mathcal{U}_0, N_1, f_1\rangle$ with the following properties.*

1. *$\langle \mathcal{U}_0, N_0, f_0\rangle \preceq \langle \mathcal{U}_0, N_1, f_1\rangle$, $|N_0| = |N_1|$ and $x \in N_1$.*
2. *$\mathrm{dom}\,(\bar{f}_1)$ is a definably closed subset of $\mathcal{M}_{\mathcal{U}_0, N_1}$.*

Proof. There is a unique extension $f^* : \mathcal{M}_{\mathcal{U}_0,N_0} \xrightarrow{part} \mathcal{M}_{\mathcal{U}_0,N_0}$ of $\bar{f}_0$ to a partial elementary map whose domain is the definable closure of $\mathrm{dom}\,(\bar{f}_0)$. Clearly, $\mathrm{dom}\,(f^*)$ is definably closed in $\prod_{\mathcal{U}_0} \mathcal{M}_n$, $|\mathrm{dom}\,(f^*)| \leq |\mathrm{dom}\,(\bar{f}_0)| + \aleph_0 \leq |N_0| < 2^{\aleph_0}$ and we can apply Proposition 2.3 with $\mathcal{U}_0$, $N_0 \cup \{x\}$, f^* and $f_0 \cap N_0$ to find an adequate triple $\langle \mathcal{U}_0, N_1, f_1\rangle$ with the above properties. In particular, $\bar{f}_1 = f^*$ and $\mathrm{dom}\,(f^*) \subseteq \mathcal{M}_{\mathcal{U}_0,N_0} \subseteq \mathcal{M}_{\mathcal{U}_0,N_1}$. By the first part of Lemma 2.2, $\mathcal{M}_{\mathcal{U}_0,N_0}$ is an elementary substructures of $\mathcal{M}_{\mathcal{U}_0,N_1}$ and this shows that $\mathrm{dom}\,(\bar{f}) = \mathrm{dom}\,(f^*)$ is definably closed in $\mathcal{M}_{\mathcal{U}_0,N_1}$. □

We are now ready to state and prove the Main Lemma needed in the proof of Theorem 1.3. This lemma allows us to enlarge the domain of our partial functions and make sure that the final automorphism will be external.

Lemma 2.5. *Assume that the intersection of less than continuum-many dense open subsets of Cantor Space ${}^\omega 2$ is non-empty. Let $\langle \mathcal{U}_0, N_0, f_0\rangle$ be an adequate triple with $\mathrm{dom}\,(\bar{f}_0)$ definably closed in $\mathcal{M}_{\mathcal{U}_0,N_0}$, $\vec{w}, \vec{z} \in N_0 \cap \prod_{n<\omega} \mathcal{M}_n$ with $\vec{z} \notin \mathrm{dom}\,(f_0)$. There is an adequate triple $\langle \mathcal{U}_1, N_1, f_1\rangle$ with the following properties.*

1. *$\langle \mathcal{U}_0, N_0, f_0\rangle \preceq \langle \mathcal{U}_1, N_1, f_1\rangle$ and $|N_1| = |N_0|$.*
2. *$\vec{z} \in \mathrm{dom}\,(f_1)$ and $\bar{f}_1([\vec{z}]_{\mathcal{U}_1}) \neq [\vec{w}]_{\mathcal{U}_1}$.*

Proof. We write $\bar{z} = [\vec{z}]_{\mathcal{U}_0}$ and $\mathcal{M}^* = \mathcal{M}_{\mathcal{U}_0,N_0}$. For each $n < \omega$, we equip the model $\mathcal{M}_n$ with the discrete topology and the product $K = \prod_{i<2}(\prod_{n<\omega} \mathcal{M}_n)$ with the corresponding product topology. This means K is a perfect non-empty, compact metrizable, zero-dimensional topological space and homeomorphic to Cantor Space ${}^\omega 2$ (see [2, Theorem 7.4]).

Define T to be the set of all tuples $\langle A, m, \varphi, \langle \vec{x}_0, \ldots, \vec{x}_{k-1}\rangle\rangle$ with the following properties.

- (i) $A \in N_0 \cap \mathcal{U}_0$, $m < \omega$ and $\vec{x}_0, \ldots, \vec{x}_{k-1} \in \mathrm{dom}\,(f_0) \cap N_0$.
- (ii) $\varphi \equiv \varphi(u_0, u_1, v_0, \ldots, v_{k-1})$ is an $\mathcal{L}$-formula and

$$\mathcal{M}^* \models (\exists a, b)\ \varphi(a, b, [\vec{x}_0]_{\mathcal{U}_0}, \ldots, [\vec{x}_{k-1}]_{\mathcal{U}_0}).$$

Given $\vec{t} = \langle A, m, \varphi, \langle \vec{x}_0, \ldots, \vec{x}_{k-1}\rangle\rangle \in T$ with $f_0(\vec{x}_i) = \vec{y}_i$, the above definition implies that the set

$$D_{\vec{t}} = \{\langle \vec{a}, \vec{b}\rangle \in K \mid (\exists n \in A \setminus m)\ \mathcal{M}_n \models \varphi(\vec{a}(n), \vec{b}(n), \vec{y}_0(n), \ldots, \vec{y}_{k-1}(n))\}$$

is a dense and open subset of K. The set T has cardinality less than $2^{\aleph_0}$ and our assumptions imply that there is a pair $\langle\vec{a}_*,\vec{b}_*\rangle \in \bigcap_{\vec{t}\in T} D_{\vec{t}}$.

Next, we define U to be the set consisting of triples $\vec{u} = \langle A,\varphi,\langle\vec{y}_0,\ldots,\vec{y}_{k-1}\rangle\rangle$ such that $A \in \mathcal{U}_0 \cap N_0$, $\vec{y}_0,\ldots,\vec{y}_{k-1} \in N_0 \cap \prod_{n<\omega}\mathcal{M}_n$ and $\varphi \equiv \varphi(v,\bar{x}_0,\ldots,\bar{x}_{k-1})$ is an element of $\mathrm{tp}_{\mathcal{M}^*}(\bar{z}/\operatorname{dom}(\bar{f}_0))$ with $\bar{f}_0(\bar{x}_i) = [\vec{y}_i]_{\mathcal{U}_0}$. Given such an $\vec{u}\in U$, we define

$$X_{\vec{u}} = \{n \in A \mid \mathcal{M}_n \models [\vec{a}_*(n) \neq \vec{b}_*(n) \wedge \varphi(\vec{a}_*(n),\vec{y}_0(n),\ldots,\vec{y}_{k-1}(n)) \wedge \varphi(\vec{b}_*(n),\vec{y}_0(n),\ldots,\vec{y}_{k-1}(n))]\}$$

and $\mathcal{D} = \{X_{\vec{u}} \mid \vec{u}\in U\}$. Clearly, $\mathcal{D}$ is closed under taking finite intersections.

Assume, toward a contradiction, that there is an $\vec{u} = \langle A,\varphi,\langle\vec{y}_0,\ldots,\vec{y}_{k-1}\rangle\rangle \in U$ as above and $m<\omega$ with $X_{\vec{u}} \subseteq m$. By assumptions, $\bar{z}\notin\operatorname{dom}(\bar{f}_0)$ and $\operatorname{dom}(\bar{f}_0)$ is a definably closed subset of $\mathcal{M}^*$. This shows $\mathcal{M}^* \models (\exists a)\ \psi(\bar{z},a,\bar{x}_0,\ldots,\bar{x}_{k-1})$ with

$$\psi(u_0,u_1,\vec{v}) \equiv [u_0 \neq u_1 \wedge \varphi(u_0,\vec{v}) \wedge \varphi(u_1,\vec{v})].$$

Given $\vec{x}_0,\ldots,\vec{x}_{k-1} \in N_0\cap\prod_{n<\omega}\mathcal{M}_n$ with $\bar{x}_i = [\vec{x}_i]_{\mathcal{U}_0}$, our assumption implies that the tuple $\vec{t} = \langle A,m,\psi,\langle\vec{x}_0,\ldots,\vec{x}_{k-1}\rangle\rangle$ is an element of T. Since $\langle\vec{a}_*,\vec{b}_*\rangle \in D_{\vec{t}}$, we can find a natural number $n\in A\setminus m$ with $\mathcal{M}_n \models \psi(\vec{a}_*(n),\vec{b}_*(n),\vec{y}_0(n),\ldots,\vec{y}_{k-1}(n))$ and hence $n\in X_{\vec{u}}\setminus m$, a contradiction. This argument shows that there is a non-principal ultrafilter $\mathcal{U}_1$ over ω with $\mathcal{D}\subseteq\mathcal{U}_1$ and hence such that

$$\{n\in A \mid \vec{a}_*(n) \neq \vec{b}_*(n)\} \in \mathcal{U}_1$$

for all $A\in N_0\cap\mathcal{U}_0$. Clearly this implies $N_0\cap\mathcal{U}_0 \subseteq \mathcal{U}_1$ and $[\vec{a}_*]_{\mathcal{U}_1} \neq [\vec{b}_*]_{\mathcal{U}_1}$. Therefore, we may assume $[\vec{a}_*]_{\mathcal{U}_1} \neq [\vec{w}]_{\mathcal{U}_1}$.

Let $i : \mathcal{M}_{\mathcal{U}_0,N_0} \to \prod_{\mathcal{U}_1}\mathcal{M}_n$ be the elementary embedding given by Lemma 2.2. Suppose $\vec{x}_0,\ldots,\vec{x}_{k-1} \in N_0\cap\operatorname{dom}(f_0)$ and $\prod_{\mathcal{U}_1}\mathcal{M}_n \models \varphi([\vec{z}]_{\mathcal{U}_1},[\vec{x}_0]_{\mathcal{U}_1},\ldots,[\vec{x}_{k-1}]_{\mathcal{U}_1})$ with $f_0(\vec{x}_i) = \vec{y}_i$. This means

$$\vec{u} = \langle\omega,\varphi(v,[\vec{x}_0]_{\mathcal{U}_0},\ldots,[\vec{x}_{k-1}]_{\mathcal{U}_0}),\langle\vec{y}_0,\ldots,\vec{y}_{k-1}\rangle\rangle \in U$$

and hence $X_{\vec{u}}\in\mathcal{D}\subseteq\mathcal{U}_1$. In particular,

$$\{n<\omega \mid \mathcal{M}_n \models \varphi(\vec{a}_*(n),\vec{y}_0(n),\ldots,\vec{y}_{k-1}(n))\} \in \mathcal{U}_1$$

and therefore $\prod_{\mathcal{U}_1}\mathcal{M}_n \models \varphi([\vec{a}_*]_{\mathcal{U}_1},[\vec{y}_0]_{\mathcal{U}_1},\ldots,[\vec{y}_{k-1}]_{\mathcal{U}_1})$. This implication shows that there is a partial elementary map $f^* : \prod_{\mathcal{U}_1}\mathcal{M}_n \xrightarrow{part} \prod_{\mathcal{U}_1}\mathcal{M}_n$ whose domain is the image of $\operatorname{dom}(\bar{f}_0)\cup\{\bar{z}\}$ under i that satisfies $f^*([\vec{z}]_{\mathcal{U}_1}) = [\vec{a}_*]_{\mathcal{U}_1}$ and $f^*([\vec{x}]_{\mathcal{U}_1}) = [\vec{y}]_{\mathcal{U}_1}$ for all $\vec{x}\in N_0\cap\operatorname{dom}(f_0)$ with $f_0(\vec{x}) = \vec{y}$. We can apply Proposition 2.3 with $\mathcal{U}_1$, N_0, f^* and $f_0\cap N_0$ to obtain the desired adequate triple $\langle\mathcal{U}_1,N_1,f_1\rangle$. □

Corollary 2.6. *Assume that the intersection of less than continuum-many dense open subsets of Cantor Space ${}^\omega 2$ is non-empty. Let $\langle\mathcal{U}_0,N_0,f_0\rangle$ be an adequate triple such that $\operatorname{dom}(\bar{f}_0)$ is definably closed in $\mathcal{M}_{\mathcal{U}_0,N_0}$ and $\vec{z}\in N_0\cap\prod_{n<\omega}\mathcal{M}_n$. Then there is an adequate triple $\langle\mathcal{U}_1,N_1,f_1\rangle$ such that $\langle\mathcal{U}_0,N_0,f_0\rangle \preceq \langle\mathcal{U}_1,N_1,f_1\rangle$, $|N_1| = |N_0|$ and $[\vec{z}]_{\mathcal{U}_1}\in\operatorname{ran}(\bar{f}_1)$ holds.*

Proof. We may assume $[\vec{z}]_{\mathcal{U}_0}\notin\operatorname{ran}(\bar{f}_0)$. By the elementarity of $\bar{f}_0$, $\operatorname{ran}(\bar{f}_0)$ is definably closed in $\mathcal{M}_{\mathcal{U}_0,N_0}$. We apply Proposition 2.3 with $\mathcal{U}_0$, N_0, $\bar{f}_0^{-1}$ and $p=\emptyset$ to obtain an adequate triple $\langle\mathcal{U}_0,M_0,g_0\rangle$ with $\bar{g}_0 = \bar{f}_0^{-1}$, $N_0\subseteq M_0$ and $|M_0| = |N_0|$. In particular, $\vec{z}\notin\operatorname{dom}(g_0)$ and $\operatorname{dom}(\bar{g}_0)$ is definably closed in $\mathcal{M}_{\mathcal{U}_0,M_0}$. Next, we use Lemma 2.5 with $\langle \mathcal{U}_0, M_0, g_0 \rangle$ and $\vec{z}$ to find an adequate triple $\langle \mathcal{U}_1, M_1, g_1 \rangle$ with $\langle \mathcal{U}_0, M_0, g_0 \rangle \preceq \langle \mathcal{U}_1, M_1, g_1 \rangle$, $|M_1| = |N_0|$ and $\vec{z} \in \operatorname{dom}(g_1)$.

Let $\vec{x} \in N_0 \cap \operatorname{dom}(f_0) \subseteq M_0$ with $f_0(\vec{x}) = \vec{y}$. Then $\vec{y} \in M_0 \cap \operatorname{dom}(g_0)$ and therefore $g_0(\vec{y}) = \vec{x}'$ for some $\vec{x}' \in M_0$ with $[\vec{x}]_{\mathcal{U}_0} = \bar{g}_0([\vec{y}]_{\mathcal{U}_0}) = [\vec{x}']_{\mathcal{U}_0}$. We can conclude $\{n < \omega \mid \vec{x}(n) = \vec{x}'(n)\} \in M_0 \cap \mathcal{U}_0 \subseteq \mathcal{U}_1$ and $\bar{g}_1^{-1}([\vec{x}]_{\mathcal{U}_1}) = [\vec{y}]_{\mathcal{U}_1}$. This allows us to use Proposition 2.3 with $\mathcal{U}_1$, M_1, $\bar{g}_1^{-1}$ and f_0 to obtain the desired adequate triple $\langle \mathcal{U}_1, N_1, f_1 \rangle$. □

3. Approximations of External Automorphisms

In this section, we use the above results to inductively construct a non-principal ultrafilter $\mathcal{U}$ over ω such that the corresponding ultraproduct $\prod_{\mathcal{U}} \mathcal{M}_n$ has an external automorphism. In the following, we also fix the following objects.

(1) An enumeration $\langle c_\alpha \mid \alpha < 2^{\aleph_0} \rangle$ of all subsets of ω.
(2) An enumeration $\langle \vec{\sigma}_\alpha \mid \alpha < 2^{\aleph_0} \rangle$ of all sequences in $\prod_{n<\omega} \operatorname{Aut}(\mathcal{M}_n)$.
(3) An enumeration $\langle \vec{x}_\alpha \mid \alpha < 2^{\aleph_0} \rangle$ of all sequences in $\prod_{n<\omega} \mathcal{M}_n$.

The next definition and the following lemma show how external automorphisms are constructed in a routine way from ascending sequences of adequate triples.

Definition 3.1. A sequence $\langle \langle \mathcal{U}_\alpha, N_\alpha, f_\alpha, \vec{z}_\alpha \rangle \mid \alpha < 2^{\aleph_0} \rangle$ is an *approximation of an external automorphism* if the following statements hold true for all $\alpha < 2^{\aleph_0}$.

(1) The triple $\langle \mathcal{U}_\alpha, N_\alpha, f_\alpha \rangle$ is adequate, N_α has cardinality $|\alpha| + |N_0|$, $\operatorname{dom}(\bar{f}_\alpha)$ is definably closed in $\mathcal{M}^*_\alpha = \mathcal{M}_{\mathcal{U}_\alpha, N_\alpha}$ and $c_\alpha, \vec{x}_\alpha, \vec{\sigma}_\alpha \in N_\alpha$.
(2) $\vec{z}_\alpha \notin \operatorname{dom}(f_\alpha)$ and, if $\vec{x}_\alpha \notin \operatorname{dom}(f_\alpha)$, then $\vec{x}_\alpha = \vec{z}_\alpha$.
(3) If $\beta < \alpha$, then $\langle \mathcal{U}_\beta, N_\beta, f_\beta \rangle \preceq \langle \mathcal{U}_\alpha, N_\alpha, f_\alpha \rangle$.
(4) If $\vec{w} \in \prod_{n<\omega} \mathcal{M}_n$ with $\vec{w}(n) = \vec{\sigma}_\alpha(n)(\vec{z}_\alpha(n))$ for all $n < \omega$, then
$$\bar{f}_{\alpha+1}([\vec{z}_\alpha]_{\mathcal{U}_{\alpha+1}}) \neq [\vec{w}]_{\mathcal{U}_{\alpha+1}}.$$
(5) $\vec{z}_\alpha \in N_\alpha \cap \operatorname{dom}(f_{\alpha+1})$ and $[\vec{x}_\alpha]_{\mathcal{U}_{\alpha+1}} \in \operatorname{ran}(\bar{f}_{\alpha+1})$.

Lemma 3.2. *Let $\langle \langle \mathcal{U}_\alpha, N_\alpha, f_\alpha, \vec{z}_\alpha \rangle \mid \alpha < 2^{\aleph_0} \rangle$ be an approximation of an external automorphism. Define $\mathcal{U} = \bigcup_{\alpha<2^{\aleph_0}} N_\alpha \cap \mathcal{U}_\alpha$ and $f = \bigcup_{\alpha<2^{\aleph_0}} f_\alpha \cap N_\alpha$.*

(1) *$\mathcal{U}$ is a non-principal ultrafilter over ω and $f : \prod_{n<\omega} \mathcal{M}_n \to \prod_{n<\omega} \mathcal{M}_n$ is a total function.*
(2) *There is an external automorphism $\bar{f}$ of $\prod_{\mathcal{U}} \mathcal{M}_n$ with $\bar{f}([\vec{x}]_{\mathcal{U}}) = [\vec{y}]_{\mathcal{U}}$ for all $\vec{x} \in \prod_{n<\omega} \mathcal{M}_n$ with $f(\vec{x}) = \vec{y}$.*

Proof. Our construction directly implies that $\mathcal{U}$ is a filter and non-principal. It is an ultrafilter, because every $c \subseteq \omega$ is of the form $c = c_\alpha \in N_\alpha$ for some $\alpha < 2^{\aleph_0}$.

Given $\vec{x} \in \prod_{n<\omega} \mathcal{M}_n$, there is an $\alpha < 2^{\aleph_0}$ with $\vec{x} = \vec{x}_\alpha \in N_\alpha$. If $\vec{x} \in \operatorname{dom}(f_\alpha)$, then $f_\alpha(\vec{x}) \in N_\alpha$, $\langle f_\alpha(\vec{x}), \vec{x} \rangle \in f_\alpha \cap N_\alpha \subseteq f$ and $\vec{x} \in \operatorname{dom}(f)$. Conversely, if $\vec{x} \notin \operatorname{dom}(f_\alpha)$, then $\vec{x} = \vec{z}_\alpha \in \operatorname{dom}(f_{\alpha+1}) \cap N_\alpha$ and we can conclude $\vec{x} \in \operatorname{dom}(f)$ as above. This shows that f is a total function.

By Lemma 2.2, there are elementary embeddings $i_\alpha : \mathcal{M}^*_\alpha \to \prod_{\mathcal{U}} \mathcal{M}_n$ that satisfy $i_\alpha([\vec{x}]_{\mathcal{U}_\alpha}) = [\vec{x}]_{\mathcal{U}}$ for all $\alpha < 2^{\aleph_0}$ and $\vec{x} \in N_\alpha \cap \prod_{n<\omega} \mathcal{M}_n$ and a directed system
$$\langle i_{\alpha,\beta} : \mathcal{M}^*_\alpha \to \mathcal{M}^*_\beta \mid \alpha \leq \beta < 2^{\aleph_0} \rangle$$
of elementary embeddings with $i_\alpha = i_\beta \circ i_{\alpha,\beta}$ for all $\alpha \leq \beta < 2^{\aleph_0}$. Moreover, we have $i_{\alpha,\beta} \circ \bar{f}_\alpha = (\bar{f}_\beta \circ i_{\alpha,\beta}) \restriction \operatorname{dom}(\bar{f}_\alpha)$ for all $\alpha \leq \beta < 2^{\aleph_0}$. This shows that

there is a total elementary map $\bar{f} : \prod_{\mathcal{U}} \mathcal{M}_n \to \prod_{\mathcal{U}} \mathcal{M}_n$ with $\bar{f}([\vec{x}]_{\mathcal{U}}) = [\vec{y}]_{\mathcal{U}}$ for all $\vec{x} \in \prod_{n<\omega} \mathcal{M}_n$ with $f(\vec{x}) = \vec{y}$.

Given $\vec{x} \in \prod_{n<\omega} \mathcal{M}_n$, there is an $\alpha < 2^{\aleph_0}$ with $\vec{x} = \vec{x}_\alpha$ and this implies $[\vec{x}]_{\mathcal{U}_{\alpha+1}} \in \operatorname{ran}\left(\bar{f}_{\alpha+1}\right)$. Since $\vec{x} \in N_{\alpha+1}$, the above computations show that $[\vec{x}]_{\mathcal{U}} \in \operatorname{ran}\left(\bar{f}\right)$ holds. This proves that $\bar{f}$ is surjective.

Finally, assume, toward a contradiction, that $\bar{f}$ is an internal automorphism. Then we can find an $\alpha < 2^{\aleph_0}$ such that $\bar{f}([\vec{x}]_{\mathcal{U}}) = [\vec{y}]_{\mathcal{U}}$ holds for all $\vec{x}, \vec{y} \in \prod_{n<\omega} \mathcal{M}_n$ with $\vec{y}(n) = \vec{\sigma}_\alpha(n)(\vec{x}(n))$ for all $n < \omega$. By definition, $\bar{f}_{\alpha+1}([\vec{z}_\alpha]_{\mathcal{U}_{\alpha+1}}) \neq [\vec{w}]_{\mathcal{U}_{\alpha+1}}$ with $\vec{w}(n) = \vec{\sigma}_\alpha(n)(\vec{z}_\alpha(n))$ and this means $\bar{f}([\vec{z}_\alpha]_{\mathcal{U}}) \neq [\vec{w}]_{\mathcal{U}}$, a contradiction. □

If Martin's Axiom for Cohen-Forcing holds, then we can apply Lemma 2.5 and Corollary 2.6 continuum-many times to construct an approximation of an external automorphism. The next lemma completes the proof of Theorem 1.3.

Lemma 3.3. *Assume that the intersection of less than continuum-many dense open subsets of Cantor Space* ${}^\omega 2$ *is non-empty. If* $\langle \mathcal{U}_*, N_*, f_* \rangle$ *is an adequate triple, then there is an approximation of an external automorphism* $\langle \langle \mathcal{U}_\alpha, N_\alpha, f_\alpha, \vec{z}_\alpha \rangle \mid \alpha < 2^{\aleph_0} \rangle$ *with* $\langle \mathcal{U}_*, N_*, f_* \rangle \preceq \langle \mathcal{U}_0, N_0, f_0 \rangle$.

Proof. We inductively construct the approximating sequence. Assume that we already constructed $\langle \langle \mathcal{U}_\alpha, N_\alpha, f_\alpha, \vec{z}_\alpha \rangle \mid \alpha < \gamma \rangle$ with $\gamma < 2^{\aleph_0}$ such that the statements (1), (2) and (3) of Definition 3.1 hold for all $\alpha < \gamma$ and the statements (4) and (5) of Definition 3.1 hold for all $\alpha < \gamma$ with $\alpha + 1 < \gamma$.

If $\gamma = 0$, then we can use Lemma 2.4 and our cardinality assumptions to construct a tuple $\langle \mathcal{U}_0, N_0, f_0, \vec{z}_0 \rangle$ with $\langle \mathcal{U}_*, N_*, f_* \rangle \preceq \langle \mathcal{U}_0, N_0, f_0 \rangle$ that satisfies the statements (1) and (2) of Definition 3.1.

If $\gamma = \beta + 1$, then $\vec{z}_\beta \notin \operatorname{dom}(f_\beta)$, $\operatorname{dom}\left(\bar{f}_\beta\right)$ is definably closed in $\mathcal{M}^*_\beta$ and there is $\vec{w} \in N_\beta \cap \prod_{n<\omega} \mathcal{M}_n$ with $\vec{w}(n) = \vec{\sigma}_\beta(n)(\vec{z}_\beta(n))$ for all $n < \omega$. In this situation, we can apply Lemma 2.4, Lemma 2.5 and Corollary 2.6 to produce an adequate triple $\langle \mathcal{U}_\gamma, N_\gamma, f_\gamma \rangle$ with the desired properties.

Let γ be a limit ordinal and define $\mathcal{D} = \bigcup_{\alpha<\gamma} N_\alpha \cap \mathcal{U}_\alpha$. This collection of subsets is closed under finite intersections and contains only infinite sets. Let $\mathcal{U}_\gamma$ be a non-principal ultrafilter over ω extending $\mathcal{D}$. As in the proof of Lemma 3.2, there are elementary embeddings $i_\alpha : \mathcal{M}^*_\alpha \to \prod_{\mathcal{U}_\gamma} \mathcal{M}_n$ and $i_{\alpha,\beta} : \mathcal{M}^*_\alpha \to \mathcal{M}^*_\beta$ such that $i_\alpha = i_\beta \circ i_{\alpha,\beta}$ for all $\alpha \leq \beta < \gamma$. Lemma 2.2 shows that there is partial elementary map $f^* : \prod_{\mathcal{U}_\gamma} \mathcal{M}_n \xrightarrow{part} \prod_{\mathcal{U}_\gamma} \mathcal{M}_n$ with domain $\{[\vec{x}]_{\mathcal{U}_\gamma} \mid \vec{x} \in \bigcup_{\alpha<\gamma} \operatorname{dom}(f_\alpha)\}$ that satisfies $i_\alpha \circ \bar{f}_\alpha = (f^* \circ i_\alpha) \restriction \operatorname{dom}\left(\bar{f}_\alpha\right)$ for all $\alpha < \gamma$. Apply Proposition 2.3 with $\mathcal{U}_\gamma$, $\bigcup_{\alpha<\gamma} N_\alpha$, f^* and $\bigcup_{\alpha<\gamma} f_\alpha \cap N_\alpha$ and Lemma 2.4, to obtain an adequate triple $\langle \mathcal{U}_\gamma, N_\gamma, f_\gamma \rangle$ with the desired properties.

By our assumptions, $\prod_{\mathcal{U}_\gamma} \mathcal{M}_n$ is an infinite model and the results of [4] show that it has cardinality $2^{\aleph_0}$. This gives us an $\vec{y} \in \prod_{n<\omega} \mathcal{M}_n$ with $[\vec{y}]_{\mathcal{U}_\gamma} \notin \operatorname{dom}\left(\bar{f}_\gamma\right)$ and therefore $\vec{y} \notin \operatorname{dom}(f_\gamma)$. By elementarity, we can assume $\vec{y} \in N_\gamma$. If $\vec{x}_\gamma \notin \operatorname{dom}(f_\gamma)$, then we define $\vec{z}_\gamma = \vec{x}_\gamma$. Otherwise, we define $\vec{z}_\gamma = \vec{y}$. □

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