PARTITION PROPERTIES FOR SIMPLY DEFINABLE COLOURINGS

PHILIPP LÜCKE

Abstract. We study partition properties for uncountable regular cardinals that arise by restricting partition properties defining large cardinal notions to classes of simply definable colourings. We show that both large cardinal assumptions and forcing axioms imply that there is a homogeneous closed unbounded subset of ω_1 for every colouring of the finite sets of countable ordinals that is definable by a Σ_1-formula that only uses the cardinal ω_1 and real numbers as parameters. Moreover, it is shown that certain large cardinal properties cause analogous partition properties to hold at the given large cardinal and these implications yield natural examples of inaccessible cardinals that possess strong partition properties for Σ_1-definable colourings and are not weakly compact. In contrast, we show that Σ_1-definability behaves fundamentally different at ω_2 by showing that various large cardinal assumptions and *Martin's Maximum* are compatible with the existence of a colouring of pairs of elements of ω_2 that is definable by a Σ_1-formula with parameter ω_2 and has no uncountable homogeneous set. Our results will also allow us to derive tight bounds for the consistency strengths of various partition properties for definable colourings. Finally, we use the developed theory to study the question whether certain homeomorphisms that witness failures of weak compactness at small cardinals can be simply definable.

1. Introduction

Many important results in contemporary set theory show that canonical extensions of the axioms of *Zermelo–Fraenkel set theory* ZFC by large cardinal assumptions or forcing axioms cause small uncountable cardinals to satisfy strong fragments of large cardinal properties. For example, a classical result of Baumgartner shows that the *Proper Forcing Axiom* PFA implies the non-existence of $\aleph_2$-Aronszajn trees (see [5, Theorem 7.2]) and hence this axiom causes the second uncountable cardinal ω_2 to possess a strong fragment of weak compactness. Another example of such a result is given by a theorem of Woodin that shows that the existence of a Woodin cardinal δ causes the first uncountable cardinal ω_1 to possess non-trivial fragments of almost hugeness by showing that, in a generic extension $\mathrm{V}[G]$ of the ground model V, there is an elementary embedding $j : \mathrm{V} \longrightarrow M$ with critical point ω_1^{V} that satisfies $j(\omega_1^{\mathrm{V}}) = \delta$ and $({}^{<\delta}M)^{\mathrm{V}[G]} \subseteq M$ (see [24, Theorem 2.5.8]).

In this paper, we focus on large cardinal properties defined with the help of partition properties and fragments of these properties that arise through restrictions of the considered colourings. Remember that, if X is a set and $k < \omega$, then we let $[X]^k$ denote the collection of all k-element subsets of X and, given a function c with domain $[X]^k$, a subset H of X is *c-homogeneous* if $c \restriction [H]^k$ is constant. A classical result of Erdős and Tarski then shows that an uncountable cardinal κ is weakly compact if and only if for every function $c : [\kappa]^2 \longrightarrow 2$, there is a c-homogeneous subset of κ of cardinality κ. The other large cardinal property defined through partition properties that is relevant for this paper is the concept of *Ramseyness* introduced by Erdős and Hajnal in [13]. They defined an infinite cardinal κ to be *Ramsey* if for every function $c : [\kappa]^{<\omega} \longrightarrow \gamma$ that sends elements of the collection $[\kappa]^{<\omega}$ of all finite subsets of κ to elements of an ordinal $\gamma < \kappa$, there is a subset H of κ of cardinality κ that is $(c \restriction [\kappa]^k)$-homogeneous for all $k < \omega$.

The work presented in this paper studies the fragments of the above partition properties that are obtained by restricting these properties to definable colourings. Similar restrictions have already been studied in [4], [6] and [27], where large cardinal properties are restricted to objects that are *locally* definable, i.e. subsets of $\mathrm{H}(\kappa)$ that are definable over the structure $\langle \mathrm{H}(\kappa), \in \rangle$. In contrast, we will focus on partitions that are *globally* definable, i.e. subsets of $\mathrm{H}(\kappa)$ that are definable over

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$\langle \mathrm{V}, \in \rangle$. Our results will show that canonical extensions of ZFC by large cardinal assumptions or forcing axioms cause strong partion properties for simply definable colourings to hold at ω_1 and that several well-known large cardinal properties provide examples of inaccessible cardinals that are not weakly compact but possess strong partition properties for simply definable colourings. In contrast, we show that neither large cardinal assumptions nor forcing axioms yield similar partition properties for ω_2.

Before we formulate the relevant partition properties, we make two observations that suggest that the validity of partition principles for simply definable functions can be considered intuitively plausible and also foundationally desirable. First, we will later show that the axioms of ZFC already prove such partition properties for colourings that are very simply definable, i.e. functions defined by formulas that only use bounded quantifiers and parameters contained in $\mathrm{H}(\kappa) \cup \{\kappa\}$ (see Theorem 2.1 below). Therefore it is reasonable to expect that canonical extensions of ZFC expand this implication to larger classes of simply definable partitions. Second, if we look at the colourings that witness failures of weak compactness of small cardinals, then the constructions of these functions rely on complicated objects, like κ-Aronszajn trees or well-orderings of the collection $\mathrm{H}(\kappa)$ of all sets of hereditary cardinality less than κ, that can, in general, only be obtained through applications of the *Axiom of Choice* AC. For example, the canonical colouring $c : [\omega_1]^2 \longrightarrow 2$ witnessing the failure of the weak compactness of ω_1 is constructed by using AC to find an injection $\iota : \omega_1 \longrightarrow \mathbb{R}$ of ω_1 into the real line $\mathbb{R}$ and then setting

$$c(\{\alpha, \beta\}) = 1 \iff \iota(\alpha) < \iota(\beta)$$

for all $\alpha < \beta < \omega_1$. Moreover, it is well-known that these applications of AC are actually necessary to derive failures of weak compactness at accessible cardinals, because the axioms of ZF are consistent with the statement that ω_1 is weakly compact (see [19]). This suggests that the partitions witnessing failures of weak compactness of small cardinals should be viewed as complicated objects and therefore it seems natural to expect canonical extensions of ZFC to imply that these functions are not simply definable.

In the following, we formulate the principles studied in this paper. Remember that a formula in the language $\mathcal{L}_\in = \{\in\}$ of set theory is a Σ_0*-formula* if it is contained in the smallest collection of $\mathcal{L}_\in$-formulas that contains all atomic formulas and is closed under negations, conjunctions and bounded quantification. Moreover, a $\mathcal{L}_\in$-formula is a Σ_{n+1}*-formula* for some $n < \omega$ if it is of the form $\exists x \ \neg\varphi$ for some Σ_n-formula φ. Note that the class of all formulas that are ZFC-provably equivalent to a Σ_{n+1}-formula is closed under existential quantification, bounded quantification, conjunctions and disjunctions. Finally, given sets $z_0, \ldots, z_{m-1}$, we say that a class X is $\Sigma_n(z_0, \ldots, z_{m-1})$*-definable* if there is a Σ_n-formula $\varphi(v_0, \ldots, v_m)$ with the property that $X = \{x \mid \varphi(x, z_0, \ldots, z_{m-1})\}$ holds.[1]

Definition 1.1. Given a cardinal κ, $0 < k < \omega$ and sets $z_0, \ldots, z_{m-1}$, a function c with domain $[\kappa]^k$ is a $\Sigma_n(z_0, \ldots, z_{m-1})$*-partition* if there is a Σ_n-formula $\varphi(v_0, \ldots, v_{k+m+1})$ with the property that for all $\alpha_0 < \ldots < \alpha_{k-1} < \kappa$, the value $c(\{\alpha_0, \ldots, \alpha_{k-1}\})$ is the unique set y such that $\varphi(\alpha_0, \ldots, \alpha_{k-1}, y, \kappa, z_0, \ldots, z_{m-1})$ holds.

It is easy to see that if κ is a cardinal and $n > 0$, then a function c with domain $[\kappa]^k$ is a $\Sigma_n(z_0, \ldots, z_{m-1})$-partition if and only if the set c is $\Sigma_n(\kappa, z_0, \ldots, z_{m-1})$-definable. Moreover, since we allow the cardinal κ as a parameter in the definition of the graphs of our partitions, these sets will in fact be Δ_n-definable, i.e. there also is a Π_n-formula (i.e. a negated Σ_n-formula) that defines the function c in the above way. In addition, the same argument shows that, if we instead consider Π_n-definable partitions, then we end up with the same class of functions.

The next definition shows how we restrict weak compactness to the definable context.

Definition 1.2. Let κ be an uncountable regular cardinal.

(i) Given sets $z_0, \ldots, z_{m-1}$, the cardinal κ has the $\Sigma_n(z_0, \ldots, z_{m-1})$*-colouring property* if for all $\Sigma_n(z_0, \ldots, z_{m-1})$-partitions $c : [\kappa]^2 \longrightarrow 2$, there is a c-homogeneous set of cardinality κ.

[1] In the case $m = 0$, we will simply say that X is Σ_n*-definable*. Similar naming conventions are used throughout this paper, i.e. whenever some property is defined relative to some finite lists of parameters, then the omission of parameters in the name of the given property refers to the *lightface* (i.e. parameter-free) instance of the property. In the case of the properties defined in Definitions 1.2 and 1.3 below, the reader should be careful to distinguish these lightface properties from the corresponding boldface properties defined there.

(ii) The cardinal κ has the *$\mathbf{\Sigma}_n$-colouring property*[2] if it has the $\Sigma_n(z_0, \ldots, z_{m-1})$-colouring property for all $m < \omega$ and all $z_0, \ldots, z_{m-1} \in \mathrm{H}(\kappa)$.

The results of this paper will show that the assumption V = HOD implies that all cardinals with the Σ_2-colouring property are already weakly compact. Since all extensions of ZFC that we consider in this paper are compatible with the assumption V = HOD, this result shows that the above property is most interesting for $n \leq 1$. The restriction of parameters to the set $\mathrm{H}(\kappa) \cup \{\kappa\}$ in the second part of the above definition is supposed to prevent partitions witnessing failures of weak compactness to be used as parameters in our definitions. Note that the class of sets that are definable by a Σ_1-formula with parameters in $\mathrm{H}(\kappa) \cup \{\kappa\}$ was already studied in detail in [29] and there it was shown that for certain uncountable regular cardinals κ, canonical extensions of ZFC provide a strong structure theory for this rich class of objects.

We will later show that every uncountable regular cardinal has the $\mathbf{\Sigma}_0$-colouring property and this statement cannot be strengthened to $n = 1$, because cardinals with the $\mathbf{\Sigma}_1$-colouring property will turn out to be inaccessible with high Mahlo-degree in Gödel's constructible universe L. But the results of this paper will allow us to show that successors of regular cardinals, successors of singular cardinals of countable cofinality and non-weakly compact inaccessible cardinals can all consistently possess the $\mathbf{\Sigma}_n$-colouring property for all $n < \omega$. Moreover, we will show that many canonical extensions of ZFC cause ω_1 to have the $\mathbf{\Sigma}_1$-colouring property. Finally, ZFC alone proves that several types of non-weakly compact large cardinals have this property. In contrast, we will show that the influence of large cardinal assumptions and forcing axioms on Σ_1-definability at ω_2 is completely different from the effect of these extensions of ZFC on Σ_1-definability at ω_1 by showing that these assumptions are compatible with a failure of the Σ_1-colouring property at ω_2. These arguments will also allow us to answer one of the main questions left open by the results of [29] by showing that the existence of a $\Sigma_1(\omega_2)$-definable well-ordering of the real numbers is compatible with the existence of various very large cardinal assumptions (see [29, Question 7.5]). Finally, we will show that the $\mathbf{\Sigma}_2$-colouring property provably fails for all successors of singular strong limits cardinals of uncountable cofinality.

In the proofs of the positive results mentioned above, we will often derive the following much stronger partition property for definable colourings.

Definition 1.3. Let κ be an uncountable regular cardinal.

(i) Given sets $z_0, \ldots, z_{m-1}$, the cardinal κ has the *$\Sigma_n(z_0, \ldots, z_{m-1})$-club property* if for every $0 < k < \omega$, every $\alpha < \kappa$ and every $\Sigma_n(z_0, \ldots, z_{m-1})$-partition $c : [\kappa]^k \longrightarrow \alpha$, there is a c-homogeneous set that is closed and unbounded in κ.
(ii) The cardinal κ has the *$\mathbf{\Sigma}_n$-club property* if it has the $\Sigma_n(z_0, \ldots, z_{m-1})$-club property for all $m < \omega$ and all $z_0, \ldots, z_{m-1} \in \mathrm{H}(\kappa)$.

For $n > 0$, the $\mathbf{\Sigma}_n$-club property can easily be seen as a strengthening of the restriction of the partition property defining Ramsey cardinals to definable colourings, because, if $c : [\kappa]^{<\omega} \longrightarrow \alpha$ is a function with $\alpha < \kappa$ that is definable by a Σ_n-formula with parameter $z \in \mathrm{H}(\kappa)$, then the corresponding restrictions $c \restriction [\kappa]^k$ are all $\Sigma_n(z)$-partitions and hence this property yields a club in κ that is $(c \restriction [\kappa]^k)$-homogeneous for all $0 < k < \omega$. We will present more justification for this view by showing that this implication also holds true when we consider alternative characterizations of Ramseyness through the existence of certain iterable models containing subsets of κ and the

[2] This name was chosen to avoid conflicts with the definitions of [4] and [6], where *$\mathbf{\Sigma}_n$-weakly compact cardinals* and the *$\mathbf{\Sigma}_n$-partition property* were introduced. If κ is an inaccessible cardinal with the $\mathbf{\Sigma}_1$-colouring property, then the fact that the set $\mathrm{H}(\kappa)$ is $\Sigma_1(\kappa)$-definable implies that κ has the $\mathbf{\Sigma}_1$-partition property (see [4, Definition 2.9]). Moreover, if κ is an inaccessible cardinal with the $\mathbf{\Sigma}_2$-colouring property, then the set $\{\mathrm{H}(\kappa)\}$ is $\Sigma_2(\kappa)$-definable and therefore κ has the $\mathbf{\Sigma}_\omega$-partition property. In addition, if V = L holds and κ is a cardinal with the $\mathbf{\Sigma}_1$-colouring property, then the set $\{\mathrm{H}(\kappa)\}$ is $\Sigma_1(\kappa)$-definable, Corollary 3.2 below shows that κ is inaccessible and hence κ has the $\mathbf{\Sigma}_\omega$-partition property. Finally, if V = L holds and κ is a cardinal with the $\mathbf{\Sigma}_\omega$-partition property, then there is a subset A of κ with the property that the set $\{A\}$ is $\Sigma_1(\kappa)$-definable and whenever a function $c : [\kappa]^2 \longrightarrow 2$ is definable over $\langle \mathrm{L}_\kappa, \in \rangle$ and λ is an ordinal greater than κ with $\mathrm{L}_\lambda[A] \models \mathrm{ZFC}^-$, then $\mathrm{L}_\lambda[A]$ contains a c-homogeneous subset of κ of cardinality κ. In combination with Lemma 3.5 below, this shows that the assumption V = L implies that every cardinal with the $\mathbf{\Sigma}_1$-colouring and the $\mathbf{\Sigma}_\omega$-partition property lies above an inaccessible cardinal with the $\mathbf{\Sigma}_\omega$-partition property. In particular, the $\mathbf{\Sigma}_\omega$-partition property does not provably imply the $\mathbf{\Sigma}_1$-colouring property.

restrictions of these properties to definable subsets. In fact, we will show that in the *Dodd–Jensen core model* K^{DJ}, the $\mathbf{\Sigma}_1$-club property is equivalent to the restriction of Ramseyness to Σ_1-definable subsets of κ in the above sense. In another direction, we will show that for all $n > 0$, the validity of the $\mathbf{\Sigma}_n$-club property is equivalent to the non-existence of bistationary (i.e. stationary and costationary) subsets A of κ with the property that the corresponding set $\{A\}$ is definable by a Σ_n-formula with parameters in $\mathrm{H}(\kappa) \cup \{\kappa\}$.

In the next section, we will show that all uncountable regular cardinals provably have the $\mathbf{\Sigma}_0$-club property and earlier remarks show that this statement cannot be extended to $n = 1$. Moreover, we will later show that the existence of a cardinal with the $\mathbf{\Sigma}_1$-club property implies the existence of $0^{\#}$. A short argument will allow us to show that a cardinal with the $\mathbf{\Sigma}_1$-club property is either equal to ω_1 or a limit cardinal. Moreover, our results will show that many canonical extensions of ZFC cause ω_1 to have the $\mathbf{\Sigma}_1$-club property, several large cardinal notions imply this property at the given large cardinal and the existence of an accessible regular limit cardinal with this property is consistent. Finally, we will show that no cardinal greater than ω_1 has the $\mathbf{\Sigma}_2$-club property and that the statement that ω_1 has the $\mathbf{\Sigma}_n$-club property for all $n < \omega$ is equiconsistent with the existence of a measurable cardinal.

We end this introduction by outlining the content of this paper. As a motivation for the later results of this paper, we show that all uncountable regular cardinals have the $\mathbf{\Sigma}_0$-club property in Section 2. In Section 3, we derive a long list of basic results on the $\mathbf{\Sigma}_n$-colouring property and present two alternative characterizations of this property that are also fragments of properties characterizing weakly compact cardinals. These results will allow us to determine the consistency strength of the $\mathbf{\Sigma}_n$-colouring property in many important cases. Section 4 contains an analogous investigation of the $\mathbf{\Sigma}_n$-club property that provides the consistency strength of all consistent instances of this property. In Section 5, we use results from [29] to show that both large cardinal assumptions and forcing axioms imply that ω_1 has the $\mathbf{\Sigma}_1$-club property. In contrast, the results of Section 6 show that both of these assumptions are compatible with a failure of the Σ_2-colouring property at ω_2. Section 7 contains various examples of non-weakly compact limit cardinals that provably have the $\mathbf{\Sigma}_1$-club property. In Section 8, we will use results from [9], [10] and [34] to study the $\mathbf{\Sigma}_n$-colouring property at successors of singular cardinals. Section 9 contains the results that originally motivated the work of this paper. These results deal with the question whether certain homeomorphisms witnessing failures of weak compactness can be simply definable and connect this question with the $\mathbf{\Sigma}_n$-colouring property. We conclude this paper in Section 10 with some questions raised by its results.

2. Σ_0-definable partitions

As a motivation for the main results of this paper, we show that all uncountable regular cardinals are weakly compact with respect to Σ_0-definable colourings. In fact, we will prove to following stronger statement.

Theorem 2.1. *Every uncountable regular cardinal has the $\mathbf{\Sigma}_0$-club property.*

In order to prove this result, we introduce certain equivalence relations on the classes $[\mathrm{Ord} \setminus \xi]^{<\omega}$ of finite sets of ordinals greater than some fixed ordinal ξ. Given $0 < l < \omega$ and $\xi \in \mathrm{Ord}$, we let E^ξ_l denote the unique equivalence relation on $[\mathrm{Ord} \setminus \xi]^{<\omega}$ with the property that for all $a, b \in [\mathrm{Ord} \setminus \xi]^{<\omega}$, we have $E^\xi_l(a, b)$ if and only if the following statements hold:

(i) $|a| = |b|$.
(ii) Let $\alpha_1 < \ldots < \alpha_k$ be the monotone enumeration of a and let $\beta_1 < \ldots < \beta_k$ be the monotone enumeration of b. Set $\alpha_0 = \beta_0 = \xi$. For all $i < k$, then there are ordinals μ_i, ν_i and ρ_i such that the following statements hold:
  (a) $\alpha_{i+1} = \alpha_i + \omega^l \cdot \mu_i + \rho_i$.
  (b) $\beta_{i+1} = \beta_i + \omega^l \cdot \nu_i + \rho_i$.
  (c) $\rho_i < \omega^l$.
  (d) $\min\{\mu_i, \nu_i\} = 0$ implies $\mu_i = \nu_i = 0$.

Note that we have $E^\chi_{l+1} \subseteq E^\xi_l$ for all $0 < l < \omega$ and $\xi \leq \chi \in \mathrm{Ord}$.

Proposition 2.2. *If $0 < k < \omega$, $0 < l < \omega$, $\xi \in \mathrm{Ord}$, $a \in [\mathrm{Ord} \setminus \xi]^{k+1}$, $b \in [\mathrm{Ord} \setminus \xi]^k$ and $\alpha \in a$ with $E^\xi_{l+1}(a \setminus \{\alpha\}, b)$, then there is $\xi \leq \beta \in \mathrm{Ord} \setminus b$ with $E^\xi_l(a, b \cup \{\beta\})$.*

Proof. Let $\alpha_1 < \ldots < \alpha_k$ be the monotone enumeration of $a \setminus \{\alpha\}$, let $\beta_1 < \ldots < \beta_k$ be the monotone enumeration of b and set $\alpha_0 = \beta_0 = \xi$.

Case 1: $\alpha = \xi$. Pick μ, ν and ρ such that $\alpha_1 = \xi + \omega^{l+1} \cdot \mu + \rho$ and $\beta_1 = \xi + \omega^{l+1} \cdot \nu + \rho$. Since $\alpha \notin a$, we have $\alpha_1 > \xi$ and either $\mu > 0$ or $\rho > 0$. This implies that $\beta_1 > \xi$ and $\xi \notin b$. If we set $\beta = \xi$, then $E^\xi_{l+1}(a, b \cup \{\beta\})$ and therefore $E^\xi_l(a, b \cup \{\beta\})$.

Case 2: $\alpha > \alpha_k$. Pick $\sigma, \tau \in \mathrm{Ord}$ with $\alpha = \alpha_k + \omega^l \cdot \sigma + \tau$ and $\tau < \omega^l$. If we set $\beta = \beta_k + \omega^l \cdot \sigma + \tau > \beta_k$, then $E^\xi_l(a, b \cup \{\beta\})$ holds.

Case 3: $\alpha_i < \alpha < \alpha_{i+1}$ for some $i \leq k$. Pick $\mu, \nu, \rho \in \mathrm{Ord}$ such that $\alpha_{i+1} = \alpha_i + \omega^{l+1} \cdot \mu + \rho$, $\beta_{i+1} = \beta_i + \omega^{l+1} \cdot \nu + \rho$, $\rho < \omega^{l+1}$ and $\min\{\mu, \nu\} = 0$ implies $\mu = \nu = 0$.

Subcase 3.1: $\alpha \geq \alpha_i + \omega^{l+1} \cdot \mu$. Pick $\sigma < \rho$ and $0 < \tau \leq \rho$ with $\alpha = \alpha_i + \omega^{l+1} \cdot \mu + \sigma$ and $\rho = \sigma + \tau$. Set $\beta = \beta_i + \omega^{l+1} \cdot \nu + \sigma$. Then $\alpha_{i+1} = \alpha + \tau$ and $\beta_{i+1} = \beta + \tau > \beta$. This shows that $E^\xi_{l+1}(a, b \cup \{\beta\})$ and therefore $E^\xi_l(a, b \cup \{\beta\})$.

Subcase 3.2: $\alpha < \alpha_i + \omega^{l+1} \cdot \mu$. Since $\alpha_i < \alpha < \alpha_i + \omega^{l+1} \cdot \mu$, we know that $\mu > 0$ in this case. Then we can find $\mu_0 < \mu$, $\mu_1 \leq \mu$ and $\sigma < \omega^{l+1}$ with $\mu = \mu_0 + 1 + \mu_1$ and $\alpha = \alpha_i + \omega^{l+1} \cdot \mu_0 + \sigma$. Pick ordinals π and τ such that $\sigma = \omega^l \cdot \pi + \tau$ and $\tau < \omega^l$. Then $\alpha = \alpha_i + \omega^l \cdot (\omega \cdot \mu_0 + \pi) + \tau$ and

$$\begin{aligned} \alpha + \omega^{l+1} \cdot (1 + \mu_1) + \rho \ &= \ \alpha_i + \omega^{l+1} \cdot \mu_0 + \sigma + \omega^{l+1} + \omega^{l+1} \cdot \mu_1 + \rho \\ &= \ \alpha_i + \omega^{l+1} \cdot (\mu_0 + 1 + \mu_1) + \rho \ = \ \alpha_{i+1}, \end{aligned} \tag{1}$$

because $\sigma < \omega^{l+1}$ implies that $\sigma + \omega^{l+1} = \omega^{l+1}$.

Subcase 3.2.1: $\omega \cdot \mu_0 + \pi = 0$. Then $\tau > 0$ and $\alpha = \alpha_i + \tau$. Set $\beta = \beta_i + \tau > \beta_i$. Since $\mu > 0$ implies $\nu > 0$, we have $\tau + \omega^{l+1} \cdot \nu = \omega^{l+1} \cdot \nu$,

$$\beta + \omega^{l+1} \cdot \nu + \rho \ = \ \beta_i + \tau + \omega^{l+1} \cdot \nu + \rho \ = \ \beta_i + \omega^{l+1} \cdot \nu + \rho \ = \ \beta_{i+1}$$

and $\beta < \beta_{i+1}$. In combination with (1), this shows that $E^\xi_{l+1}(a, b \cup \{\beta\})$ and we can conclude that $E^\xi_l(a, b \cup \{\beta\})$.

Subcase 3.2.2: $\omega \cdot \mu_0 + \pi > 0$. Set $\beta = \beta_i + \omega^l + \tau$. Since $\omega^l + \tau < \omega^{l+1}$ and $\mu > 0$ implies $\nu > 0$, we then have

$$\beta + \omega^{l+1} \cdot \nu + \rho = \beta_i + \omega^l + \tau + \omega^{l+1} \cdot \nu + \rho = \beta_i + \omega^{l+1} \cdot \nu + \rho = \beta_{i+1}.$$

This allows us to conclude that $\beta_i < \beta < \beta_{i+1}$ and $E^\xi_l(a, b \cup \{\beta\})$ holds. □

We now use the above proposition to show that for all Σ_0-formulas, there are indices l and ξ such that the validity of the given formula is invariant across all E^ξ_l-equivalence classes.

Lemma 2.3. *For every Σ_0-formula $\varphi(v_0, \ldots, v_K)$, every natural number $k \leq K$ and every injection $\iota : k+1 \longrightarrow K+1$, there is a natural number $0 < l_{\varphi,\iota} < \omega$ such that*

$$\varphi(y_0, \ldots, y_K) \ \longleftrightarrow \ \varphi(z_0, \ldots, z_K)$$

holds for all sets $y_0, \ldots, y_K, z_0, \ldots, z_K$ such that there are $\xi \in \mathrm{Ord}$ and $a, b \in [\mathrm{Ord} \setminus \xi]^{k+1}$ satisfying the following statements:

- *(i) $E^\xi_{l_{\varphi,\iota}}(a, b)$.*
- *(ii) If $\alpha_0 < \ldots < \alpha_k$ is the monotone enumeration of a and $\beta_0 < \ldots < \beta_k$ is the monotone enumeration of b, then $\alpha_i = y_{\iota(i)}$ and $\beta_i = z_{\iota(i)}$ for all $i \leq k$.*
- *(iii) If $j \leq K \setminus \mathrm{ran}(\iota)$, then $y_j = z_j$ and $\mathrm{tc}(\{y_j\}) \cap \mathrm{Ord} \subseteq \xi$.*

Proof. We prove the above statement by induction on the complexity of φ.

First, assume that φ is atomic and set $l_{\varphi,\iota} = 1$. Then some easy case distinctions show that the above assumptions (ii) and (iii) imply the desired equivalence for φ. In the case of negations and conjunctions, the above statement follows directly from the induction hypothesis if we set $l_{\neg\varphi,\iota} = l_{\varphi,\iota}$ and $l_{\varphi \wedge \psi,\iota} = \max\{l_{\varphi,\iota}, l_{\psi,\iota}\}$. Finally, assume that $\varphi \equiv \exists x \in v_j \ \psi(v_0, \ldots, v_K, x)$ and the above statement holds for $\psi(v_0, \ldots, v_{K+1})$. Given $i \leq k+1$, let $\tau_i : i+1 \longrightarrow k+2$ denote the unique order-preserving function with $i \notin \mathrm{ran}(\tau_i)$ and let $\iota_i : k+2 \longrightarrow K+2$ denote the unique injection with $\iota_i(i) = K+1$ and $\iota(h) = (\iota_i \circ \tau_i)(h)$ for all $h \leq k$. Next, given $i \leq k$, let

$\psi_i(v_0,\ldots,v_K)$ denote the formula obtained from ψ by replacing all occurrences of the variable v_{K+1} with the variable $v_{\iota(i)}$. Define

$$l_{\varphi,\iota} \ = \ \max\{l_{\psi,\iota},l_{\psi_0,\iota},\ldots,l_{\psi_k,\iota},l_{\psi,\iota_0}+1,\ldots,l_{\psi,\iota_{k+1}}+1\}$$

and fix sets $y_0,\ldots,y_K,z_0,\ldots,z_K$, an ordinal ξ and sets $a,b\in[\mathrm{Ord}\setminus\xi]^{k+1}$ that satisfy the above statements (i)-(iii) with respect to ι and $l_{\varphi,\iota}$. Now, assume that there is an $y_{K+1}\in y_j$ such that $\psi(y_0,\ldots,y_{K+1})$ holds. First, if either $j\notin\mathrm{ran}(\iota)$ or $y_{K+1}\in\xi$, then we know that $E^\xi_{l_{\psi,\iota}}(a,b)$, $K+1\notin\mathrm{ran}(\iota)$ and $\mathrm{tc}(\{y_{K+1}\})\cap\mathrm{Ord}\subseteq\xi$. Therefore our induction hypothesis implies that $\psi(z_0,\ldots,z_K,y_{K+1})$ holds in this case. Next, if $y_{K+1}=\alpha_i$ for some $i\leq k$, then $E^\xi_{l_{\psi_i},\iota}(a,b)$ and our induction hypothesis implies that $\psi(z_0,\ldots,z_K,\beta_i)$ holds. Finally, assume that $j\in\mathrm{ran}(\iota)$, $\xi\leq y_{K+1}\notin a$ and y_{K+1} is the i-th element in the monotone enumeration of $a\cup\{y_{K+1}\}$. Then $E^\xi_{l_{\psi,\iota_i}+1}(a,b)$ and Proposition 2.2 yields a $\xi\leq\beta_{k+1}\in\mathrm{Ord}\setminus b$ with $E^\xi_{l_{\psi,\iota_i}}(a\cup\{y_{K+1}\},b\cup\{\beta_{k+1}\})$. In particular, our induction hypothesis implies that $\psi(z_0,\ldots,z_K,\beta_{k+1})$ holds. In all of the above cases, we can conclude that $\varphi(y_0,\ldots,y_K)$ implies $\varphi(z_0,\ldots,z_K)$. Moreover, the same arguments show that $\varphi(z_0,\ldots,z_K)$ also implies $\varphi(y_0,\ldots,y_K)$. □

Proof of Theorem 2.1. Let κ be an uncountable regular cardinal, let z be an element of $\mathrm{H}(\kappa)$, let $\alpha<\kappa$, let $0<k<\omega$ and let $c:[\kappa]^k\longrightarrow\alpha$ be a $\Sigma_0(z)$-partition. Then there is a Σ_0-formula $\varphi(v_0,\ldots,v_{k+1})$ with the property that for $\alpha_0<\ldots<\alpha_{k-1}<\kappa$, $c(\{\alpha_0,\ldots,\alpha_{k-1}\})$ is the unique ordinal γ such that $\varphi(\alpha_0,\ldots,\alpha_{k-1},\kappa,\gamma,z)$ holds. Pick an ordinal $\alpha+\omega^\omega<\xi<\kappa$ with $\mathrm{tc}(\{z\})\cap\mathrm{Ord}\subseteq\xi$, let H be the set of all multiplicatively indecomposable ordinals in the interval $[\xi,\kappa]$ and let ι denote the identity function on $k+1$. Then $\kappa\in H$, $C=H\cap\kappa$ is a club in κ and $E^\xi_{l_{\varphi,\iota}}(a,b)$ holds for all $a,b\in[H]^{k+1}$. But then Lemma 2.3 shows that, if $\alpha_0<\ldots<\alpha_{k-1}$ is the monotone enumeration of $a\in[C]^k$ and $\beta_0<\ldots<\beta_{k-1}$ is the monotone enumeration of $b\in[C]^k$, then

$$\varphi(\alpha_0,\ldots,\alpha_{k-1},\kappa,\gamma,z) \ \longleftrightarrow \ \varphi(\beta_0,\ldots,\beta_{k-1},\kappa,\gamma,z)$$

for all $\gamma<\alpha$ and therefore $c(a)=c(b)$. □

3. The $\mathbf{\Sigma}_n$-colouring property

In the remainder of this paper, we always use n to denote a natural number greater than 0. Note that, since sets of the form $\mathrm{H}(\kappa)$ are closed under the pairing functions, this assumptions allows us to only consider Σ_n-formulas that use a single parameter from $\mathrm{H}(\kappa)$ when we verify that an uncountable regular cardinal κ has the $\mathbf{\Sigma}_n$-colouring property.

This section contains a number of basic results about the $\mathbf{\Sigma}_n$-colouring property that generalize fundamental results about weakly compact cardinals to the definable setting. These results will allow us to show that for all $0<n<\omega$, there is a natural connection between the $\mathbf{\Sigma}_n$-colouring property and a certain large cardinal property, in the sense that, first, the large cardinal property implies the $\mathbf{\Sigma}_n$-colouring property, second, that the $\mathbf{\Sigma}_n$-colouring property implies that the given cardinal has the large cardinal property in L and third, that it is possible to use forcing to turn such a large cardinal into either the successor of a regular cardinal with the $\mathbf{\Sigma}_n$-colouring property or into an accessible regular limit cardinal with this property. For $n\geq 2$, the corresponding large cardinal property will turn out to be weak compactness. In contrast, our results will show that the $\mathbf{\Sigma}_1$-colouring property corresponds to a large cardinal property strictly between Mahloness and weak compactness. Finally, our results will also allow us to present several ways to establish the consistency of failures of definable weak compactness.

The following result transfers the fact that weakly compact cardinals are inaccessible to the definable setting.

Proposition 3.1. *Let κ be an uncountable regular cardinal and let z be a set such that κ has the $\Sigma_n(z)$-colouring property. If $f:\kappa\longrightarrow{}^{<\kappa}2$ is a $\Sigma_n(\kappa,z)$-definable function and $\gamma<\kappa$, then the set $\{f(\alpha)\restriction\gamma\mid\alpha<\kappa\}$ has cardinality less than κ.*

Proof. Assume, towards a contradiction, that there is a $\gamma<\kappa$ with the property that the set $\{f(\alpha)\restriction\gamma\mid\alpha<\kappa\}$ has cardinality at least κ. Let δ be minimal with this property.

Claim. *The set $\{\delta\}$ is $\Sigma_n(\kappa,z)$-definable.*

Proof of the Claim. By the minimality of δ, the set $\{f(\alpha) \restriction \gamma \mid \alpha < \kappa, \ \gamma < \delta\}$ has cardinality less than κ and the set $\{f(\alpha) \restriction \delta \mid \alpha < \kappa, \ \delta \subseteq \mathrm{dom}(f(\alpha))\}$ has cardinality at least κ. Hence δ is the unique element of κ with the property that there exist an ordinal $\beta < \kappa$, a map $s : \beta \longrightarrow {}^{<\delta}2$ and a map $u : \kappa \longrightarrow \kappa$ such that the following statements hold:

(i) If $\alpha < \kappa$ and $\gamma < \delta$, then $f(\alpha) \restriction \gamma \in \mathrm{ran}(s)$.
(ii) If $\alpha < \kappa$, then $\delta \subseteq \mathrm{dom}((f \circ u)(\alpha))$.
(iii) The map
$$v : \kappa \longrightarrow {}^{\delta}2; \ \alpha \longmapsto (f \circ u)(\alpha) \restriction \delta$$
is an injection.

By our assumptions on f, this characterization yields a $\Sigma_n(\kappa, z)$-definition of δ. □

As mentioned above, the set $\{f(\alpha) \restriction \delta \mid \alpha < \kappa, \ \delta \subseteq \mathrm{dom}(f(\alpha))\}$ has cardinality κ and this shows that there is a unique injection $i : \kappa \longrightarrow \kappa$ with the property that for all $\alpha < \kappa$, the image $i(\alpha)$ is the minimal $\beta < \kappa$ with $\delta \subseteq \mathrm{dom}(f(\beta))$ and $f(\beta) \restriction \delta \neq f(i(\bar{\alpha})) \restriction \delta$ for all $\bar{\alpha} < \alpha$. Then the Σ_n-Recursion Theorem implies that i is $\Sigma_n(\kappa, z)$-definable and this shows that the injection
$$\iota : \kappa \longrightarrow {}^{\delta}2; \ \alpha \longmapsto (f \circ i)(\alpha) \restriction \delta$$
is definable in the same way. Set
$$\Delta(\alpha, \beta) \ = \ \min\{\gamma < \delta \mid \iota(\alpha)(\gamma) \neq \iota(\beta)(\gamma)\}$$
for all $\alpha < \beta < \kappa$ and let $c : [\kappa]^2 \longrightarrow 2$ denote the unique map satisfying
$$c(\{\alpha, \beta\}) = 0 \iff \iota(\alpha)(\Delta(\alpha, \beta)) < \iota(\beta)(\Delta(\alpha, \beta))$$
for all $\alpha < \beta < \kappa$. Then c is $\Sigma_n(\kappa, z)$-definable and our assumptions yield a c-homogeneous set H that is unbounded in κ. Given $\gamma < \delta$, define
$$H_\gamma \ = \ \{\alpha \in H \mid \min\{\Delta(\alpha, \beta) \mid \alpha < \beta \in H\} = \gamma\}$$
Since $H = \bigcup\{H_\gamma \mid \gamma < \delta\}$, there is a $\gamma_* < \delta$ with H_{γ_*} unbounded in κ. Then we can find $\alpha_0, \alpha_1 \in H_{\gamma_*}$ and $\beta_0, \beta_1 \in H$ with $\alpha_0 < \beta_0 < \alpha_1 < \beta_1$ and $\gamma_* = \Delta(\alpha_0, \beta_0) = \Delta(\alpha_1, \beta_1)$. By the definition of H_{γ_*}, we then have
$$\iota(\alpha_0) \restriction \gamma_* \ = \ \iota(\beta_0) \restriction \gamma_* \ = \ \iota(\alpha_1) \restriction \gamma_* \ = \ \iota(\beta_1) \restriction \gamma_*$$
and therefore $\iota(\beta_0)(\gamma_*) = \iota(\alpha_1)(\gamma_*)$, because otherwise we would have $\Delta(\alpha_0, \beta_0) = \Delta(\beta_0, \alpha_1) = \gamma_*$ and the homogeneity of H would imply that the ordinals $\iota(\alpha_0)(\gamma_*)$, $\iota(\beta_0)(\gamma_*)$ and $\iota(\alpha_1)(\gamma_*)$ are pairwise different elements of the set $\{0, 1\}$. But then
$$\Delta(\beta_0, \beta_1) \ = \ \Delta(\alpha_1, \beta_1) \ = \ \gamma_* \ = \ \Delta(\alpha_0, \beta_0)$$
and this allows us to conclude that the ordinals $\iota(\alpha_0)(\gamma_*)$, $\iota(\beta_0)(\gamma_*)$ and $\iota(\beta_1)(\gamma_*)$ are pairwise different elements of the set $\{0, 1\}$, a contradiction. □

Corollary 3.2. *Let κ be an uncountable regular cardinal and let z be a set such that κ has the $\Sigma_n(z)$-colouring property. If A is a subset of κ with the property that the set $\{A\}$ is $\Sigma_n(\kappa, z)$-definable, then κ is inaccessible in* $\mathrm{L}[A]$.

Proof. Assume that the above conclusion fails. Let ι denote the $<_{\mathrm{L}[A]}$-least injection of κ into some ${}^{\nu}2$ with $\nu < \kappa$ in $\mathrm{L}[A]$. By our assumptions, the sets $\{\nu\}$ and $\{\iota\}$ are both $\Sigma_n(\kappa, z)$-definable and hence there is a $\Sigma_n(\kappa, z)$-definable injection from κ into ${}^{\nu}2$, contradicting Proposition 3.1. □

Proposition 3.1 also allows us to show that a small partial order can force a failure of the $\mathbf{\Sigma}_1$-colouring property at the successor of an uncountable regular cardinal. In particular, large cardinal axioms do not imply that successors of uncountable regular cardinals have the $\mathbf{\Sigma}_1$-colouring property. The results of Section 5 will show that the situation for ω_1 is completely different.

Corollary 3.3. *If ν is an uncountable cardinal with $\nu = \nu^{<\nu}$, then there is a $<\nu$-closed partial order $\mathbb{P}$ satisfying the ν^+-chain condition with*

(2) $$\mathbb{1}_{\mathbb{P}} \Vdash \text{“}\textit{The cardinal } \nu^+ \textit{ does not have the } \mathbf{\Sigma}_1\textit{-colouring property”}.$$

Proof. Pick an injection $\iota : \nu^+ \longrightarrow {}^{\nu}2$. Given $\alpha < \nu^+$, fix a surjection $s_\alpha : \nu \longrightarrow \iota[\alpha + 1]$ and let x_α denote the unique element of ${}^{\nu}2$ satisfying $x_\alpha(\prec\gamma, \delta\succ) = s_\alpha(\gamma)(\delta)$ for all $\gamma, \delta < \nu$.[3] Define

[3]We use $\prec\cdot, \ldots, \cdot\succ$ to denote (iterated applications of) the Gödel pairing function.

$A = \{x_\alpha \mid \alpha < \nu^+\} \subseteq {}^\nu 2$. By [28, Theorem 1.6], there is a $<\nu$-closed partial order $\mathbb{P}$ satisfying the ν^+-chain condition with the property that whenever G is $\mathbb{P}$-generic over V, then A is a $\mathbf{\Sigma}^1_1$-definable subset of ${}^\nu 2$ in V$[G]$, i.e. in V$[G]$, there is a subtree T of ${}^{<\nu}2 \times {}^{<\nu}\nu$ with

$$A \ = \ \{x \in {}^\nu 2 \mid \exists y \in {}^\nu \nu \ \forall \gamma < \nu \ \langle x \restriction \gamma, \ y \restriction \gamma \rangle \in T\}. \tag{3}$$

Let G be $\mathbb{P}$-generic over V and work in V$[G]$. Pick a subtree T of ${}^{<\nu}2 \times {}^{<\nu}\nu$ witnessing the $\mathbf{\Sigma}^1_1$-definability of A. Since $\mathbb{P}$ is $<\nu$-closed in V, our assumptions imply that $\nu = \nu^{<\nu}$ holds and hence T is an element of H(ν^+). Moreover, the equality (3) directly implies that the set A is $\Sigma_1(\nu, T)$-definable.

Claim. *If $\alpha < \nu^+$, then $\iota(\alpha)$ is the unique element z of ${}^\nu 2 \setminus \iota[\alpha]$ with the property that there exists an element x of A and a function $s : \nu \longrightarrow {}^\nu 2$ with $\mathrm{ran}(s) = \iota[\alpha] \cup \{z\}$ and $x(\prec\gamma, \delta\succ) = s(\gamma)(\delta)$ for all $\gamma, \delta < \nu$.*

Proof of the Claim. Fix $\alpha < \nu^+$. Then $x_\alpha \in A$ and $s_\alpha : \nu \longrightarrow {}^\nu 2$ witness that $\iota(\alpha) \in {}^\nu 2 \setminus \iota[\alpha]$ possesses the desired property. Now, pick $z \in {}^\nu 2 \setminus \iota[\alpha]$, $x \in A$ and $s : \nu \longrightarrow {}^\nu 2$ such that $\mathrm{ran}(s) = \iota[\alpha] \cup \{z\}$ and $x(\prec\gamma, \delta\succ) = s(\gamma)(\delta)$ for all $\gamma, \delta < \nu$. Then $x = x_{\bar\alpha}$ for some $\bar\alpha < \nu^+$ and therefore $s(\gamma)(\delta) = x_{\bar\alpha}(\prec\gamma, \delta\succ) = s_{\bar\alpha}(\gamma)(\delta)$ for all $\gamma, \delta < \nu$. This shows that $s = s_{\bar\alpha}$ and $\iota[\alpha] \cup \{z\} = \mathrm{ran}(s_{\bar\alpha}) = \iota[\bar\alpha + 1]$. Since ι is an injection, we can now conclude that $\alpha = \bar\alpha$ and $z = \iota(\alpha)$. □

In combination with the Σ_1-Recursion Theorem and the fact that A is $\Sigma_1(\nu, T)$-definable in V$[G]$, the above claim shows that the injection $\iota : \nu^+ \longrightarrow {}^\nu 2$ is $\Sigma_1(\nu^+, \nu, T)$-definable in V$[G]$. But then Proposition 3.1 shows that (2) holds. □

Our next aim is to generalize the characterizations of weak compactness through the tree property (see [22, Theorem 7.8]) and the existence of certain elementary embeddings (see [17]) to the definable setting. In order to derive these characterizations, we rely on the following lemma whose statement can be extracted from the proof of the classical *Ramification Lemma* (see, for example, [22, Lemma 7.2]).

Lemma 3.4. *Let κ be an uncountable regular cardinal and let $c : [\kappa]^2 \longrightarrow 2$ be a function.*

- *(i) There exists a sequence $\langle <_\alpha \mid \alpha < \kappa \rangle$ such that the following statements hold for all $\alpha < \kappa$:*
  - *(a) $<_\alpha$ is a binary relation on α that refines the $\in$-relation, the structure $\langle \alpha, <_\alpha \rangle$ is a tree and, if α is a limit ordinal, then $<_\alpha = \bigcup \{<_{\bar\alpha} \mid \bar\alpha < \alpha\}$.*
  - *(b) If $\alpha < \beta < \kappa$, then $<_\alpha = <_\beta \restriction (\alpha \times \alpha)$.*
  - *(c) The set b_α of all predecessors of α in $\langle \alpha + 1, <_{\alpha+1} \rangle$ is a $\subseteq$-maximal element of the collection B_α of all branches[4] b in $\langle \alpha, <_\alpha \rangle$ with the property that $c(\{\alpha_0, \alpha_1\}) = c(\{\alpha_0, \alpha\})$ holds for all $\alpha_0, \alpha_1 \in b$ with $\alpha_0 < \alpha_1$.*
- *(ii) If $\langle <_\alpha \mid \alpha < \kappa \rangle$ is a sequence with the properties listed in (i) and $\alpha < \kappa$, then b_α is the unique $\subseteq$-maximal element of B_α.*
- *(iii) There is a unique sequence $\langle <^c_\alpha \mid \alpha < \kappa \rangle$ with the properties listed in (i).*

Proof. *(i)*: We construct the desired sequence by induction. Pick $0 < \alpha < \kappa$ and assume that we already constructed a sequence $\langle <_{\bar\alpha} \mid \bar\alpha < \alpha \rangle$ of relations with the above properties. First, assume that $\alpha \in \mathrm{Lim}$ and define $<_\alpha = \bigcup \{<_{\bar\alpha} \mid \bar\alpha < \alpha\}$. Then our induction hypothesis implies that $<_\alpha$ refines the $\in$-relation, the structure $\langle \alpha, <_\alpha \rangle$ is a tree and $<_{\bar\alpha} = <_\alpha \restriction (\bar\alpha \times \bar\alpha)$ for all $\bar\alpha < \alpha$. Now, assume that $\alpha = \bar\alpha + 1$. Then $\emptyset \in B_{\bar\alpha} \neq \emptyset$ and, by definition, the collection $B_{\bar\alpha}$ is closed under increasing unions. Hence Zorn's Lemma implies that $B_{\bar\alpha}$ contains a $\subseteq$-maximal element $b \subseteq \bar\alpha$. If we define $<_\alpha = <_{\bar\alpha} \cup \{\langle \beta, \bar\alpha \rangle \mid \beta \in b\}$, then $<_\alpha$ refines the $\in$-relation, the structure $\langle \alpha, <_\alpha \rangle$ is a tree, $<_{\bar\alpha} = <_\alpha \restriction (\bar\alpha \times \bar\alpha)$ and $b \in B_{\bar\alpha}$ is the set of all predecessors of $\bar\alpha$ in $\langle \alpha, <_\alpha \rangle$.

(ii): Assume, towards a contradiction, that there is an $\alpha < \kappa$ and $\subseteq$-maximal elements b^0 and b^1 of B_α with $b^0 \neq b^1$. Since $\langle \alpha, <_\alpha \rangle$ is a tree, we can find $\alpha_0 \in b^0$ and $\alpha_1 \in b^1$ such that $\alpha_0 \notin b^1$, $\alpha_1 \notin b^0$ and both elements have the same predecessors in $\langle \alpha, <_\alpha \rangle$. Pick $i < 2$ with $\alpha_i < \alpha_{1-i}$. Then $b_{\alpha_0} = b_{\alpha_1}$ and $\alpha_i \notin b_{\alpha_{1-i}} \subseteq b^{1-i}$. In particular, if $\beta \in b_{\alpha_{1-i}}$, then $\beta \in b^0 \cap b^1$ and therefore $c(\{\beta, \alpha_i\}) = c(\{\beta, \alpha\}) = c(\{\beta, \alpha_{1-i}\})$. Since $b_{\alpha_{1-i}} \subseteq \alpha_i$, this shows

[4] Remember that, given a tree $\mathbb{T} = \langle T, <_\mathbb{T} \rangle$, a subset of T is a *branch in* $\mathbb{T}$ if it is $\lhd$-downwards-closed and linearly ordered by $\lhd$.

that $c(\{\beta_0,\beta_1\}) = c(\{\beta_0,\alpha_{1-i}\})$ holds for all $\beta_0,\beta_1 \in b_{\alpha_{1-i}} \cup \{\alpha_i\}$ with $\beta_0 < \beta_1$, contradicting the $\subseteq$-maximality of $b_{\alpha_{1-i}}$.

(iii): This statement follows directly from the clauses (a) and (c) in (i) and the uniqueness of the sequences b_α proven in (ii). □

We are now ready to derive the alternative characterizations of the $\Sigma_n(z)$-colouring property. Remember that, given an infinite cardinal κ, a *weak κ-model* is a transitive model M of ZFC^- of size κ with $\kappa \in M$.[5]

Lemma 3.5. *The following statements are equivalent for every uncountable regular cardinal κ and every set z:*

(i) *κ has the $\Sigma_n(z)$-colouring property.*

(ii) *If $\iota : \kappa \longrightarrow {}^{<\kappa}2$ is a $\Sigma_n(\kappa,z)$-definable injection, then there is an $x \in {}^\kappa 2$ with the property that the set $\{\alpha < \kappa \mid \exists \beta < \kappa \ x \restriction \alpha \subseteq \iota(\beta)\}$ is unbounded in κ.*

(iii) *If $A \subseteq \kappa$ with the property that $\{A\}$ is $\Sigma_n(\kappa,z)$-definable, then there is a weak κ-model M, a transitive set N and an elementary embedding $j : M \longrightarrow N$ such that $A \in M$, $\operatorname{crit}(j) = \kappa$, κ is inaccessible in M and $\mathrm{H}(\kappa)^M \in M$.*

Proof. *(i) $\Longrightarrow$ (ii)*: Assume that (i) holds and let $\iota : \kappa \longrightarrow {}^{<\kappa}2$ be a $\Sigma_n(\kappa,z)$-definable injection. Remember that the lexicographic ordering $<_{lex}$ of ${}^{<\kappa}2$ is the unique linear ordering of ${}^{<\kappa}2$ with the property that for all $s,t \in {}^{<\kappa}2$, we have $s <_{lex} t$ if either $s \subsetneq t$ or there is an ordinal $\alpha \in \operatorname{dom}(s) \cap \operatorname{dom}(t)$ with $s \restriction \alpha = t \restriction \alpha$ and $s(\alpha) < t(\alpha)$. Given $s,t,u,v \in {}^{<\kappa}2$ with $s \subseteq t \cap v$ and $t <_{lex} u <_{lex} v$, a short computation shows that $s \subseteq u$ holds. Let $c : [\kappa]^2 \longrightarrow 2$ denote the unique function with the property that for all $\alpha < \beta < \kappa$, we have $c(\{\alpha,\beta\}) = 0$ if and only if $\iota(\alpha) <_{lex} \iota(\beta)$. Then c is a $\Sigma_n(z)$-partition and our assumption yields a c-homogeneous subset H of κ of cardinality κ.

Claim. *Given $\gamma < \kappa$, there is $\gamma < \alpha_\gamma \in H$ and $t_\gamma \in {}^\gamma 2$ with $t_\gamma \subseteq \iota(\alpha)$ for all $\alpha_\gamma < \alpha \in H$.*

Proof of the Claim. By Proposition 3.1, there is a sequence $t_\gamma \in {}^\gamma 2$ with the property that the set $H_\gamma = \{\alpha \in H \mid t_\gamma \subseteq \iota(\alpha)\}$ has cardinality κ. Define $\alpha_\gamma = \min(H_\gamma)$, fix $\alpha_\gamma < \alpha \in H$ and pick $\alpha < \beta \in H_\gamma$. Then $t_\gamma \subseteq \iota(\alpha_\gamma) \cap \iota(\beta)$ and we either have $\iota(\alpha_\gamma) <_{lex} \iota(\alpha) <_{lex} \iota(\beta)$ or $\iota(\beta) <_{lex} \iota(\alpha) <_{lex} \iota(\alpha_\gamma)$. By the above remarks, we can conclude that $t_\gamma \subseteq \iota(\alpha)$. □

Pick $\gamma < \delta < \kappa$ and $\max\{\alpha_\gamma,\alpha_\delta\} < \alpha \in H$. Then the above claim yields $t_\gamma \subseteq t_\delta \subseteq \iota(\alpha)$ and this implies that $x = \bigcup\{t_\gamma \mid \gamma < \kappa\}$ is an element of ${}^\kappa 2$ with the property that the set $\{\alpha < \kappa \mid \exists \beta < \kappa \ x \restriction \alpha \subseteq \iota(\beta)\}$ is unbounded in κ. This shows that (ii) holds with respect to ι.

(ii) $\Longrightarrow$ (i): Assume that (ii) holds. Let $c : [\kappa]^2 \longrightarrow 2$ be a $\Sigma_n(z)$-partition and let $\langle <^c_\alpha \mid \alpha < \kappa\rangle$ be the unique sequence of binary relations corresponding to the function c as in Lemma 3.4. Define $<_c = \bigcup\{<^c_\alpha \mid \alpha < \kappa\}$. Then $\langle \kappa, <_c\rangle$ is a tree and we have $<_c \restriction (\kappa \times \alpha) = <^c_\alpha$ for all $\alpha < \kappa$. Note that, in order to verify the $\subseteq$-maximality of the branch b_α in the collection B_α for some $\alpha < \kappa$, we only need to consider extensions of b_α by single ordinals less than α. Together with our assumptions on the definability of c and the uniqueness of the sequence $\langle <_\alpha \mid \alpha < \kappa\rangle$ provided by Lemma 3.4, this observation allows us to conclude that the set $\{<_c\}$ is $\Sigma_n(\kappa,z)$-definable.

Claim. *Every $\alpha < \kappa$ has at most two direct successors in $\langle \kappa, <_c\rangle$.*

Proof of the Claim. Otherwise, we can find $\alpha < \beta_0 < \beta_1 < \kappa$ such that β_0 and β_1 are both direct successors of α in $\langle \kappa, <_c\rangle$ and $c(\{\alpha,\beta_0\}) = c(\{\alpha,\beta_1\})$. Since $b_{\beta_0} = b_{\beta_1} = b_\alpha \cup \{\alpha\}$, our assumptions imply that $c(\{\alpha_0,\alpha_1\}) = c(\{\alpha_0,\beta_1\})$ holds for all $\alpha_0,\alpha_1 \in b_{\beta_1} \cup \{\beta_0\}$. But this contradicts the maximality of b_{β_1}. □

Now let $\iota : \kappa \longrightarrow {}^{<\kappa}2$ denote the unique injection with $\operatorname{dom}(\iota(\beta)) = \beta + 1$ and

$$\iota(\beta)(\alpha) = 1 \iff (\alpha <_c \beta \ \vee \ \alpha = \beta)$$

for all $\alpha \leq \beta < \kappa$. Then the function ι is $\Sigma_n(\kappa,z)$-definable and our assumption (ii) yields $x \in {}^\kappa 2$ with the property that the set $\{\alpha < \kappa \mid \exists \beta < \kappa \ x \restriction \alpha \subseteq \iota(\beta)\}$ is unbounded in κ. Define $K = \{\alpha < \kappa \mid x(\alpha) = 1\}$. Then $\alpha <_c \beta$ for all $\alpha,\beta \in K$ with $\alpha < \beta$.

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[5]By ZFC$^-$, we mean the usual axioms of ZFC without the power set axiom, however including the Collection scheme instead of the Replacement scheme. Note that $\mathrm{H}(\kappa)$ is a model of this theory for every uncountable regular cardinal κ.

Claim. *The set K is unbounded in κ.*

Proof of the Claim. First, assume that K has a maximal element $\alpha < \kappa$. Then the above claim shows that there is a $\beta \in \mathrm{Lim} \cap \kappa$ such that all direct successor of α in $\langle \kappa, <_c \rangle$ are elements of β. Pick $\gamma < \kappa$ that is minimal with the property that $x \restriction \beta \subseteq \iota(\gamma)$. Then $\beta \leq \gamma$ and then the minimality of γ implies that $\iota(\gamma)(\bar{\gamma}) = 0 = x(\bar{\gamma})$ for all $\beta \leq \bar{\gamma} < \gamma$, because otherwise $\iota(\gamma)(\bar{\gamma}) = 1$ would imply that $\iota(\bar{\gamma}) = \iota(\gamma) \restriction (\bar{\gamma}+1)$ and hence $x \restriction \beta \subseteq \iota(\bar{\gamma})$. This shows that $\iota(\gamma) \restriction \gamma = x \restriction \gamma$. Since $\iota(\gamma)(\alpha) = x(\alpha) = 1$, we can conclude that γ is a direct successor of α in $\langle \kappa, <_c \rangle$ that is not contained in β, a contradiction.

Now, assume that K is a cofinal subset of $\alpha \in \mathrm{Lim} \cap \kappa$. Pick $\beta_0 < \kappa$ minimal with $x \restriction \alpha \subseteq \iota(\beta_0)$. Then $\alpha \leq \beta_0$ and, since $K \subseteq \alpha$, the minimality of β_0 implies that $x \restriction \beta_0 = \iota(\beta_0) \restriction \beta_0$. Next, pick $\beta_1 < \kappa$ minimal with $x \restriction (\beta_0 + 1) \subseteq \iota(\beta_1)$. Then $\beta_0 \leq \beta_1$ and $x(\beta_0) = 0 < 1 = \iota(\beta_1)(\beta_1)$ implies that $\beta_0 < \beta_1$. Then the minimality of β_1 and $K \subseteq \alpha$ imply $x \restriction \beta_1 = \iota(\beta_1) \restriction \beta_1$. In particular, we have $b_{\beta_0} = K = b_{\beta_1}$. Given $\alpha_0 \in K$, there is $\alpha_1 \in K$ with $\alpha_0 < \alpha_1$ and the above equalities imply that $c(\{\alpha_0, \beta_0\}) = c(\{\alpha_0, \alpha_1\}) = c(\{\alpha_0, \beta_1\})$. This shows that $c(\{\alpha_0, \alpha_1\}) = c(\{\alpha_0, \beta_1\})$ holds for all $\alpha_0, \alpha_1 \in b_{\beta_1} \cup \{\beta_0\}$, contradicting the maximality of b_{β_1}. □

If we define

$$f : K \longrightarrow 2;\ \alpha \longmapsto c(\{\alpha, \min(K \setminus (\alpha+1))\}),$$

then the above claim yields an unbounded subset H of K with $f \restriction H$ is constant. Since $\alpha <_c \beta$ holds for all $\alpha, \beta \in K$ with $\alpha < \beta$, we know that $c(\{\alpha, \beta\}) = c(\{\alpha, \gamma\})$ for all $\alpha, \beta, \gamma \in K$ with $\alpha < \beta \leq \gamma$. In particular, if $\alpha, \beta \in H$ with $\alpha < \beta$, then $c(\{\alpha, \beta\}) = c(\{\alpha, \min(K \setminus (\alpha+1))\}) = f(\alpha)$. This shows that H is c-homogeneous and hence (i) holds with respect to c.

<u>*(ii)* $\Longrightarrow$ *(iii)*:</u> Assume that (ii) holds and pick $A \subseteq \kappa$ such that the set $\{A\}$ is $\Sigma_n(\kappa, z)$-definable. Since we already know that (i) also holds, we can use Corollary 3.2 to show that κ is inaccessible in $\mathrm{L}[A]$ and hence $({}^{<\kappa}2)^{\mathrm{L}[A]} \subseteq \mathrm{L}_\kappa[A] = \mathrm{H}(\kappa)^{\mathrm{L}[A]}$. Let $\theta > \kappa$ be the minimal ordinal such that $\mathrm{L}_\theta[A] \models \mathrm{ZFC}^- +$ "$\mathcal{P}(\kappa)$ *exists*", let b be the $<_{\mathrm{L}[A]}$-minimal bijection between κ and $\mathcal{P}(\kappa)^{\mathrm{L}_\theta[A]}$ in $\mathrm{L}[A]$ and let $\vartheta > \theta$ be minimal with the property that $b \in \mathrm{L}_\vartheta[A] \models \mathrm{ZFC}^- +$ "$\mathcal{P}(\kappa)$ *exists*". Then the sets $\{\mathrm{L}_\theta[A]\}$, $\{\mathrm{L}_\vartheta[A]\}$ and $\{b\}$ are all $\Sigma_1(\kappa, A)$-definable and therefore our assumption implies that they are also $\Sigma_n(\kappa, z)$-definable. Define

$$B_t \ = \ \left(\bigcap\{b(\alpha) \mid t(\alpha) = 1\}\right) \ \cap \ \left(\bigcap\{\kappa \setminus b(\alpha) \mid t(\alpha) = 0\}\right) \ \in \ \mathcal{P}(\kappa)^{\mathrm{L}_\vartheta[A]}$$

for all $t \in ({}^{<\kappa}2)^{\mathrm{L}[A]}$ and let $\mathcal{B}$ denote the set of all $t \in ({}^{<\kappa}2)^{\mathrm{L}[A]}$ with $|B_t|^{\mathrm{L}_\vartheta[A]} = \kappa$.

Claim. *The set $\mathcal{B}$ has cardinality κ.*

Proof of the Claim. Assume not. Then there is a minimal $\beta < \kappa$ with $\mathcal{B} \subseteq {}^{<\beta}2$. Let $f : \kappa \longrightarrow {}^{\beta}2$ denote the unique function with

$$f(\gamma)(\alpha) = 1 \iff \alpha \in b(\gamma)$$

for all $\gamma < \kappa$ and $\alpha < \beta$. Then $\mathrm{ran}(f) \subseteq \mathrm{L}[A]$ and for all $\gamma < \kappa$, we have $\gamma \in B_{f(\gamma)}$ and $|B_{f(\gamma)}| < \kappa$. This shows that $|\mathrm{ran}(f)| = \kappa$. Let $\iota : \kappa \longrightarrow \mathrm{ran}(f)$ denote the monotone enumeration of $\mathrm{ran}(f)$ with respect to $<_{\mathrm{L}[A]}$. Then the set $\{\beta\}$, the function f and the function ι are all definable over the structure $\langle \mathrm{L}_\vartheta[A], \in \rangle$ by a formula with parameters A and b. But this shows that ι is a $\Sigma_n(\kappa, z)$-definable injection from κ into ${}^{\beta}2$, contradicting Proposition 3.1. □

Now, let $\iota : \kappa \longrightarrow {}^{<\kappa}2$ denote the monotone enumeration of $\mathcal{B}$ with respect to $<_{\mathrm{L}[A]}$. As above, we know that ι is $\Sigma_n(\kappa, z)$-definable and hence the assumption (ii) yields an $x \in {}^{\kappa}2$ with $\{\alpha < \kappa \mid \exists \beta < \kappa \ x \restriction \alpha \subseteq \iota(\beta)\}$ is unbounded in κ. If we define

$$U \ = \ \{B \subseteq \kappa \mid \exists \gamma < \kappa \ B_{x \restriction \gamma} \subseteq B\},$$

then it is easy to see that U is a non-principal, $<\kappa$-complete filter on κ that measures every subset of κ contained in $\mathrm{L}_\theta[A]$. In particular, this implies that the ultrapower $\mathrm{Ult}(\mathrm{L}_\theta[A], U \cap \mathcal{P}(\kappa)^{\mathrm{L}_\theta[A]})$ (that uses only functions $f : \kappa \longrightarrow \mathrm{L}_\theta[A]$ contained in $\mathrm{L}_\theta[A]$) is well-founded and, if we let N denote its transitive collapse, then the corresponding elementary embedding $j : \mathrm{L}_\theta[A] \longrightarrow N$ has critical point κ. In particular, (iii) holds with respect to A.

(iii) $\Longrightarrow$ (ii): Assume that (iii) holds and let $\iota : \kappa \longrightarrow {}^{<\kappa}2$ be a $\Sigma_n(\kappa, z)$-definable injection. Define

$$A \ = \ \{\prec\alpha, \gamma, \iota(\alpha)(\gamma)\succ \mid \alpha < \kappa, \ \gamma \in \mathrm{dom}(\iota(\gamma))\}.$$

Then the set $\{A\}$ is also $\Sigma_n(\kappa, z)$-definable and (iii) yields a weak κ-model M, a transitive set N and an elementary embedding $j : M \longrightarrow N$ such that $\mathrm{crit}\,(j) = \kappa$, $A \in M$ and κ is inaccessible in M. Since κ is inaccessible in M and $\mathrm{H}(\kappa)^M \in M$, elementarity implies that $\mathrm{H}(\kappa)^N \subseteq M$. Moreover, $A \in M$ implies that ι is an element of M. Set $t = j(\iota)(\kappa)$ and assume, towards a contradiction, that $\mathrm{dom}(t) < \kappa$. Then $t = j(t) \in \mathrm{H}(\kappa)^N \subseteq M$ and elementarity yields an $\alpha < \kappa$ with $\iota(\alpha) = t$. But then $j(\iota)(\alpha) = t = j(\iota)(\kappa)$, a contradiction. This shows that $\mathrm{dom}(t) \geq \kappa$ and $x = t \restriction \kappa \in {}^{\kappa}2$. If $\gamma < \kappa$, then $x \restriction \gamma \in M$ and elementarity yields an $\alpha < \kappa$ with $x \restriction \gamma \subseteq \iota(\alpha)$. Therefore x witnesses that (ii) holds with respect to ι. □

We now use the above characterizations to strengthen the conclusion of Corollary 3.2 and isolate the large cardinal properties that correspond to the $\mathbf{\Sigma}_n$-colouring properties.

Corollary 3.6. *Let κ be an uncountable regular cardinal and let z be a set such that κ has the $\Sigma_n(z)$-colouring property. If A is a subset of κ with the property that the set $\{A\}$ is $\Sigma_n(\kappa, z)$-definable, then κ is a Mahlo cardinal in $\mathrm{L}[A]$.*

Proof. Assume that the above conclusion fails. Since Corollary 3.2 implies that κ is inaccessible in $\mathrm{L}[A]$, a result of Todorčević (see [37, Theorem 6.1.4]) shows that $\mathrm{L}[A]$ contains a special κ-Aronszajn tree (see [37, Definition 6.1.1]). By using $<_{\mathrm{L}[A]}$ and $\prec\cdot,\cdot\succ$ to code the $<_{\mathrm{L}[A]}$-least special κ-Aronszajn tree in $\mathrm{L}[A]$ into an element of $\mathcal{P}(\kappa)^{\mathrm{L}[A]}$, we find $B \subseteq \kappa$ with the property that the set $\{B\}$ is $\Sigma_1(\kappa, A)$-definable and every weak κ-model that contains B also contains a special κ-Aronszajn tree. Then the set $\{B\}$ is $\Sigma_n(\kappa, z)$-definable and Lemma 3.5 yields a weak κ-model M, a transitive set N and an elementary embedding $j : M \longrightarrow N$ with $\mathrm{crit}\,(j) = \kappa$ and $B \in M$. Then M contains a special κ-Aronszajn tree $\mathbb{T}$ and every element of the κ-th level of $j(\mathbb{T})$ in N induces a cofinal branch through $\mathbb{T}$. Since $\mathbb{T}$ is special, this contradicts the regularity of κ. □

Corollary 3.7. *Let κ be an uncountable regular cardinal, let $x \in \mathrm{H}(\kappa) \cap \mathcal{P}(\kappa)$ and let $z \in \mathrm{H}(\kappa^+)^{\mathrm{L}[x]}$. If κ has the $\Sigma_n(x, z)$-colouring property, then κ has the $\Sigma_n(z)$-colouring property in $\mathrm{L}[x]$.*

Proof. Pick $A \in \mathcal{P}(\kappa)^{\mathrm{L}[x]}$ such that $\{A\}$ is $\Sigma_n(\kappa, z)$-definable in $\mathrm{L}[x]$. Then $\{A\}$ is $\Sigma_n(\kappa, x, z)$-definable. Let $\theta > \kappa$ be minimal with $z, A \in \mathrm{L}_\theta[x] \models \mathrm{ZFC}^-$, let b be the $<_{\mathrm{L}[x]}$-least bijection between κ and $\mathrm{L}_\theta[x]$ in $\mathrm{L}[x]$, let $\vartheta > \theta$ be minimal such that $b \in \mathrm{L}_\vartheta[x] \models \mathrm{ZFC}^-$ and let c be the $<_{\mathrm{L}[x]}$-least bijection between κ and $\mathrm{L}_\vartheta[x]$ in $\mathrm{L}[x]$. Define

$$B \ = \ \{\prec\alpha, \beta\succ \mid \alpha, \beta < \kappa, \ c(\alpha) \in c(\beta)\} \ \in \ \mathcal{P}(\kappa)^{\mathrm{L}[x]}.$$

Then the set $\{B\}$ is $\Sigma_1(\kappa, x, z, A)$-definable and therefore it is also $\Sigma_n(\kappa, x, z)$-definable. Using Lemma 3.5, we can find a weak κ-model M, a transitive set N and an elementary embedding $j : M \longrightarrow N$ with $\mathrm{crit}\,(j) = \kappa$ and $B \in M$. Then $A \in \mathrm{L}_\vartheta[x] \in M$ and $j(x) = x$. If we define

$$E \ = \ \{\langle\alpha, \beta\rangle \in \kappa \times \kappa \mid b(\alpha) \in b(\beta)\},$$

then we know that $E \in \mathrm{L}_\vartheta[x] \in M$ and $j(E) \in \mathrm{L}_{j(\vartheta)}[x] \subseteq \mathrm{L}[x]$. Moreover, since the function $j(b)$ is the transitive collapse of $\langle j(\kappa), j(E)\rangle$, we know that $j(b)$ is also contained in $\mathrm{L}[x]$. Finally, an easy computation shows that

$$j \restriction \mathrm{L}_\theta[x] = j(b) \circ b^{-1} : \mathrm{L}_\theta[x] \longrightarrow \mathrm{L}_{j(\theta)}[x]$$

is an elementary embedding contained in $\mathrm{L}[x]$. By Lemma 3.5, these computations show that κ has the $\Sigma_n(z)$-colouring property in $\mathrm{L}[x]$. □

The concept introduced in the next definition will allow us to further strengthen the above conclusions. Moreover, it will enable us to show that for all $n \geq 2$, the $\mathbf{\Sigma}_n$-colouring property is equivalent to the $\mathbf{\Sigma}_2$-colouring property.

Definition 3.8. Given sets $z_0, \ldots, z_{m-1}$, a class A *has a good* $\Sigma_n(z_0, \ldots, z_{m-1})$*-well-ordering* if there is a well-ordering $\lhd$ of a class B such that $A \subseteq B$ and the class

$$I(\lhd) \ = \ \{\{y \mid y \lhd x\} \mid x \in B\}$$

of all proper initial segments of $\lhd$ is $\Sigma_n(z_0,\ldots,z_{m-1})$-definable.

It is easy to see that the canonical well-ordering of the constructible universe witnesses that the class L has a good Σ_1-well-ordering. More generally, for every set of ordinals x, the class $\mathrm{L}[x]$ has a good $\Sigma_1(x)$-well-ordering. Moreover, in the *Dodd–Jensen core model* K^{DJ}, there is a good $\Sigma_1(\kappa)$-well-ordering of $\mathcal{P}(\kappa)$ for every uncountable cardinal κ (see [30, Lemma 1.10.]). Finally, it can also easily be shown that the canonical well-ordering of the collection HOD_z of all hereditarily z-ordinal-definable witnesses that HOD_z has a good $\Sigma_2(z)$-well-ordering (see the proof of [20, Lemma 13.25] for details). Since the assumption $\mathrm{V} = \mathrm{HOD}$ is compatible with the various large cardinal assumptions and forcing axioms consider in this paper, this shows that the existence a good Σ_2-well-ordering of V is also consistent with these extensions of ZFC and this will imply that it is most interesting to study the influence of these extensions of ZFC on the $\mathbf{\Sigma}_1$-colouring property. Moreover, this fact will allow us to show that the $\mathbf{\Sigma}_2$-colouring property implies all higher partition properties. This implication will be an easy consequence of the following observation.

Proposition 3.9. *Let X be a class of sets of ordinals with the property that both X and $\mathrm{V}\setminus X$ are $\Sigma_n(y)$-definable for some set y. If z is a set with the property that $\mathrm{HOD}_z \cap X \neq \emptyset$, then there is an $A \in \mathrm{HOD}_z \cap X$ such that the set $\{A\}$ is $\Sigma_n(y,z)$-definable.*

Proof. Let $\lhd$ denote the canonical well-ordering of HOD_z and let A be the $\lhd$-least element in $\mathrm{HOD}_z \cap X$. By the above remarks, the class $I(\lhd)$ is definable by a Σ_2-formula with parameter z. Then A is the unique element of X with the property that there is a $D \in I(\lhd)$ with $D \cup \{A\} \in I(\lhd)$ and $D \cap X = \emptyset$. By our assumptions, this shows that the set $\{A\}$ is $\Sigma_n(y,z)$-definable. □

The following corollary now shows that the validity of the $\mathbf{\Sigma}_2$-colouring property at a cardinal κ is equivalent to the assumption that for every function $c : [\kappa]^2 \longrightarrow 2$ that is ordinal definable with parameters from $\mathrm{H}(\kappa)$, there is a c-homogeneous set that is unbounded in κ.

Corollary 3.10. *The following statements are equivalent for every uncountable regular cardinal κ and every set z:*

(i) *κ has the $\Sigma_2(z)$-colouring property.*
(ii) *κ has the $\Sigma_n(z)$-colouring property for all $n < \omega$.*
(iii) *For every function $c : [\kappa]^2 \longrightarrow 2$ that is an element of HOD_z, there is a c-homogeneous set that is unbounded in κ.*

Proof. <u>*(iii)* $\Longrightarrow$ *(ii)* $\Longrightarrow$ *(i)*:</u> Both implications are trivial.

<u>*(i)* $\Longrightarrow$ *(iii)*:</u> Assume, towards a contradiction, that there is a function $c : [\kappa]^2 \longrightarrow 2$ that is an element of HOD_z and has the property that every c-homogeneous set is bounded in κ. Given $A \subseteq \kappa$, we let $c_A : [\kappa]^2 \longrightarrow 2$ denote the unique function with the property that for all $\alpha < \beta < \kappa$, we have $c_A(\{\alpha,\beta\}) = 1$ if and only if $\prec\alpha,\beta\succ \in A$. Let X denote the set of all $A \subseteq \kappa$ with the property that every c_A-homogeneous set is bounded in κ. Then both X and $\mathrm{V}\setminus X$ are $\Sigma_2(\kappa)$-definable. Now, if we set

$$A \ = \ \{\prec\alpha,\beta\succ \mid \alpha < \beta < \kappa, \ c(\{\alpha,\beta\}) = 1\},$$

then $c_A = c$ and hence our assumptions imply that $A \in \mathrm{HOD}_z \cap X \neq \emptyset$. In this situation, we can apply Proposition 3.9 to find $B \in \mathrm{HOD}_z \cap X$ with the property that the set $\{B\}$ is $\Sigma_2(\kappa,z)$-definable. But this shows that c_B is a $\Sigma_2(z)$-partition and, since B is an element of X, this function witnesses that (i) fails. □

Next, we show that the $\mathbf{\Sigma}_2$-colouring property is equivalent to weak compactness in certain canonical models of set theory.

Proposition 3.11. *Let κ be an uncountable regular cardinal with the property that there is a good $\Sigma_n(\kappa,z)$-well-ordering of $\mathcal{P}(\kappa)$ for some set z. If κ has the $\Sigma_2(z)$-colouring property, then κ is weakly compact.*

Proof. Assume that κ is not weakly compact. Let $\lhd$ be a well-ordering of some class containing $\mathcal{P}(\kappa)$ such that the induced class $I(\lhd)$ of all proper initial segments of $\lhd$ is $\Sigma_n(\kappa,z)$-definable. Define the functions $c_A : [\kappa]^2 \longrightarrow 2$ for $A \subseteq \kappa$ as in the proof of Corollary 3.10. As above, let X denote the set of all elements A of $\mathcal{P}(\kappa)$ with the property that all c_A-homogeneous subsets of κ are bounded in κ. Then our assumption implies that X is non-empty. Let B denote the

$\lhd$-least element of X. Then $B \in X$ is the unique subset of κ with the property that there exists a $D \in I(\lhd)$ with $D \cap X = \emptyset$ and $D \cup \{B\} \in I(\lhd)$. Since both X and $\mathrm{V} \setminus X$ are $\Sigma_2(\kappa)$-definable, our assumptions imply that the set $\{B\}$ is $\Sigma_{n+1}(\kappa, z)$-definable and hence c_B is a $\Sigma_{n+1}(z)$-partition. Using Corollary 3.10, we can conclude that κ does not have the $\Sigma_2(z)$-colouring property. □

Corollary 3.12. *Let κ be an uncountable regular cardinal.*

(i) If there is a set z such that $\mathrm{V} = \mathrm{HOD}_z$ and κ has the $\Sigma_2(z)$-colouring property, then κ is weakly compact.

(ii) If κ has the $\Sigma_2(z)$-colouring property for some $z \in \mathrm{H}(\kappa) \cap \mathcal{P}(\kappa)$, then κ is weakly compact in $\mathrm{L}[z]$. □

The above results show that there is a natural correspondence between the $\mathbf{\Sigma}_2$-colouring property and weak compactness. In contrast, the results of this paper will show that the $\mathbf{\Sigma}_1$-colouring property corresponds to a large cardinal property that is weaker than weak compactness but stronger than Mahloness. In the following, we strengthen earlier results by showing that cardinals with the $\mathbf{\Sigma}_1$-colouring property possess a high degree of Mahloness in the constructible universe. An upper bound for the consistency strength of the $\mathbf{\Sigma}_1$-colouring property will be given by Theorem 7.1 below.

Proposition 3.13. *Let κ be an uncountable regular cardinal and let z be a set with the property that there exists a good $\Sigma_n(\kappa, z)$-well-ordering of $\mathrm{H}(\kappa)$. If $\nu < \kappa$ is a cardinal with $2^\nu \geq \kappa$, then κ does not have the $\Sigma_n(\nu, z)$-colouring property.*

Proof. By our assumptions, we can use the good $\Sigma_n(\kappa, z)$-well-ordering of $\mathrm{H}(\kappa)$ to construct a $\Sigma_n(\kappa, \nu, z)$-definable injection of κ into ${}^{\nu}2$. By Proposition 3.1, the existence of such an injection contradicts the $\Sigma_n(\nu, z)$-colouring property. □

The following lemma generalizes the simultaneous reflection of stationary subsets of weakly compact cardinals to our definable setting. Note that the assumptions of the next lemma are satisfied in L for every uncountable regular cardinal.

Lemma 3.14. *Let κ be an uncountable regular cardinal with the $\mathbf{\Sigma}_1$-colouring property. Assume that for some $z \in \mathrm{H}(\kappa)$, the set $\{\mathrm{H}(\kappa)\}$ is $\Sigma_1(\kappa, z)$-definable and $\mathcal{P}(\kappa)$ has a good $\Sigma_1(\kappa, z)$-well-ordering. If $s : \kappa \longrightarrow \mathcal{P}(\kappa)$ is a $\Sigma_1(\kappa, z)$-definable function with the property that $s(\alpha)$ is stationary in κ for all $\alpha < \kappa$, then the set*

$$S = \{\mu < \kappa \mid \mu \textit{ is a regular cardinal with } s(\alpha) \cap \mu \textit{ stationary in } \mu \textit{ for all } \alpha < \mu\}$$

is also stationary in κ.

Proof. Assume that the above conclusion fails. Note that the assumption that the set $\{\mathrm{H}(\kappa)\}$ is $\Sigma_1(\kappa, z)$-definable implies that the set $\{S\}$ is definable in the same way. Let $\lhd$ be a well-ordering of a class B such that $\mathcal{P}(\kappa) \subseteq B$ and the corresponding class $I(\lhd)$ is $\Sigma_1(\kappa, z)$-definable. Let C denote the $\lhd$-least club in κ with $C \cap S = \emptyset$. Then the set $\{C\}$ is also $\Sigma_1(\kappa, z)$-definable. By Lemma 3.5, there is a weak κ-model M, a transitive set N and an elementary embedding $j : M \longrightarrow N$ with critical point κ and $C, S \in M$. Since $\kappa \in j(C)$ and κ is regular in N, elementarity yields an $\alpha < \kappa$ and a club subset D of κ in N with $D \cap j(s)(\alpha) = \emptyset$. But $j(s)(\alpha) \cap \kappa = j(s(\alpha)) \cap \kappa = s(\alpha)$ and hence D witnesses that $s(\alpha)$ is not stationary in κ, a contradiction. □

Remember that, given an inaccessible cardinal κ and an ordinal $\delta \leq \kappa^+$, the cardinal κ is *δ-Mahlo* if there is a sequence $\langle A_\gamma \mid \gamma < \delta \rangle$ of stationary subsets of κ such that the following statements hold for all $\gamma < \delta$:

(i) $A_0 = \{\alpha < \kappa \mid \alpha \textit{ is regular}\}$.

(ii) If $\gamma = \beta + 1$, then $A_\gamma = \{\alpha \in A_\beta \mid A_\beta \cap \alpha \textit{ is stationary in } \alpha\}$.

(iii) If γ is a limit ordinal of cofinality less than κ, then there is a strictly increasing sequence $\langle \beta_\alpha \mid \alpha < \mathrm{cof}(\gamma) \rangle$ that is cofinal in γ with $A_\gamma = \bigcap \{A_{\beta_\alpha} \mid \alpha < \mathrm{cof}(\gamma)\}$.

(iv) If γ is a limit ordinal of cofinality κ, then there is a strictly increasing sequence $\langle \beta_\alpha \mid \alpha < \kappa \rangle$ that is cofinal in γ with $A_\gamma = \triangle \{A_{\beta_\alpha} \mid \alpha < \kappa\}$.

A cardinal κ is then called *hyper-Mahlo* if it is κ-Mahlo. Note that, given two sequences $\langle A_\beta \mid \beta < \alpha \rangle$ and $\langle B_\beta \mid \beta < \alpha \rangle$ of subsets of κ that satisfy the above four statements and some $\beta < \alpha$, the sets A_β and B_β only differ by a non-stationary subset of κ. In particular, if κ is an inaccessible cardinal that is not δ-Mahlo for some $\delta \leq \kappa^+$, then there is a $\gamma < \delta$, such that κ is γ-Mahlo and not $(\gamma+1)$-Mahlo.

Theorem 3.15. *Let κ be an uncountable regular cardinal with the $\mathbf{\Sigma}_1$-colouring property. Assume that for some $z \in \mathrm{H}(\kappa)$, the set $\{\mathrm{H}(\kappa)\}$ is $\Sigma_1(\kappa, z)$-definable and $\mathcal{P}(\kappa)$ has a good $\Sigma_1(\kappa, z)$-well-ordering. Define σ to be the supremum of all ordinals δ with the property that there is a subset E of $\kappa \times \kappa$ such that $\langle \kappa, E\rangle$ is a well-ordering of order-type δ and the set $\{E\}$ is $\Sigma_1(\kappa, w)$-definable for some $w \in \mathrm{H}(\kappa)$. Then κ is a σ-Mahlo cardinal.*

Proof. Let $\lhd$ be a well-ordering of a class B with $\mathcal{P}(\kappa) \subseteq B$ such that the class $I(\lhd)$ is $\Sigma_1(\kappa, z)$-definable. Assume that the above conclusion fails. Since Proposition 3.13 shows that κ is inaccessible, the above remarks show that there is a $\delta < \sigma$ such that κ is δ-Mahlo and not $(\delta+1)$-Mahlo. Pick $E \subseteq \kappa \times \kappa$ such that $\langle \kappa, E\rangle$ is a well-ordering of order-type at least δ and the set $\{E\}$ is $\Sigma_1(\kappa, w)$-definable for some $w \in \mathrm{H}(\kappa)$. Then we can find $\lambda \leq \kappa$ and $y \in \mathrm{H}(\kappa)$ such that the set $\{\delta\}$ is $\Sigma_1(\kappa, y)$-definable and there is a $\Sigma_1(\kappa, y)$-definable bijection $b : \lambda \longrightarrow \delta$. Let $\langle A_\gamma \mid \gamma \leq \delta\rangle$ denote the unique sequence of subsets of κ such that for all $\gamma \leq \delta$, the above statements (i) and (ii) as well as the following two statements hold:

(iii)′ If $\gamma \in \mathrm{Lim}$ with $\mathrm{cof}(\gamma) < \kappa$ and c_γ is the $\lhd$-least subset of λ of cardinality less than κ with the property that $b[c_\gamma]$ is a cofinal subset of γ of order-type $\mathrm{cof}(\gamma)$, then $A_\gamma = \bigcap\{A_{b(\beta)} \mid \beta \in c_\gamma\}$.

(iv)′ If $\gamma \in \mathrm{Lim}$ with $\mathrm{cof}(\gamma) = \kappa$ and c_γ is the $\lhd$-least subset of κ such that $b \restriction b_\gamma$ is strictly increasing and $b[c_\gamma]$ is a cofinal subset of γ of order-type κ, then $A_\gamma = \triangle\{A_{b(\beta^\gamma_\alpha)} \mid \alpha < \kappa\}$, where $\langle \beta^\gamma_\alpha \mid \alpha < \kappa\rangle$ denotes the monotone enumeration of c_γ.

Then the sequence $\langle A_\gamma \mid \gamma \leq \delta\rangle$ also satisfies the above properties (iii) and (iv). Therefore, our assumptions imply that A_γ is a stationary subset of κ for every $\gamma < \delta$ and that there is a club D in κ that is disjoint from A_δ. Moreover, by combining the Σ_1-Recursion Theorem with the fact that the set $\{\mathrm{H}(\kappa)\}$ and the function b are both $\Sigma_1(\kappa, y, z)$-definable, we know that the set $\{\langle A_\gamma \mid \gamma \leq \delta\rangle\}$ is $\Sigma_1(\kappa, y, z)$-definable and therefore the function

$$s : \lambda \longrightarrow \mathcal{P}(\kappa);\ \beta \longmapsto A_{b(\beta)}$$

is definable in the same way. If we now let S denote the set of all regular cardinals $\mu < \kappa$ with the property that $s(\alpha) \cap \mu$ is stationary in μ for all $\alpha < \min\{\lambda, \mu\}$, then Lemma 3.14 shows that S is stationary in κ.

Claim. *For all $\gamma \leq \delta$, there is a club C_γ in κ with $C_\gamma \cap S \subseteq A_\gamma$.*

Proof of the Claim. We prove the claim by induction on $\gamma \leq \delta$. First, since $S \subseteq A_0$, we can define $C_0 = \kappa$. Now, if $\gamma < \delta$ and C_γ is already constructed, then we define $C_{\gamma+1} = C_\gamma \cap (b^{-1}(\gamma), \kappa)$. Given $\mu \in C_{\gamma+1} \cap S$, we then have $\mu \in A_\gamma$, $b^{-1}(\gamma) < \min\{\lambda, \mu\}$ and hence $A_\gamma \cap \mu = s(b^{-1}(\gamma)) \cap \mu$ is stationary in μ. This shows that $C_{\gamma+1} \cap S \subseteq A_{\gamma+1}$. Next, if $\gamma \in \mathrm{Lim} \cap (\delta+1)$ with $\mathrm{cof}(\gamma) < \kappa$ and C_β is defined for all $\beta < \gamma$, then we define $C_\gamma = \bigcap\{C_{b(\beta)} \mid \beta \in c_\gamma\}$. Then the definition of A_γ directly implies that $C_\gamma \cap S \subseteq A_\gamma$. Finally, assume that $\gamma \in \mathrm{Lim} \cap (\delta+1)$ with $\mathrm{cof}(\gamma) = \kappa$ and C_β is defined for all $\beta < \gamma$. Set $C_\gamma = \triangle\{C_{b(\beta^\gamma_\alpha)} \mid \alpha < \kappa\}$. Given $\mu \in C_\gamma \cap S$, we then have

$$\mu \ \in\ C_{b(\beta^\gamma_\alpha)} \cap S \ \subseteq\ A_{b(\beta^\gamma_\alpha)}$$

for all $\alpha < \mu$ and this allows us to conclude that $\mu \in A_\gamma = \triangle\{A_{b(\beta^\gamma_\alpha)} \mid \alpha < \kappa\}$. □

But, now we have $\emptyset \neq C_\delta \cap D \cap S \subseteq A_\delta \cap D$, a contradiction. □

Note that the assumptions of Theorem 3.15 are satisfied if κ has the $\mathbf{\Sigma}_1$-colouring property and there is an $A \subseteq \kappa$ such that $\mathcal{P}(\kappa) \subseteq \mathrm{L}[A]$ and the set $\{A\}$ is $\Sigma_1(\kappa, z)$-definable for some $z \in \mathrm{H}(\kappa)$. Moreover, Proposition 3.18 below shows that we can force over L to show that the conclusion of the above theorem can fail if one discards the assumption that there is a good Σ_1-well-ordering of $\mathcal{P}(\kappa)$.

Corollary 3.16. *Let κ be an uncountable regular cardinal with the $\mathbf{\Sigma}_1$-colouring property.*

(i) If there is $A \subseteq \kappa$ such that $\mathrm{V} = \mathrm{L}[A]$ and the set $\{A\}$ is $\Sigma_1(\kappa, z)$-definable for some $z \in \mathrm{H}(\kappa)$, then κ is a hyper-Mahlo cardinal.

(ii) If $x \in \mathrm{H}(\kappa) \cap \mathcal{P}(\kappa)$, then κ is a hyper-Mahlo cardinal in $\mathrm{L}[x]$. □

In the remainder of this section, we will show that the validity of the $\mathbf{\Sigma}_n$-colouring property at the successor of a regular cardinal is equiconsistent with both the existence of an inaccessible cardinal with the $\mathbf{\Sigma}_n$-colouring property and the existence of an accessible limit cardinal with this property. By the above results, this will show that, in the case $n = 2$, all of these corresponding theories are equiconsistent to the existence of a weakly compact cardinal. In the case $n = 1$, the above computations and Theorem 7.1 below will show that the consistency strength of the given theories lies strictly between the existence of a hyper-Mahlo cardinal and a weakly compact cardinal. Finally, our results will also show that the $\mathbf{\Sigma}_2$-colouring property does not imply Mahloness for inaccessible cardinals.

Lemma 3.17. *Let κ be an uncountable regular cardinal with the $\mathbf{\Sigma}_n$-colouring property and let $\mathbb{P} \in \mathrm{H}(\kappa)$ be a partial order. If G is $\mathbb{P}$-generic over* V*, then κ has the $\mathbf{\Sigma}_n$-colouring property in* $\mathrm{V}[G]$.

Proof. Fix $z \in \mathrm{H}(\kappa)^{\mathrm{V}[G]}$, $A \in \mathcal{P}(\kappa)^{\mathrm{V}[G]}$ and a Σ_n-formula $\varphi(v_0, v_1, v_2)$ such that A is the unique set in $\mathrm{V}[G]$ with the property that $\varphi(\kappa, z, A)$ holds in $\mathrm{V}[G]$. Since $\mathbb{P}$ is an element of $\mathrm{H}(\kappa)^{\mathrm{V}}$, there is a $\mathbb{P}$-name $\dot{z}$ in $\mathrm{H}(\kappa)^{\mathrm{V}}$ with $z = \dot{z}^G$. Pick a condition p in G that forces the above statements about $\dot{z}$ to hold true and fix a bijection b between a cardinal $\nu < \kappa$ and the set of all conditions below p in $\mathbb{P}$. Set

$$B \;=\; \{\prec\alpha,\gamma\succ \mid \alpha<\kappa,\ \gamma<\nu,\ b(\gamma) \Vdash_{\mathbb{P}}^{\mathrm{V}} \text{“}\exists x\ [\varphi(\check{\kappa},\dot{z},x) \ \wedge\ \check{\alpha}\in x]\text{”}\} \;\in\; \mathcal{P}(\kappa)^{\mathrm{V}}.$$

A careful review of the definition of the forcing relation (see, for example, [23, Section VII.3]) shows that for every Σ_n-formula $\psi(v_0, \ldots, v_{m-1})$, there is a Σ_n-formula $\psi(v_0, \ldots, v_{m+1})$ such that the axioms of ZFC^- prove that for every partial order $\mathbb{P}$, every p in $\mathbb{P}$ and all $\tau_0, \ldots, \tau_{n-1}$, the statement $\psi(\tau_0, \ldots, \tau_{n-1}, \mathbb{P}, p)$ holds if and only if the sets $\tau_0, \ldots, \tau_{n-1}$ are $\mathbb{P}$-names with $p \Vdash_{\mathbb{P}} \varphi(\tau_0, \ldots, \tau_{n-1})$. In particular, the set B is $\Sigma_n(\kappa, \nu, b, \dot{z}, \mathbb{P})$-definable in V. Moreover, if $\alpha < \kappa$ and $\gamma < \nu$, then $\prec\alpha,\gamma\succ$ is not contained in B if and only if there is a $\delta < \nu$ with $b(\delta) \leq_{\mathbb{P}} b(\gamma)$ and

$$b(\delta) \Vdash_{\mathbb{P}}^{\mathrm{V}} \text{“}\exists x\ [\varphi(\check{\kappa},\dot{z},x) \wedge \check{\alpha} \notin x]\text{”}.$$

This shows that the set $\kappa \setminus B$ is also $\Sigma_n(\kappa, \nu, b, \dot{z}, \mathbb{P})$-definable in V and this allows us to conclude that the set $\{B\}$ is definable in the same way. By our assumption and Lemma 3.5, there is an elementary embedding $j : M \longrightarrow N$ with critical point κ in V such that M is a weak κ-model, N is transitive, $\dot{z}, B, \mathbb{P} \in M$ and κ is inaccessible in M. Since $j \restriction \mathbb{P} = \mathrm{id}_{\mathbb{P}}$, this embedding has a canonical lift $j_G : M[G] \longrightarrow N[G]$ in $\mathrm{V}[G]$ (see [8, Proposition 9.1]). But then A consists of all $\alpha < \kappa$ with the property that there is a $\gamma < \nu$ with $b(\gamma) \in G$ and $\prec\alpha,\gamma\succ \in B$. This shows that A is an element of $M[G]$. Since κ is inaccessible in $M[G]$, Lemma 3.5 shows that κ has the $\mathbf{\Sigma}_n$-colouring property in $\mathrm{V}[G]$. □

Proposition 3.18. *Let κ be an uncountable regular cardinal.*

(i) Let $\mu < \kappa$ be an infinite regular cardinal and let $\mathbb{P} \in \{\mathrm{Add}(\mu,\kappa), \mathrm{Col}(\mu,{<}\kappa)\}$. If either $\mu = \omega$ or $\mathrm{V} = \mathrm{L}$ holds, then the set $\{\mathbb{P}\}$ is $\Sigma_1(\kappa,\mu)$-definable.

(ii) Let $\mathbb{P}$ be a weakly homogeneous partial order, let G be $\mathbb{P}$-generic over V *and let A be a subset of κ in $\mathrm{V}[G]$. If there is a set $z \in \mathrm{V}$ with the property that the set $\{\mathbb{P}\}$ is $\Sigma_n(\kappa, z)$-definable in* V *and the set $\{A\}$ is $\Sigma_n(\kappa, z)$-definable in $\mathrm{V}[G]$, then A is an element of* V *and the set $\{A\}$ is $\Sigma_n(\kappa, z)$-definable in* V.

(iii) If κ has the $\mathbf{\Sigma}_2$-colouring property, then in a generic extension $\mathrm{V}[G]$ of V *with $\mathrm{H}(\kappa)^{\mathrm{V}[G]} \subseteq \mathrm{V}$, the cardinal κ has the $\mathbf{\Sigma}_2$-colouring property and is not Mahlo.*

Proof. (i): Our assumptions imply $\mathrm{Add}(\mu,\kappa) = \mathrm{Add}(\mu,\kappa)^{\mathrm{L}}$ and $\mathrm{Col}(\mu,{<}\kappa) = \mathrm{Col}(\mu,{<}\kappa)^{\mathrm{L}}$ and this shows that $\mathbb{P}$ is definable over $\langle \mathrm{L}_\kappa, \in\rangle$ by a formula with parameter μ. Since the set $\{\mathrm{L}_\kappa\}$ is $\Sigma_1(\kappa)$-definable, we know that the set $\{\mathbb{P}\}$ is $\Sigma_1(\kappa,\mu)$-definable.

(ii): Pick a Σ_n-formula $\varphi(v_0, v_1, v_2)$ such that A is the unique set in $\mathrm{V}[G]$ with the property that $\varphi(A, \kappa, z)$ holds. Then the weak homogeneity of $\mathbb{P}$ in V implies

$$A \;=\; \{\alpha<\kappa \mid \mathbb{1}_{\mathbb{P}} \Vdash^{\mathrm{V}} \text{“}\exists X\ [\check{\alpha}\in X \ \wedge\ \varphi(X,\check{\kappa},\check{z})\text{”}]\} \;\in\; \mathrm{V}$$

and, by the remarks made in the proof of Lemma 3.17, this shows that the set $\{A\}$ is $\Sigma_n(\kappa, z, \mathbb{P})$-definable in V. By our assumptions on $\mathbb{P}$, this shows that $\{A\}$ is $\Sigma_n(\kappa, z)$-definable in V.

(iii): Let S denote the set of all singular limit ordinals less than κ. Then S is a fat stationary subset of κ and the canonical partial order $\mathbb{C}(S)$ that shoots a club through S using bounded

closed subsets of S is $<\kappa$-distributive (see [1, Section 1]). Moreover, the set $\{\mathbb{C}(S)\}$ is $\Sigma_2(\kappa)$-definable and [16, Section 3.5, Theorem 1] implies that $\mathbb{C}(S)$ is weakly homogeneous. Let G be $\mathbb{C}(S)$-generic over V and let A be a subset of κ in $\mathrm{V}[G]$ such that the set $\{A\}$ is $\Sigma_2(\kappa, z)$-definable for some $z \in \mathrm{H}(\kappa)^{\mathrm{V}[G]}$. By (ii) and the above remarks, we know that $A, z \in \mathrm{V}$ and the set $\{A\}$ is $\Sigma_2(\kappa, z)$-definable in V. Hence our assumptions allow us to use Lemma 3.5 to find a weak κ-model M, a transitive set N and an elementary embedding $j : M \longrightarrow N$ in V such that $A \in M$, $\mathrm{crit}\,(j) = \kappa$, κ inaccessible in M and $\mathrm{H}(\kappa)^M \in M$. Since these properties of M, N and j are upwards absolute to $\mathrm{V}[G]$, Lemma 3.5 shows that κ has the $\mathbf{\Sigma}_2$-colouring property in $\mathrm{V}[G]$. □

Lemma 3.19. *Let κ be an inaccessible cardinal with the $\mathbf{\Sigma}_n$-colouring property, let $\mu < \kappa$ be an infinite regular cardinal, let $\mathbb{P} \in \{\mathrm{Add}(\mu, \kappa), \mathrm{Col}(\mu, {<}\kappa)\}$ and let G be $\mathbb{P}$-generic over V. If either $\mu = \omega$, or $\mathrm{V} = \mathrm{L}$ holds, or κ is weakly compact in V, then κ has the $\mathbf{\Sigma}_n$-colouring property in $\mathrm{V}[G]$.*

Proof. Fix $A \in \mathcal{P}(\kappa)^{\mathrm{V}[G]}$ with the property that the set $\{A\}$ is $\Sigma_n(\kappa, z)$-definable in $\mathrm{V}[G]$ for some $z \in \mathrm{H}(\kappa)^{\mathrm{V}[G]}$. Then there is a regular cardinal $\mu < \nu < \kappa$ and $H \in \mathrm{V}[G]$ such that $z \in \mathrm{V}[H]$, H is either a $\mathrm{Add}(\mu, \nu)$- or $\mathrm{Col}(\mu, \nu)$-generic over V and $\mathrm{V}[G]$ is a $\mathbb{P}$-generic extension of $\mathrm{V}[H]$. By Proposition 3.18, we know that A is an element of $\mathrm{V}[H]$ and the set $\{A\}$ is $\Sigma_n(\kappa, z, \mathbb{P})$-definable in $\mathrm{V}[H]$. Moreover, Lemma 3.17 implies that κ has the $\mathbf{\Sigma}_n$-colouring property in $\mathrm{V}[H]$.

If either $\mu = \omega$ or $\mathrm{V} = \mathrm{L}$, then $\mathbb{P} \in \{\mathrm{Add}(\mu, \kappa)^{\mathrm{L}}, \mathrm{Col}(\mu, {<}\kappa)^{\mathrm{L}}\}$ and the proof of the first part of Proposition 3.18 shows that the set $\{\mathbb{P}\}$ is $\Sigma_1(\kappa, \mu)$-definable in $\mathrm{V}[H]$. In the remaining case, if κ is weakly compact in V, then our assumptions on $\mathbb{P}$ imply that κ is weakly compact in $\mathrm{V}[H]$. In all cases, the cardinal κ has the $\Sigma_n(\kappa, z, \mathbb{P})$-partition property in $\mathrm{V}[H]$ and Lemma 3.5 shows that $\mathrm{V}[H]$ contains a weak κ-model M, a transitive set N and an elementary embedding $j : M \longrightarrow N$ such that $A \in M$, $\mathrm{crit}\,(j) = \kappa$, κ is inaccessible in M and $\mathrm{H}(\kappa)^M \in M$. Since these properties of M, N and j are upwards absolute to $\mathrm{V}[G]$, another application of Lemma 3.5 shows that, in $\mathrm{V}[G]$, the cardinal κ has the $\mathbf{\Sigma}_n$-colouring property with respect to A. □

4. The $\mathbf{\Sigma}_n$-club property

In this section, we will provide an analysis of the $\mathbf{\Sigma}_n$-club property that parallels the investigation of the $\mathbf{\Sigma}_n$-colouring property in the last section. In particular, we will show that ω_1 is the only uncountable cardinal that can consistently possess the Σ_2-club property and the only successor cardinal that can consistently have the $\mathbf{\Sigma}_1$-club property. In contrast, the results of this paper will show that ω_1 can consistently possess the $\mathbf{\Sigma}_n$-club property for all $n < \omega$, several well-known large cardinal properties imply the $\mathbf{\Sigma}_1$-club property and the existence of an accessible limit cardinal with the $\mathbf{\Sigma}_1$-club property is consistent. Moreover, we will show that the $\mathbf{\Sigma}_2$-club property implies all higher club properties. Finally, we will again establish a natural connection between these properties and large cardinal properties. Our results will show that the $\mathbf{\Sigma}_2$-club property is naturally connected with measurability through the inner model HOD and that it is possible to use the *Dodd–Jensen core model* K^{DJ} to connect the $\mathbf{\Sigma}_1$-club property to a large cardinal property that implies the existence of sharps for reals and is a consequence of ω_1*-iterability* (see Definition 4.14).

The implications derived in the following proposition are the starting point of our analysis of the $\mathbf{\Sigma}_n$-club properties.

Proposition 4.1. *Let κ be an uncountable regular cardinal and let z be a set such that κ has the $\Sigma_n(z)$-club property.*

(i) If $c : [\kappa]^{<\omega} \longrightarrow \alpha$ is a $\Sigma_n(\kappa, z)$-definable function with $\alpha < \kappa$, then there is a closed unbounded subset C of κ that is $(c \upharpoonright [\kappa]^k)$-homogeneous for all $0 < k < \omega$.

(ii) If A is a subset of κ with the property that the set $\{A\}$ is $\Sigma_n(\kappa, z)$-definable, then either A contains a closed unbounded subset of κ or A is disjoint from such a subset.

Proof. (i): Given $0 < k < \omega$, the induced function $c \upharpoonright [\kappa]^k$ is again $\Sigma_n(\kappa, z)$-definable and hence our assumption yields a $(c \upharpoonright [\kappa]^k)$-homogeneous club C_k in κ. Then $C = \bigcap_{0<k<\omega} C_k$ is a club in κ that is $(c \upharpoonright [\kappa]^k)$-homogeneous for all $0 < k < \omega$.

(ii): If $c : [\kappa]^1 \longrightarrow 2$ denotes the unique function with

$$c(a) = 1 \iff a \subseteq A,$$

then c is a $\Sigma_n(z)$-partition and hence our assumptions yield a c-homogeneous club C in κ. But then either $C \subseteq A$ or $A \cap C = \emptyset$ holds. □

By allowing the introduction of elements of the given cardinal as additional parameters, it is also possible to obtain the reversals of the above implications. The resulting characterizations provided an analog of Lemma 3.5 for the $\mathbf{\Sigma}_n$-club property.

Lemma 4.2. *The following statements are equivalent for every uncountable regular cardinal κ and every set z:*

(i) For all $m < \omega$ and $\gamma_0, \ldots, \gamma_{m-1} < \kappa$, the cardinal κ has the $\Sigma_n(\gamma_o, \ldots, \gamma_{m-1}, z)$-club property.

(ii) For all $m < \omega$, $\gamma_0, \ldots, \gamma_{m-1} < \kappa$ and $c : [\kappa]^{<\omega} \longrightarrow \alpha$ with $\alpha < \kappa$, if the function c is $\Sigma_n(\kappa, \gamma_0, \ldots, \gamma_{m-1}, z)$-definable, then there is a closed unbounded subset C of κ that is $(c \restriction [\kappa]^k)$-homogeneous for all $0 < k < \omega$.

(iii) For all $m < \omega$, $\gamma_0, \ldots, \gamma_{m-1} < \kappa$ and $A \subseteq \kappa$, if the set $\{A\}$ is $\Sigma_n(\kappa, \gamma_0, \ldots, \gamma_{m-1}, z)$-definable, then either A contains a closed unbounded subset of κ or A is disjoint from such a subset.

(iv) For all $m < \omega$, $\gamma_0, \ldots, \gamma_{m-1} < \kappa$ and $c : \kappa \longrightarrow \alpha$ with $\alpha < \kappa$, if the function c is $\Sigma_n(\kappa, \gamma_0, \ldots, \gamma_{m-1}, z)$-definable, then c is constant on a closed unbounded subset of κ.

Proof. <u>*(i)* $\Longrightarrow$ *(ii)* + *(iii)*</u>: These implications are given by Proposition 4.1.

<u>*(ii)* $\Longrightarrow$ *(i)*</u>: Assume (ii) and fix a $\Sigma_n(\gamma_0, \ldots, \gamma_{m-1})$-partition $c : [\kappa]^k \longrightarrow \alpha$ with $0 < k < \omega$ and $\alpha < \kappa$. Let $c_* : [\kappa]^{<\omega} \longrightarrow \alpha$ denote the unique function with $c_*(a) = c(a)$ for all $a \in [\kappa]^k$ and $c(a) = 0$ for all $a \in [\kappa]^{<\omega}$ with $|a| \neq k$. Then c_* is $\Sigma_n(\kappa, \gamma_0, \ldots, \gamma_{m-1}, z)$-definable and our assumption yields a $(c_* \restriction [\kappa]^k)$-homogeneous club C in κ. Then C is also c-homogeneous.

<u>*(iii)* $\Longrightarrow$ *(iv)*</u>: Assume (iii) and fix $m < \omega$, $\gamma_0, \ldots, \gamma_{m-1} < \kappa$ and a $\Sigma_n(\kappa, \gamma_0, \ldots, \gamma_{m-1}, z)$-definable function $c : \kappa \longrightarrow \alpha$ with $\alpha < \kappa$. Given $\xi < \alpha$, define $A_\xi = \{\beta < \kappa \mid c(\beta) = \xi\}$. For every $\xi < \alpha$, the set $\{A_\xi\}$ is $\Sigma_n(\kappa, \gamma_0, \ldots, \gamma_{m-1}, \xi, z)$-definable and therefore (iii) yields a club C_ξ in κ that is either contained in A_ξ or disjoint from A_ξ. Pick $\beta \in \bigcap\{C_\xi \mid \xi < \alpha\}$. Then $\beta \in A_{c(\beta)} \cap C_{c(\beta)} \neq \emptyset$. It follows that $C_{c(\beta)} \subseteq A_{c(\beta)}$ and therefore c is constant on $C_{c(\beta)}$ with value $c(\beta)$.

<u>*(iv)* $\Longrightarrow$ *(i)*</u>: Assume (iv) and fix $m < \omega$, $\gamma_0, \ldots, \gamma_{m-1} < \kappa$ and a $\Sigma_n(\kappa, \gamma_0, \ldots, \gamma_{m-1}, z)$-partition $c : [\kappa]^k \longrightarrow \alpha$ with $0 < k < \omega$ and $\alpha < \kappa$. In this situation, we can use (iv) to inductively construct

- a sequence $\langle c^l_a : (\max(a), \kappa) \longrightarrow \alpha \mid l < k, \ a \in [\kappa]^l \rangle$ of functions,
- a sequence $\langle \xi^l_a < \alpha \mid l < k, \ a \in [\kappa]^l \rangle$ of ordinals, and
- a sequence $\langle \varphi_l(v_0, \ldots, v_{m+l+3}) \mid l < k \rangle$ of Σ_n-formulas

such that the following statements hold:

(a) If $a \in [\kappa]^{k-1}$ and $\max(a) < \beta < \kappa$, then $c^{k-1}_a(\beta) = c(a \cup \{\beta\})$.

(b) If $l < k$ and $\beta_0 < \ldots < \beta_{l-1} < \beta < \kappa$, then $c^l_{\{\beta_0, \ldots, \beta_{l-1}\}}(\beta)$ is the unique ordinal ξ such that $\varphi_l(\kappa, \beta_0, \ldots, \beta_{l-1}, \beta, \xi, \gamma_0, \ldots, \gamma_{m-1}, z)$ holds.

(c) If $l < k$ and $a \in [\kappa]^l$, then ξ^l_a is the unique element of α whose preimage under c^l_a contains a closed unbounded subset of κ.

(d) If $0 < l < k$, $a \in [\kappa]^{l-1}$ and $\max(a) < \beta < \kappa$, then $c^{l-1}_a(\beta) = \xi^l_{a \cup \{\beta\}}$.

Given $l < k$ and $a \in [\kappa]^l$, pick a club C_a in κ with $c^l_a[C_a] = \{\xi^l_a\}$. Define

$$C \ = \ \triangle\{\bigcap\{C^l_a \mid l < k, \ a \in [\beta]^l\} \mid \beta < \kappa\}.$$

Pick $\beta_0, \ldots, \beta_{k-1} \in C \cap \mathrm{Lim}$ with $\beta_0 < \ldots < \beta_{k-1}$ and set $a_l = \{\beta_0, \ldots, \beta_{l-1}\}$ for all $l \leq k$. Then $\beta_l \in C^l_{a_l}$ and $c^l_{a_l}(\beta_l) = \xi^l_{a_l}$ for all $l < k$. Moreover, if $0 < l \leq k$, then $\xi^l_{a_l} = c^{l-1}_{a_{l-1}}(\beta_{l-1})$. In combination, this allows us to conclude that $c(a_k) = c^{k-1}_{a_{k-1}}(\beta_{k-1}) = \xi^0_\emptyset$ and this shows that the club C is c-homogeneous. □

Corollary 4.3. *The following statements are equivalent for every uncountable regular cardinal κ:*

(i) κ has the $\mathbf{\Sigma}_n$-club property.

(ii) For all $A \subseteq \kappa$, if the set $\{A\}$ is $\Sigma_n(\kappa, z)$-definable for some $z \in \mathrm{H}(\kappa)$, then either A contains a closed unbounded subset of κ or A is disjoint from such a subset. □

The above computations now allows us to prove the restrictions on the possible types of cardinals possessing the club property that were mentioned above. Remember that, given regular cardinals $\mu < \nu$, we let S^{ν}_{μ} denote the set of all limit ordinals $\lambda < \nu$ with $\mathrm{cof}(\lambda) = \mu$.

Proposition 4.4. *(i) The set $\{S^{\nu^+}_{\nu}\}$ is $\Sigma_1(\nu^+, \nu)$-definable for all infinite regular cardinals ν.*
(ii) The set $\{S^{\omega_{k+1}}_{\omega_k}\}$ is $\Sigma_1(\omega_{k+1})$-definable for all $k < \omega$.
(iii) If ν is an uncountable cardinal, then ν^+ does not have the $\mathbf{\Sigma}_1$-club property.
(iv) Regular cardinals greater than ω_1 do not have the Σ_2-club property.

Proof. *(i)*: Fix an infinite regular cardinal ν and $\gamma \in \mathrm{Lim} \cap \nu^+$. If there is a strictly increasing cofinal function $s : \nu \longrightarrow \gamma$, then $\mathrm{cof}(\gamma) = \nu$. In the other case, if there is a limit ordinal $\lambda < \nu$ and a strictly increasing cofinal function $s : \lambda \longrightarrow \gamma$, then $\mathrm{cof}(\gamma) < \nu$. These two implications yield a $\Sigma_1(\nu^+, \nu)$-definition of the set $\{S^{\nu^+}_{\nu}\}$.

(ii): Given $k < \omega$, the cardinal ω_k is the unique ordinal λ with the property that there is a transitive model M of $\mathrm{ZFC}^- +$ "ω_ω *exists*" such that $\omega_{k+1} = \omega^M_{k+1}$ and $\omega_k = \lambda$. This observation shows that the set $\{\omega_k\}$ is $\Sigma_1(\omega_{k+1})$-definable and, in combination with (i), this yields the desired statement.

(iii): Assume, towards a contradiction, that there is an uncountable cardinal ν such that the cardinal ν^+ has the $\mathbf{\Sigma}_1$-club property. Since ν is uncountable, we can combine (i) with Proposition 4.1(ii) to see that ν is singular. Let z denote the set of all uncountable regular cardinals less than ν. Then the set $S^{\nu^+}_{\omega}$ consists of all limit ordinals $\lambda < \nu^+$ with the property that there is no strictly increasing cofinal function $s : \mu \longrightarrow \lambda$ with $\mu \in z$. This shows that the set $\{S^{\nu^+}_{\omega}\}$ is $\Sigma_1(\nu^+, z)$-definable, contradicting Proposition 4.1(ii).

(iv): If $\kappa > \omega_1$ is a regular cardinal, then the set $\{S^{\kappa}_{\omega}\}$ is $\Sigma_2(\kappa)$-definable and therefore Proposition 4.1(ii) implies that κ does not have the Σ_2-club property. □

Note that in general, if ν is an infinite cardinal, then the set $\{\nu\}$ need not be $\Sigma_1(\nu^+)$-definable. For example, [30, Corollary 3.3] shows that it is consistent that for some measurable cardinal δ, the sets $\{\delta\}$ and $\{\delta^+\}$ are not $\Sigma_1(\delta^{++})$-definable.

Proposition 4.5. *The following statements are equivalent for every uncountable regular cardinal κ:*

(i) κ has the $\mathbf{\Sigma}_2$-club property.
(ii) κ has the $\mathbf{\Sigma}_n$-club property for all $n < \omega$.
(iii) If $z \in \mathrm{H}(\kappa)$, then HOD_z does not contain a bistationary subset of κ.

Proof. *(iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i)*: Both implications are trivial.

(i) $\Longrightarrow$ (iii): Assume, towards a contradiction, that (iii) fails for some $z \in \mathrm{H}(\kappa)$ and let X denote the set of all bistationary subsets of κ. Then both X and $\mathrm{V} \setminus X$ are $\Sigma_2(\kappa)$-definable and hence Proposition 3.9 yields an $A \in X$ with the property that the set $\{A\}$ is $\Sigma_2(\kappa, z)$-definable. By Proposition 4.1(ii), this implies that (i) fails. □

In the remainder of this section, we investigate the consistency strength of the $\mathbf{\Sigma}_n$-club properties. In the case $n = 1$, we will show that the validity of the $\mathbf{\Sigma}_1$-club property at ω_1 is equiconsistent with both the existence of an inaccessible cardinal with this property and the existence of an accessible limit cardinal possessing this property. Moreover, we will present narrow bounds for the consistency strength of these theories.[6] In the following, many arguments rely on the notion of *good sets of indiscernibles* (see [11, Section 1]). Remember that, if κ is a cardinal and A is a subset of κ, then $I \subseteq \kappa$ is a *good set of indiscernibles for* $\langle \mathrm{L}_\kappa[A], \in, A\rangle$ if the following statements hold for all $\gamma \in I$:

(i) $\langle \mathrm{L}_\gamma[A \cap \gamma], \in, A \cap \gamma\rangle$ is an elementary substructure of $\langle \mathrm{L}_\kappa[A], \in, A\rangle$.
(ii) $I \setminus \gamma$ is a set of indiscernibles for the structure $\langle \mathrm{L}_\kappa[A], \in, A, \xi\rangle_{\xi<\gamma}$.

Then [11, Lemma 1.2] shows that a cardinal κ is Ramsey if and only if for every $A \subseteq \kappa$, there is a good set of indiscernibles for $\langle \mathrm{L}_\kappa[A], \in, A\rangle$. In the following, we will show that cardinals with $\mathbf{\Sigma}_1$-club property are Ramsey with respect to subsets of κ whose singletons are Σ_1-definable, in

[6] After a first version of this paper was circulated, Welch computed the exact consistency strength of the $\mathbf{\Sigma}_1$-club property in [39].

the sense that the club property implies the existence of good sets of indiscernibles for the corresponding structures $\langle \mathrm{L}_\kappa[A], \in, A\rangle$. Moreover, we will show that this restricted form of Ramseyness is downwards absolute to the Dodd–Jensen core model K^{DJ} and, in this inner model, it is equivalent to the $\mathbf{\Sigma}_1$-club property. These arguments will also allow us to show that the existence of a cardinal with the $\mathbf{\Sigma}_1$-club property has much higher consistency strength than the existence of a cardinal with the $\mathbf{\Sigma}_1$-colouring property.

Proposition 4.6. *Let κ be an uncountable regular cardinal and let z be a set such that κ has the $\Sigma_n(\beta_0, \ldots, \beta_{m-1}, z)$-club property for all $m < \omega$ and $\beta_0, \ldots, \beta_{m-1} < \kappa$. If A is a subset of κ such that the set $\{A\}$ is $\Sigma_n(\kappa, z)$-definable, then there is a closed unbounded subset of κ that is a good set of indiscernibles for $\langle \mathrm{L}_\kappa[A], \in, A\rangle$.*

Proof. Given $k, m < \omega$, an $\mathcal{L}_\in$-formula $\varphi(v_0, \ldots, v_{k+m-1})$ and ordinals $\beta_0, \ldots, \beta_{m-1} < \kappa$, we let $c_{\varphi,\beta_0,\ldots,\beta_{m-1}} : [\kappa]^{k+1} \longrightarrow 2$ denote the unique function with

$$c_{\varphi,\beta_0,\ldots,\beta_{m-1}}(\{\alpha_0, \ldots, \alpha_k\}) = 0 \iff \langle \mathrm{L}_\kappa[A], \in, A\rangle \models \varphi(\alpha_0, \ldots, \alpha_k, \beta_0, \ldots, \beta_{m-1})$$

for all $\alpha_0 < \ldots < \alpha_k < \kappa$. By our assumptions on A, we know that the set $\{\mathrm{L}_\kappa[A]\}$ is $\Sigma_n(\kappa, z)$-definable and hence $c_{\varphi,\beta_0,\ldots,\beta_{m-1}}$ is a $\Sigma_n(\beta_0, \ldots, \beta_{m-1}, z)$-partition. But this shows that there is a $c_{\varphi,\beta_0,\ldots,\beta_{m-1}}$-homogeneous club in κ. Given $\beta < \kappa$, this implies that there is a club C_β in κ that is $c_{\varphi,\beta_0,\ldots,\beta_{m-1}}$-homogeneous for all $k, m < \omega$, every $\mathcal{L}_\in$-formula $\varphi(v_0, \ldots, v_{k+m-1})$ and all $\beta_0, \ldots, \beta_{m-1} < \beta$. Let C denote the intersection of $\triangle\{C_\beta \mid \beta < \kappa\}$ with the club of all $\gamma < \kappa$ such that $\langle \mathrm{L}_\gamma[A \cap \gamma], \in, A \cap \gamma\rangle$ is an elementary substructure of $\langle \mathrm{L}_\kappa[A], \in, A\rangle$. Then it is easy to check that $\mathrm{Lim}(C)$ is a good set of indiscernibles for $\langle \mathrm{L}_\kappa[A], \in, A\rangle$. □

The following corollary uses the above result to show that strong anti-large cardinal assumptions imply the existence of simply definable bistationary subsets of uncountable regular cardinals.

Corollary 4.7. *If x is a real with the property that $x^\#$ does not exist and κ is an uncountable regular cardinal, then there is a bistationary subset A of κ with the property that the set $\{A\}$ is $\Sigma_1(\kappa, \beta_0, \ldots, \beta_{m-1}, x)$-definable for some $m < \omega$ and $\beta_0, \ldots, \beta_{m-1} < \kappa$.*

Proof. By our assumption, standard arguments (i.e. the relativization of [20, Corollary 18.18] for $x^\#$ discussed on [20, p. 328]) show that there is no uncountable good set of indiscernibles for $\langle \mathrm{L}_\kappa[x], \in, x\rangle$. Hence Proposition 4.6 yields $l < \omega$ and $\gamma_0, \ldots, \gamma_{l-1} < \kappa$ such that κ does not have the $\Sigma_1(\gamma_0, \ldots, \gamma_{l-1}, x)$-club property. This shows that statement (i) of Lemma 4.2 fails for x. But then statement (iii) of Lemma 4.2 also fails for x and we can find $m < \omega$ and $\beta_0, \ldots, \beta_{m-1} < \kappa$ with the property that there exists a $\Sigma_1(\kappa, \beta_0, \ldots, \beta_{m-1}, x)$-definable bistationary subset of κ. □

Corollary 4.8. *If there exists an uncountable regular cardinal with the $\mathbf{\Sigma}_1$-club property, then $x^\#$ exists for every real x.*

Proof. This implication directly follows from a combination of Proposition 4.1 with Corollary 4.7. □

In [14], Gitman provided another useful characterization of Ramseyness by showing that a cardinal κ is Ramsey if and only if for every $A \subseteq \kappa$, there is a weak κ-model M with $A \in M$ and a weakly amenable countably complete M-ultrafilter on κ (see [14, Proposition 2.8(3)]). In combination with arguments from [14], the above results already show that the $\mathbf{\Sigma}_1$-club property implies the restriction of this property to Σ_1-definable singletons. We will later show that, in canonical inner models, this restricted property is actually equivalent to the $\mathbf{\Sigma}_1$-club property. This will allow us to show that the $\mathbf{\Sigma}_1$-club property is downwards absolute to the Dodd–Jensen core model K^{DJ}.

In order to prove these results, we start by noticing that the proof of the forward direction of [14, Proposition 2.8(3)] in [14, Section 4] also provides a proof of the following statement.

Lemma 4.9. *Let κ be an uncountable regular cardinal and let A be a subset of κ with the property that κ is an inaccessible cardinal in $\mathrm{L}[A]$. If there is a good set of indiscernibles for $\langle \mathrm{L}_\kappa[A], \in, A\rangle$ of cardinality κ, then there is a weak κ-model M with $A \in M$ and a weakly amenable countably complete M-ultrafilter on κ.*

By combining Corollary 3.2, Proposition 4.6 and Lemma 4.9, we directly obtain the following corollary.

Corollary 4.10. *Let κ be an uncountable regular cardinal and let z be a set such that κ has the $\Sigma_n(\beta_0,\dots,\beta_{m-1},z)$-club property for all $m<\omega$ and $\beta_0,\dots,\beta_{m-1}<\kappa$. If A is a subset of κ with the property that the set $\{A\}$ is $\Sigma_n(\kappa,z)$-definable, then there is a weak κ-model M with $A\in M$ and a weakly amenable countably complete M-ultrafilter on κ.* □

The next result will later allow us to show that, in the case $n=1$, the converse of the above implication also holds true in certain canonical inner models. The arguments used in its proof are taken from the proof of [29, Lemma 6.7].

Lemma 4.11. *Let κ be an uncountable regular cardinal, let $z\in \mathrm{H}(\kappa)$ and let $\varphi(v_0,\dots,v_3)$ be a Σ_0-formula. Assume that there is a unique subset A of κ with the property that $\exists x\ \varphi(A,x,\kappa,z)$ holds. If there exist a weak κ-model M with the property that $A,\mathrm{tc}(\{z\})\in M\models\exists x\ \varphi(A,x,\kappa,z)$ and a weakly amenable M-ultrafilter U on κ such that $\langle M,\in,U\rangle$ is ω_1-iterable, then A either contains a club subset of κ or is disjoint from such a set.*

Proof. Fix $a\in M$ such that $\varphi(A,a,\kappa,z)$ holds and pick an elementary submodel $\langle N,\in,F\rangle$ of $\langle M,\in,U\rangle$ of cardinality less than κ with $\mathrm{tc}(\{z\})\cup\{A,a\}\subseteq N$. Let $\pi:N\longrightarrow M_0$ denote the corresponding transitive collapse and set $U_0=\pi[F]$. Then U_0 is a weakly amenable M_0-ultrafilter on $\pi(\kappa)$ and [22, Theorem 19.15] implies that $\langle M_0,\in,U_0\rangle$ is ω_1-iterable. Let

$$\langle\langle M_\alpha\mid\alpha\leq\kappa\rangle,\langle j_{\alpha,\beta}:M_\alpha\longrightarrow M_\beta\mid\alpha\leq\beta\leq\kappa\rangle\rangle$$

denote the corresponding iteration of length $\kappa+1$. Then we have $(j_{0,\kappa}\circ\pi)(\kappa)=\kappa$, $(j_{0,\kappa}\circ\pi)(z)=z$ and Σ_1-upwards absoluteness implies that $\exists x\ \varphi((j_{0,\kappa}\circ\pi)(A),x,\kappa,z)$ holds. This allows us to conclude that $A=(j_{0,\kappa}\circ\pi)(A)$ and $(j_{0,\alpha}\circ\pi)(A)=A\cap(j_{0,\alpha}\circ\pi)(\kappa)$ for all $\alpha<\kappa$. Define C to be the club $\{(j_{0,\alpha}\circ\pi)(\kappa)\mid\alpha<\kappa\}$ in κ. First, assume that $A\in U$. Then $(j_{0,\alpha}\circ\pi)(A)\in U_\alpha$ and hence $(j_{0,\alpha}\circ\pi)(\kappa)\in(j_{0,\alpha+1}\circ\pi)(A)\subseteq A$ for all $\alpha<\kappa$. This shows that C is a subset of A in this case. In the other case, if $A\notin U$, then the same argument shows that $A\cap C=\emptyset$. □

Lemma 4.12. *If κ is an uncountable regular cardinal such that there exists a good $\Sigma_1(\kappa,y)$-well-ordering of $\mathcal{P}(\kappa)$ for some $y\in\mathrm{H}(\kappa)$, then the following statements are equivalent:*

- *(i) κ has the $\boldsymbol{\Sigma}_1$-club property.*
- *(ii) For all $A\subseteq\kappa$ with the property that the set $\{A\}$ is $\Sigma_1(\kappa,z)$-definable for some $z\in\mathrm{H}(\kappa)$, there is a weak κ-model M with $A\in M$ and a weakly amenable countably complete M-ultrafilter on κ.*

Proof. <u>*(i)* $\Longrightarrow$ *(ii)*</u>: This implication is a direct consequence of Corollary 4.10.

<u>*(ii)* $\Longrightarrow$ *(i)*</u>: Assume that (ii) holds. Let $\lhd$ be a well-ordering of some class containing $\mathcal{P}(\kappa)$ such that the class $I(\lhd)$ is $\Sigma_1(\kappa,z)$-definable. Fix a Σ_0-formula $\varphi(v_0,\dots,v_3)$ and $z\in\mathrm{H}(\kappa)$ such that there is a unique subset A of κ with the property that $\exists x\ \varphi(A,x,\kappa,z)$ holds. Then there is an $x\in\mathrm{H}(\kappa^+)$ such that $\varphi(A,x,\kappa,z)$ holds. Let B denote the $\lhd$-least element of $\mathcal{P}(\kappa)$ with the property that, if $\alpha>\kappa$ is minimal with $\mathrm{L}[B]\models\mathrm{ZFC}^-$, then $A,\mathrm{tc}(\{z\})\in\mathrm{L}_\alpha[B]\models\exists x\ \varphi(A,x,\kappa,z)$. Then the set $\{B\}$ is $\Sigma_1(\kappa,y,z)$-definable and our assumptions yield a weak κ-model M with $B\in M$ and a weakly amenable countably complete M-ultrafilter on κ. Then $A,z\in M\models\exists x\ \varphi(A,x,\kappa,z)$ and, since countable completeness implies ω_1-iterability (see [22, Lemma 19.11 and 19.12]), we can apply Lemma 4.11 to conclude that A either contains a club in κ or is disjoint from such a set. □

Lemma 4.13. *If κ is an uncountable regular cardinal with the $\boldsymbol{\Sigma}_1$-club property, then κ is an inaccessible cardinal with the $\boldsymbol{\Sigma}_1$-club property in K^{DJ}.*

Proof. Fix $z\in\mathrm{H}(\kappa)^{\mathrm{K}^{DJ}}$ and $A\in\mathcal{P}(\kappa)^{\mathrm{K}^{DJ}}$ such that the set $\{A\}$ is $\Sigma_1(\kappa,z)$-definable in K^{DJ}. By Corollary 4.8, our assumption implies the existence of $0^\#$ and hence results of Dodd and Jensen (see [12, p. 238]) show that K^{DJ} is equal to the union of all *lower parts* of *iterable premice* in this situation. Since the class of all iterable premice is $\Sigma_1(\kappa)$-definable (see, for example, the proof of [30, Lemma 2.3]), this shows that the class K^{DJ} is also $\Sigma_1(\kappa)$-definable in this case and we can conclude that the set $\{A\}$ is $\Sigma_1(\kappa,z)$-definable in V. Therefore, we can apply Proposition 4.6 to find a club subset of κ that is a good set of indiscernibles for $\langle\mathrm{L}_\kappa[A],\in,A\rangle$. In this situation, we can apply the *Jensen Indiscernibles Lemma* (see [11, Lemma 1.3]) to find a good set of indiscernibles for $\langle\mathrm{L}_\kappa[A],\in,A\rangle$ of cardinality κ that is an element of K^{DJ}. Since Corollary 3.2 shows that κ is inaccessible in $L[A]$, we can now apply Lemma 4.9 to show that in K^{DJ}, there is a weak κ-model M with $A \in M$ and a weakly amenable countably complete M-ultrafilter on κ. But now, the results of [30, Section 2] show that the restriction of the canonical well-ordering of K^{DJ} to $\mathcal{P}(\kappa)^{K^{DJ}}$ is a good $\Sigma_1(\kappa)$-well-ordering in K^{DJ}. Therefore, the above computations allow us to use Lemma 4.12 in K^{DJ} to conclude that κ has the $\mathbf{\Sigma}_1$-club property in this model. Finally, we can apply Proposition 3.13 to show that κ is inaccessible in K^{DJ}. □

Next, we provide an upper bound for the consistency strength of the existence of an inaccessible cardinal with the $\mathbf{\Sigma}_1$-club property with the help of the following large cardinal property strengthening weak compactness that was introduced by Sharpe and Welch in [33] and extensively studied in [15].

Definition 4.14. An uncountable cardinal κ is *ω_1-iterable* if for every subset A of κ, there is a weak κ-model M and a weakly amenable M-ultrafilter U on κ such that $A \in M$ and $\langle M, \in, U\rangle$ is ω_1-iterable.

The following corollary is a direct consequence of Lemma 4.11. Note that this result is basically already proven in [29, Section 6], but only for $\Sigma_1(\kappa)$-definitions.

Corollary 4.15. *All ω_1-iterable cardinals have the $\mathbf{\Sigma}_1$-club property.* □

In the following, we show that the $\mathbf{\Sigma}_1$-club property at ω_1 can be established by collapsing an inaccessible cardinal with this property.

Lemma 4.16. *Let κ be an uncountable regular cardinal with the $\mathbf{\Sigma}_n$-club property and let $\mathbb{P} \in H(\kappa)$ be a partial order. If G is $\mathbb{P}$-generic over V, then κ has the $\mathbf{\Sigma}_n$-club property in $V[G]$.*

Proof. Fix $z \in H(\kappa)^{V[G]}$, a $\Sigma_n(z)$-partition $c : [\kappa]^k \longrightarrow \gamma$ in $V[G]$ with $\gamma < \kappa$ and a Σ_n-formula $\varphi(v_0, \ldots, v_{k+2})$ defining c in $V[G]$ as in Definition 1.1. Work in V and fix a condition p in $\mathbb{P}$, a $\mathbb{P}$-name $\dot{z} \in H(\kappa)^V$ with $z = \dot{z}^G$ and a bijection b between a cardinal ν and the set of all conditions in $\mathbb{P}$ below p. Given $\alpha_0 < \ldots < \alpha_{k-1} < \kappa$, define $c_p(\{\alpha_0, \ldots, \alpha_{k-1}\})$ to be the least ordinal of the form $\prec\beta, \delta\succ$, where $\beta < \nu$, $\delta < \gamma$ and

$$b(\beta) \Vdash^V_{\mathbb{P}} \varphi(\check{\alpha}_0, \ldots, \check{\alpha}_{k-1}, \check{\delta}, \check{\kappa}, \dot{z}).$$

Then the arguments used in the proof of Lemma 3.17 show that c_p is a $\Sigma_n(\dot{z}, \mathbb{P})$-partition. By our assumptions, a standard density argument now yields $q \in G$, $\delta < \gamma$ and a club subset C of κ in V with $q \Vdash^V_{\mathbb{P}} \varphi(\check{\alpha}_0, \ldots, \check{\alpha}_{k-1}, \check{\delta}, \check{\kappa}, \dot{z})$ for all $\alpha_0, \ldots, \alpha_{k-1} \in C$ with $\alpha_0 < \ldots < \alpha_{k-1}$. In particular, the closed unbounded set C is a c-homogeneous subset of κ in $V[G]$. □

Lemma 4.17. *Let κ be an inaccessible cardinal, let $\mathbb{P} \in \{\mathrm{Add}(\omega, \kappa), \mathrm{Col}(\omega, {<}\kappa)\}$ and let G be $\mathbb{P}$-generic over V. If κ has the $\mathbf{\Sigma}_1$-club property in V, then κ has the $\mathbf{\Sigma}_1$-club property in $V[G]$.*

Proof. Pick a subset A of κ in $V[G]$ with the property that the set $\{A\}$ is $\Sigma_1(\kappa, z)$-definable in $V[G]$ for some $z \in H(\kappa)^{V[G]}$. Then there is a regular cardinal $\nu < \kappa$ in V and $H \in V[G]$ such that H is either $\mathrm{Add}(\omega, \nu)$- or $\mathrm{Col}(\omega, \nu)$-generic over V, the set z is an element of $V[H]$ and $V[G]$ is a $\mathbb{P}$-generic extension of $V[H]$. By Proposition 3.18, we know that A is an element of $V[H]$ and the set $\{A\}$ is $\Sigma_1(\kappa, z)$-definable in $V[H]$. In this situation, Lemma 4.16 shows that κ has the $\mathbf{\Sigma}_1$-club property in $V[H]$ and hence there is a club subset C of κ in $V[H]$ that is either contained in A or disjoint from A. By Corollary 4.3, these computations show that κ has the $\mathbf{\Sigma}_1$-club property in $V[G]$. □

In the remainder of this section, we show that the validity of the $\mathbf{\Sigma}_2$-club property at ω_1 is equiconsistent with the existence of a measurable cardinal.

Proposition 4.18. *If z is a set with the property that ω_1 has the $\Sigma_2(z)$-club property, then ω_1 is a measurable cardinal in HOD_z.*

Proof. Let F_z denote the intersection of the club filter on ω_1 with HOD_z. Then F_z is an element of HOD_z. Assume, towards a contradiction, that F_z does not witness the measurability of ω_1 in HOD_z. By the closure properties of the club filter, this implies that HOD_z contains a bistationary subset of ω_1. If A_z denotes the least such subset in the canonical well-ordering of HOD_z, then the fact that this ordering is a good $\Sigma_2(z)$-well-ordering implies that the set $\{A_z\}$ is $\Sigma_2(z)$-definable. By Proposition 4.1, this contradicts our assumption. □

The next lemma shows that the existence of a measurable cardinal is also an upper bound for the consistency of the validity of the $\mathbf{\Sigma}_2$-club property at ω_1. Its proof is a small variation of a classical result of Jech, Magidor, Mitchell and Prikry from [21].

Lemma 4.19. *If κ is a measurable cardinal, then there is a generic extension $\mathrm{V}[G]$ of V with the property that $\kappa = \omega_1^{\mathrm{V}[G]}$ and no bistationary subset of ω_1 in $\mathrm{V}[G]$ is contained in $\mathrm{HOD}(\mathbb{R})^{\mathrm{V}[G]}$. In particular, in $\mathrm{V}[G]$, the cardinal ω_1 has the $\mathbf{\Sigma}_n$-club property for all $n < \omega$.*

Proof. Let U be a normal ultrafilter on κ, let $j : \mathrm{V} \longrightarrow M$ denote the canonical ultrapower embedding induced by U, let G be $\mathrm{Col}(\omega, {<}j(\kappa))$-generic over V, let G_0 denote the filter on $\mathrm{Col}(\omega, {<}\kappa)$ induced by G and let $j_G : \mathrm{V}[G_0] \longrightarrow M[G]$ denote the canonical lifting of j to $\mathrm{V}[G]$ (see [8, Proposition 9.1]). Since $\mathrm{Col}(\omega, {<}\kappa)$ satisfies the κ-chain condition in V, we know that every element of U is a stationary subset of κ in $\mathrm{V}[G_0]$.

Work in $\mathrm{V}[G_0]$. By the above remark, if A is an element of U, then the partial order $\mathbb{C}(A)$ consisting of all bounded closed subsets of A ordered by end-extension is σ-distributive and weakly homogeneous and it forces A to contain a club subset of $\kappa = \omega_1^{\mathrm{V}[G_0]}$ (see [1, Theorem 1] and [16, Section 3.5, Theorem 1]). Let $\vec{\mathbb{C}}$ denote the countable support product of forcings of the form $\mathbb{C}(A)$ with $A \in U$. Then $\vec{\mathbb{C}}$ is weakly homogeneous and the fact that CH holds allows us to use a Δ-system argument to show that $\vec{\mathbb{C}}$ satisfies the $\aleph_2$-chain condition.

Claim. *$\vec{\mathbb{C}}$ is σ-distributive in $\mathrm{V}[G_0]$.*

Proof of the Claim. Work in $\mathrm{V}[G_0]$. Fix a condition $\vec{p}$ in $\vec{\mathbb{C}}$ and a $\vec{\mathbb{C}}$-nice name τ for a countable set of ordinals. Since $\vec{\mathbb{C}}$ satisfies the $\aleph_2$-chain condition, there is a subset U_1 of U of cardinality κ such that the support of $\vec{p}$ and the supports of all conditions appearing in τ are subsets of U_1. Using the fact that $\mathrm{Col}(\omega, {<}\kappa)$ satisfies the κ-chain condition in V, we find a subset U_0 of U in V that contains U_1 and has cardinality κ in V. In this situation, the closure properties of M imply that the sets U_0, $j[U_0]$ and $j \restriction U_0$ are all contained in M and all three sets are countable in $M[G]$. Let $\vec{\mathbb{C}}_0$ denote the countable support product of all partial orders of the form $\mathbb{C}(A)$ with $A \in U_0$, let $\vec{p}_0$ denote the condition in $\vec{\mathbb{C}}_0$ corresponding to $\vec{p}$ and let τ_0 denote the canonical $\vec{\mathbb{C}}_0$-name induced by τ.

Since $\vec{\mathbb{C}}_0 \in \mathrm{H}(\kappa^+)^{\mathrm{V}[G_0]}$ and $\mathrm{Col}(\omega, {<}\kappa)$ satisfies the κ-chain condition in V, there is a $\mathrm{Col}(\omega, {<}\kappa)$-name $\dot{\mathbb{C}} \in \mathrm{H}(\kappa^+)^{\mathrm{V}}$ for a partial order with the property that $\dot{\mathbb{C}}^{G_0}$ is the suborder of $\vec{\mathbb{C}}_0$ consisting of all conditions below $\vec{p}_0$. But then $\dot{\mathbb{C}}$ is an element of M and, by [8, Theorem 14.2], there is a complete embedding $\iota : \mathrm{Col}(\omega, {<}\kappa) * \dot{\mathbb{C}} \longrightarrow \mathrm{Col}(\omega, {<}j(\kappa))$ in M that extends the identity on $\mathrm{Col}(\omega, {<}\kappa)$. Let $\vec{H}$ denote the filter on $\vec{\mathbb{C}}_0$ induced by ι and G. Moreover, given $A \in U_0$, let H_A denote the filter on $\mathbb{C}(A)$ induced by $\vec{H}$. For each $A \in U_0$, we then have $\bigcup H_A \in M[G]$, $\kappa \in j(A)$, $\bigcup H_A \subseteq A \subseteq j(A)$ and hence $\{\kappa\} \cup \bigcup H_A$ is a bounded closed subset of $j(A)$ in $M[G]$. By the above computations, there is a condition $\vec{q}$ in $j_G(\vec{\mathbb{C}}_0)$ with support $j[U_0]$ and the property that $\vec{q}(j(A)) = \{\kappa\} \cup \bigcup H_A$ for all $A \in U_0$. But then we have $\vec{q} \leq_{j_G(\vec{\mathbb{C}}_0)} j_G(\vec{r})$ for all $\vec{r} \in \vec{H}$. In particular, if $n < \omega$, then there is a condition $\vec{r}_n$ in $\vec{H}$ that decides the n-the element of τ_0 in $\mathrm{V}[G]$ and satisfies $\vec{q} \leq_{j_G(\vec{\mathbb{C}}_0)} j_G(\vec{r}_n)$. By elementarity, this yields a condition $\vec{r}$ below $\vec{p}$ in $\vec{\mathbb{C}}_0$ with

$$\vec{r} \Vdash^{\mathrm{V}[G_0]}_{\vec{\mathbb{C}}_0} \text{“}\dot{\tau}_0 = \check{c}\text{”}$$

for some countable set of ordinals c. Since $\vec{\mathbb{C}}_0$ is a complete suborder of $\vec{\mathbb{C}}$, these computations yield the statement of the claim. □

Now, let $\vec{H}$ be $\vec{\mathbb{C}}$-generic over $\mathrm{V}[G_0]$, fix a subset B of κ in $\mathrm{HOD}(\mathbb{R})^{\mathrm{V}[G_0,\vec{H}]}$ and pick $x \in \mathbb{R}^{\mathrm{V}[G_0,\vec{H}]}$ with $B \in \mathrm{HOD}_x^{\mathrm{V}[G_0,\vec{H}]}$. By the above claim, we have $\kappa = \omega_1^{\mathrm{V}[G_0,\vec{H}]}$, $x \in \mathrm{V}[G_0]$ and the homogeneity of $\vec{\mathbb{C}}$ in $\mathrm{V}[G]$ implies that $B \in \mathrm{V}[G_0]$. Moreover, since $\vec{\mathbb{C}}$ is definable in $\mathrm{V}[G_0]$ by a formula that only uses U as a parameter, we know that $B \in \mathrm{HOD}_{U,x}^{\mathrm{V}[G_0]}$. Since $\mathrm{V}[G_0]$ is a $\mathrm{Col}(\omega, {<}\kappa)$-generic extension of $\mathrm{V}[x]$, the homogeneity of $\mathrm{Col}(\omega, {<}\kappa)$ implies that B is an element of $\mathrm{V}[x]$. In this situation, standard arguments show that $\mathrm{V}[x]$ is a generic extension of V using a partial order of size less than κ in V and therefore the proof of the Levy–Solovay–Theorem (see, for example, [22, Proposition 10.15]) shows that the set $\{E \in \mathcal{P}(\kappa)^{\mathrm{V}[x]} \mid \exists D \in U \ D \subseteq E\}$ witnesses the measurability of κ in

$\mathrm{V}[x]$. By the above computations, this yields an $A \in U$ such that either $A \subseteq B$ or $A \cap B = \emptyset$ holds. But now, our constructions ensure that there is a club subset C of κ in $\mathrm{V}[G_0, \vec{H}]$ with $C \subseteq A$ and therefore A is not a bistationary subset of ω_1 in $\mathrm{V}[G_0, \vec{H}]$. □

5. Definable partitions of countable ordinals

In this short section, we show that many natural extensions of the axioms of ZFC cause ω_1 to have strong partition properties for simply definable colourings. These results are summarized in the following theorem.

Theorem 5.1. *Assume that one of the following assumptions holds:*

(i) There is a measurable cardinal above a Woodin cardinal.
(ii) There is a measurable cardinal and a precipitous ideal on ω_1.
(iii) Bounded Martin's Maximum BMM *holds and the nonstationary ideal on ω_1 is precipitous.*
(iv) Woodin's Axiom $(*)$ holds.

Then ω_1 has the $\mathbf{\Sigma}_1$-club property.

We prove the last implication stated above by providing an alternative way to establish the conclusion of Lemma 4.19.

Proposition 5.2. *Assume that* AD *holds in* $\mathrm{L}(\mathbb{R})$ *and G is $\mathbb{P}_{max}$-generic over* $\mathrm{L}(\mathbb{R})$. *Then ω_1 has the $\mathbf{\Sigma}_2$-club property in* $\mathrm{L}(\mathbb{R})[G]$.

Proof. Pick a subset A of ω_1 in $\mathrm{HOD}(\mathbb{R})^{\mathrm{L}(\mathbb{R})[G]}$. Then there is a real z such that A is contained in HOD_z. Since $\mathbb{P}_{max}$ is σ-closed and weakly homogeneous in $\mathrm{L}(\mathbb{R})$ (see [26, Lemma 2.10 and 3.4]), we know that A is an element of $\mathrm{L}(\mathbb{R})$. By a classical result of Solovay, our assumptions imply that, in $\mathrm{L}(\mathbb{R})$, the club filter on ω_1 is an ultrafilter (see [22, Theorem 28.2]). Therefore we can conclude that the set A either contains a club subset of ω_1 in $\mathrm{L}(\mathbb{R})$ or is disjoint from such a set. By Proposition 4.5, these computations yield the desired conclusion. □

The above statement directly yields the fourth implication of Theorem 5.1.

Proof of implication (iv) of Theorem 5.1. Assume that Woodin's Axiom $(*)$ holds, i.e. AD holds in $\mathrm{L}(\mathbb{R})$ and there is some G that is $\mathbb{P}_{max}$-generic over $\mathrm{L}(\mathbb{R})$ and satisfies $\mathcal{P}(\omega_1) \subseteq \mathrm{L}(\mathbb{R})[G]$. Fix $z \in \mathrm{H}(\omega_1)$ and a subset A of ω_1 with the property that the set $\{A\}$ is $\Sigma_1(\omega_1, z)$-definable. Since $\mathrm{H}(\omega_2) \subseteq \mathrm{L}(\mathbb{R})[G]$, the same formula defines $\{A\}$ in $\mathrm{L}(\mathbb{R})[G]$ and therefore Proposition 5.2 implies that A either contains a club or is disjoint from such a subset. □

We now derive the other implications of Theorem 5.1 from the results of [29].

Proof of the implications (i)–(iii) of Theorem 5.1. Note that the results of [31] and [35] show that (i) implies that $M_1^\#(A)$ exists for every subset A of ω_1 (see [32, p. 1738] and [36, p. 1660]). Moreover, [29, Theorem 2.1] shows that (ii) implies the same conclusion. Now, assume, towards a contradiction, that one of the first three assumptions listed in Theorem 5.1 holds and ω_1 does not have the $\mathbf{\Sigma}_1$-club property. Then Corollary 4.3 yields a bistationary subset A of ω_1 with the property that the set $\{A\}$ is $\Sigma_1(\omega_1, z)$-definable for some $z \in \mathrm{H}(\omega_1)$. By the above remarks, we can now apply [29, Lemma 4.11] to conclude that the set $\{A\}$ contains both an element of the club filter and the non-stationary ideal, a contradiction. □

We end this section by showing that the existence of a Woodin cardinal alone does not cause ω_1 to have the Σ_1-colouring property.

Corollary 5.3. *If M_1 exists, then, in M_1, the cardinal ω_1 does not have the Σ_1-colouring property.*

Proof. By [29, Theorem 5.2], there is a good $\Sigma_1(\omega_1)$-well-ordering of $\mathrm{H}(\omega_1)$ in M_1. Therefore, Proposition 3.13 yields the conclusion of the corollary. □

6. Definable partitions of $[\omega_2]^2$

In this section, we study simply definable colourings of finite subsets of the second uncountable cardinal and the influence of canonical extensions of ZFC on these partitions. A combination of Corollary 3.3 with Lemma 3.19 already shows that the statement that ω_2 has the $\mathbf{\Sigma}_n$-colouring property is independent from the axiom of ZFC together with large cardinal assumptions for all $0 < n < \omega$.

The following proposition shows that strong forcing axioms outright imply a failure of the $\mathbf{\Sigma}_1$-colouring property at ω_2.

Proposition 6.1. *Assume that the* Bounded Proper Forcing Axiom BPFA *holds. If $z \subseteq \omega_1$ with $\omega_1 = \omega_1^{\mathrm{L}[z]}$, then ω_2 does not have the $\Sigma_1(z)$-colouring property.*

Proof. By our assumption, there is a *ladder system* $\vec{C}$ (i.e. a sequence $\langle C_\alpha \mid \alpha \in \mathrm{Lim} \cap \omega_1 \rangle$ with the property that C_α is a cofinal subset of α of order-type ω for every countable limit ordinal α) with the property that the set $\{\vec{C}\}$ is $\Sigma_1(\omega_1, z)$-definable. By [7, Theorem 2], BPFA implies that $\mathrm{H}(\omega_2)$ has a good $\Sigma_1(\vec{C})$-well-ordering. Since this implies that $\mathrm{H}(\omega_2)$ has a good $\Sigma_1(\omega_2, z)$-well-ordering, we can apply Proposition 3.13 to conclude that ω_2 does not have the $\Sigma_1(z)$-colouring property. □

The failures of the $\mathbf{\Sigma}_1$-colouring property at ω_2 provided by Corollary 3.3 and Proposition 6.1 both make use of subsets of ω_1 that encode a great amount of information. Therefore, it is natural to consider partition properties for even smaller classes of definable partitions and ask if large cardinal assumptions or strong forcing axioms cause ω_2 to possess the Σ_1-colouring property. This question is answered negatively by the following result. It also answers one of the main questions left open by the results of [29].

Theorem 6.2. *Assume that* BPFA *holds. If there is a well-ordering $\lhd$ of the real numbers that is definable over the structure $\langle \mathrm{H}(\omega_2), \in \rangle$ by a formula with parameter $z \in \mathrm{H}(\omega_2)$, then the following statements hold:*

(i) The well-ordering $\lhd$ is $\Sigma_1(\omega_2, z)$-definable.
(ii) The cardinal ω_2 does not have the $\Sigma_1(z)$-colouring property.

Results of Asperó and Larson cited below show that the statement that the assumptions of Theorem 6.2 are satisfied for the empty set as a parameter are compatible with both large cardinal assumptions and strong forcing axioms. In particular, the existence of a $\Sigma_1(\omega_2)$-definable well-ordering of the reals is compatible with these assumptions. This answers [29, Question 7.5].

Remark 6.3. (i) If κ is supercompact, then [3, Theorem 5.2] shows that there is a semi-proper partial order $\mathbb{P} \subseteq \mathrm{H}(\kappa)$ with the property that whenever G is $\mathbb{P}$-generic over V, then PFA^{++} (see [3, Definition 5.1]) holds in $\mathrm{V}[G]$ and there is a well-ordering of $\mathrm{H}(\omega_2)^{\mathrm{V}[G]}$ that is definable over $\langle \mathrm{H}(\omega_2), \in \rangle$ by a formula without parameters.

(ii) If κ is a supercompact limit of supercompact cardinals, then [25, Theorem 7.1] yields a semi-proper partial order $\mathbb{P}$ with the property that whenever G is $\mathbb{P}$-generic over V, then $\mathsf{MM}^{+\omega}$ (see [25, Section 1]) holds in $\mathrm{V}[G]$ and there is a well-ordering of $\mathrm{H}(\omega_2)^{\mathrm{V}[G]}$ that is definable over $\langle \mathrm{H}(\omega_2), \in \rangle$ by a formula without parameters.

Note that it is not known whether the stronger forcing axiom MM^{++} (see [25, Section 1]) implies that no well-ordering of the reals is definable over $\langle \mathrm{H}(\omega_2), \in \rangle$ by a formula without parameters. This question is motivated by the open question whether MM^{++} implies Woodin's axiom $(*)$ (see [25] for a discussion).

Theorem 6.2 is a direct consequence of the following lemma. The lemma itself follows directly from arguments due to Caicedo and Veličković that are used to prove [7, Theorem 1] stating that, if BPFA holds and M is an inner model of $\mathrm{ZFC} + \mathsf{BPFA}$ with $\omega_2 = \omega_2^M$, then $\mathcal{P}(\omega_1) \subseteq M$.

Lemma 6.4. *If* BPFA *holds, then the set $\{\mathrm{H}(\omega_2)\}$ is $\Sigma_1(\omega_2)$-definable.*

Proof. The proof of [7, Theorem 1] shows that there is a finite fragment F of the theory $\mathrm{ZFC}+\mathsf{BPFA}$ with the property that $\mathrm{ZFC} + \mathsf{BPFA}$ proves that every transitive model M of F + "*ω_2 exists*" with $\omega_2 = \omega_2^M$ contains all subsets of ω_1. This shows that BPFA implies that $\mathrm{H}(\omega_2)$ is the unique set

B with the property that there is a transitive model M of F + "ω_2 *exists*" with $\omega_2 = \omega_2^M$ and $B = \mathrm{H}(\omega_2)^M$. In particular, BPFA implies that the set $\{\mathrm{H}(\omega_2)\}$ is $\Sigma_1(\omega_2)$-definable. □

Note that both large cardinal assumptions and strong forcing axioms imply that the set $\{\mathrm{H}(\omega_1)\}$ is not $\Sigma_1(\omega_1)$-definable. This follows directly from the fact that there are non-projective sets of reals that can be defined over the structure $\langle \mathrm{H}(\omega_1), \in \rangle$ and [29, Lemma 3.3] showing that these extensions of ZFC imply that every $\Sigma_1(\omega_1)$-definable set of reals is Σ^1_3-definable.

Proof of Theorem 6.2. Assume that BPFA holds and that there is a well-ordering $\lhd$ of ${}^\omega 2$ that is definable over the structure $\langle \mathrm{H}(\omega_2), \in \rangle$ by a formula with parameter $z \in \mathrm{H}(\omega_2)$. Since BPFA implies that CH fails, we know that $\langle {}^\omega 2, \lhd \rangle$ has order-type at least ω_2. Let $\iota : \omega_2 \longrightarrow {}^\omega 2$ denote the canonical enumeration of the first ω_2-many elements of $\langle {}^\omega 2, \lhd \rangle$. An application of the recursion theorem now shows that the function ι is also definable over the structure $\langle \mathrm{H}(\omega_2), \in \rangle$ by a formula with parameter z. In this situation, Lemma 6.4 shows that ι is $\Sigma_1(\omega_2, z)$-definable and we can apply Proposition 3.1 to conclude that ω_2 does not have the $\Sigma_1(z)$-colouring property. □

7. Limit cardinals

We now consider the question whether the above partition relations for definable colourings can hold at regular limit cardinals. In these considerations, we focus on inaccessible cardinals that are not weakly compact. We start by showing that there are many such cardinals with the $\mathbf{\Sigma}_1$-colouring property below weakly compact cardinals and there are many such cardinals with the $\mathbf{\Sigma}_1$-club property below weakly compact cardinals with this property.

Theorem 7.1. *Let κ be a weakly compact cardinal, let A be a subset of κ and let $\Psi(v)$ be a Π^1_1-formula with $\mathrm{V}_\kappa \models \Psi(A)$.*

- *(i) The statement $\Psi(A)$ reflects to an inaccessible cardinal less than κ with the $\mathbf{\Sigma}_1$-colouring property.*
- *(ii) If the cardinal κ has the $\mathbf{\Sigma}_1$-club property, then the statement $\Psi(A)$ reflects to an inaccessible cardinal less than κ with the $\mathbf{\Sigma}_1$-club property.*

Proof. Pick an elementary submodel M of $\mathrm{H}(\kappa^+)$ of cardinality κ with $\kappa, A \in M$ and ${}^{<\kappa}M \subseteq M$. The results of [17] now yield a transitive set N and an elementary embedding $j : M \longrightarrow N$ with critical point κ such that both the model M and the embedding j are elements of N. Then κ is inaccessible in N, $\mathrm{H}(\kappa) \subseteq M \subseteq N$, $A = j(A) \cap \kappa \in N$ and Π^1_1-downwards absoluteness implies that $(\mathrm{V}_\kappa \models \Psi(A))^N$.

Claim. *Σ_1-formulas with parameters in M are absolute between M and N.*

Proof of the Claim. Since $M \subseteq N$, it suffices to show that Σ_1-formulas with parameters from M are downwards-absolute from N to M. Fix $z_0, \ldots, z_{n-1} \in M$ and a Σ_1-formula $\varphi(v_0, \ldots, v_{n-1})$ such that $\varphi(z_0, \ldots, z_{n-1})$ holds in N. Then Σ_1-upwards absoluteness implies that $\varphi(z_0, \ldots, z_{n-1})$ holds in V and the Σ_1-Reflection Principle implies that this statement holds in $\mathrm{H}(\kappa^+)$. By the definition of M, we can conclude that $\varphi(z_0, \ldots, z_{n-1})$ holds in M. □

<u>*(i)*</u>: First, let $c : [\kappa]^2 \longrightarrow 2$ be a function in N that is $\Sigma_1(\kappa, z)$-definable in N for some $z \in \mathrm{H}(\kappa)^N$. Then $z \in M$ and the above claim implies that the same Σ_1-formula defines c in M. Since M is an elementary submodel of $\mathrm{H}(\kappa^+)$ and κ is weakly compact, M contains a c-homogeneous subset of κ that is unbounded in κ and this subset is also an element of N. These computations show that in the structure $\langle j(\mathrm{V}_\kappa), \in, A \rangle$, there is an inaccessible cardinal ν with the $\mathbf{\Sigma}_1$-colouring property with $\mathrm{V}_\nu \models \Psi(j(A) \cap \nu)$. With the help of a universal Σ_1-formula, this statement can be expressed by a first-order statement in N that only uses the parameters $j(\kappa)$ and $j(A)$. By elementarity, there is an inaccessible cardinal $\mu < \kappa$ with the $\mathbf{\Sigma}_1$-colouring property such that $\mathrm{V}_\mu \models \Psi(A \cap \mu)$ holds.

<u>*(ii)*</u>: Now, assume that κ has the $\mathbf{\Sigma}_1$-club property and let $c : [\kappa]^{<\omega} \longrightarrow \gamma$ be a function in N with $\gamma < \kappa$ that is $\Sigma_1(\kappa, z)$-definable in N for some $z \in \mathrm{H}(\kappa)^N$. As above, we can conclude that c is an element of M and $\Sigma_1(\kappa, z)$-definable in that model. Since M is an elementary submodel of $\mathrm{H}(\kappa^+)$ and κ has the $\mathbf{\Sigma}_1$-club property, elementarity implies that M contains a c-homogeneous set that is closed and unbounded in κ. As in (i), these computations show that in $\langle j(\mathrm{V}_\kappa), \in, A \rangle$, there

is an inaccessible cardinal ν with the $\mathbf{\Sigma}_1$-club property and the property that $V_\nu \models \Psi(j(A) \cap \nu)$ holds. □

Next, we show that certain regular limits of cardinals with large cardinal properties stronger than weak compactness are also examples of inaccessible cardinals with the $\mathbf{\Sigma}_1$-club property. The following lemma also shows that successors of singular cardinals ν can possess the $\Sigma_1(z)$-club property for all parameters z in $H(\nu)$.

Lemma 7.2. *Let ν be a strong limit cardinal, let $\kappa \in \{\nu, \nu^+\}$ be a regular cardinal and let $\delta < \nu$ be a measurable cardinal with $\operatorname{cof}(\nu) \neq \delta$. If A is a bistationary subset of κ and $z \in H(\delta)$, then the set $\{A\}$ is not $\Sigma_1(\kappa, z)$-definable.*

Proof. Fix a normal ultrafilter U on δ and let

$$\langle \langle M_\alpha \mid \alpha \in \mathrm{Ord}\rangle, \langle j_{\alpha,\beta} : M_\alpha \longrightarrow M_\beta \mid \alpha \leq \beta \in \mathrm{Ord}\rangle\rangle$$

denote the system of ultrapowers and elementary embeddings induced by $\langle V, \in, U\rangle$. Given $\alpha < \kappa$, set $\delta_\alpha = j_{0,\alpha}(\delta)$ and $\nu_\alpha = j_{0,\alpha}(\nu)$.

Claim. *If $\alpha < \kappa$, then $j_{0,\alpha}(\kappa) = \kappa$.*

Proof of the Claim. First, assume that $\kappa = \nu$. Since κ is inaccessible in this case, [22, Corollary 19.7(c)] directly yields the statement of the claim.

Now, assume that $\kappa = \nu^+$. Note that, in order to prove the statement of the claim, it suffices to show that $\nu_\alpha < \kappa$ holds for all $\alpha < \kappa$, because we then have $\kappa \geq (\nu_\alpha^+)^{M_\alpha} = j_{0,\alpha}(\kappa) \geq \kappa$ for all $\alpha < \kappa$. If $\operatorname{cof}(\nu) > \delta$, then our assumptions imply that $\nu^\delta = \nu$ and therefore [22, Corollary 19.7(a)] shows that $\nu_\alpha < (\nu^\delta \cdot |\alpha|)^+ = \kappa$ holds for all $\alpha < \kappa$. In the other case, assume that $\operatorname{cof}(\nu) < \delta$. This assumption implies that $\operatorname{cof}(\nu_\alpha)^{M_\alpha} = \operatorname{cof}(\nu) < \delta \leq \delta_\alpha$ for all $\alpha < \kappa$. We show $\nu_\alpha < \kappa$ by induction on $\alpha < \kappa$. Assume that $\alpha = \bar{\alpha} + 1$. Then elementarity implies that $\nu_{\bar{\alpha}}$ is a strong limit cardinal greater than δ_α in $M_{\bar{\alpha}}$. Since $j_{\bar{\alpha},\alpha} : M_{\bar{\alpha}} \longrightarrow M_\alpha$ is equal to the ultrapower embedding induced by $j_{0,\bar{\alpha}}(U)$ in $M_{\bar{\alpha}}$, an application of [22, Corollary 19.7(a)] in $M_{\bar{\alpha}}$ now shows that $j_{\bar{\alpha},\alpha}(\gamma) < \nu_{\bar{\alpha}}$ holds for all $\gamma < \nu_{\bar{\alpha}}$. By the above remarks, there is a function $c : \operatorname{cof}(\nu) \longrightarrow \nu_{\bar{\alpha}}$ in $M_{\bar{\alpha}}$ that is cofinal in $\nu_{\bar{\alpha}}$. But then elementarity implies that $j_{\bar{\alpha},\alpha}(c) : \operatorname{cof}(\nu) \longrightarrow \nu_{\bar{\alpha}}$ is cofinal in ν_α and therefore $\nu_\alpha = \nu_{\bar{\alpha}} < \kappa$. Now, assume that $\alpha \in \mathrm{Lim} \cap \kappa$. Then our induction hypothesis implies that $\mu = \sup_{\bar{\alpha} < \alpha} \nu_{\bar{\alpha}} < \kappa$. If $\gamma < \nu_\alpha$, then there is an $\bar{\alpha} < \alpha$ and $\bar{\gamma} < \nu_{\bar{\alpha}}$ with $j_{\bar{\alpha},\alpha}(\bar{\gamma}) = \gamma$. This shows that $|\nu_\alpha| \leq \mu \cdot |\alpha| < \kappa$. □

Fix a Σ_1-formula $\varphi(v_0, v_1, v_2)$ and $z \in H(\delta)$ with the property that there is a unique subset A of κ with the property that $\varphi(A, \kappa, z)$ holds.

Claim. *If $\alpha < \kappa$, then $j_{0,\alpha}(A) = A$.*

Proof. By the above claim and elementarity, we know that $\varphi(j_{0,\alpha}(A), \kappa, z)$ holds in M_α. But then Σ_1-upwards absoluteness implies that $\varphi(j_{0,\alpha}(A), \kappa, z)$ holds in V and the uniqueness of A yields the statement of the claim. □

Let C be the club subset $\{\delta_\alpha \mid \alpha < \kappa\}$ of κ. If $\delta \in A$, then $\delta_\alpha = j_{0,\alpha}(\delta) \in j_{0,\alpha}(A) = A$ for all $\alpha < \kappa$ and hence $C \subseteq A$. In the other case, if $\delta \notin A$, then the same argument shows that $A \cap C = \emptyset$. □

Corollary 7.3. *Regular limits of measurable cardinals have the $\mathbf{\Sigma}_1$-club property.* □

Next, we use ω_1-iterable cardinals to provide more examples of non-weakly compact cardinals with the $\mathbf{\Sigma}_1$-club property. Again, the results of [29, Section 6] already provide this statement for $\Sigma_1(\kappa)$-definable subsets of κ.

Theorem 7.4. *Stationary limits of ω_1-iterable cardinals have the $\mathbf{\Sigma}_1$-club property.*

Proof. Pick such a cardinal κ, a Σ_1-formula $\varphi(v_0, v_1, v_2)$ and $z \in H(\kappa)$ with the property that there is a unique subset A of κ such that $\varphi(A, \kappa, z)$ holds. By constructing a continuous elementary chain of elementary submodels of $H(\kappa^+)$ of cardinality less than κ, we find an ω_1-iterable cardinal $\delta < \kappa$ and an elementary substructure X of $H(\kappa^+)$ of cardinality δ such that $\operatorname{tc}(\{z\}) \cup \{\kappa, A\} \subseteq X$ and $\delta = \kappa \cap X$. Let $\pi : X \longrightarrow M$ denote the corresponding transitive collapse. Since δ is ω_1-iterable, there is a transitive ZFC$^-$-model N and a weakly amenable N-ultrafilter U on δ such that $\delta, M \in N$ and $\langle N, \in, U\rangle$ is ω_1-iterable. Then $z, \pi(A) \in N$ and a combination of elementarity and Σ_1-upwards absoluteness implies that $N \models \varphi(\pi(A), \delta, z)$. Let

$$\langle\langle N_\alpha \mid \alpha \in \mathrm{Ord}\rangle, \langle j_{\alpha,\beta} : N_\alpha \longrightarrow N_\beta \mid \alpha \leq \beta \in \mathrm{Ord}\rangle\rangle$$

denote the system of ultrapowers and elementary embeddings induced by $\langle N, \in, U\rangle$. Then we have $j_{0,\kappa}(\delta) = \kappa$ and elementarity implies $N_\kappa \models \varphi((j_{0,\kappa} \circ \pi)(A), \kappa, z)$. But then $\varphi((j_{0,\kappa} \circ \pi)(A), \kappa, z)$ holds in V and hence $A = (j_{0,\kappa} \circ \pi)(A)$. This shows that $z, A \in N_\kappa \models \varphi(A, \kappa, z)$ and, since there is a weakly amenable N_κ-ultrafilter U_κ on κ such that $\langle N_\kappa, \in, U_\kappa\rangle$ is ω_1-iterable, we can apply Lemma 4.11 to conclude that A either contains a club subset of κ or is disjoint from such a subset. □

8. Successors of singular cardinals

In this short section, we study the extend of definable partition properties at successors of singular cardinals. By combining Corollary 3.10 with the following result of Cummings, S. Friedman, Magidor, Rinot and Sinapova from [9], it can be shown that the consistency of the existence of a singular strong limit cardinal of countable cofinality whose successor has the $\mathbf{\Sigma}_2$-colouring property can be established from strong large cardinal assumptions.

Theorem 8.1 ([9]). *Assume that ν is a singular limit of supercompact cardinals with $\mathrm{cof}(\nu) = \omega$ and $\kappa > \nu$ is supercompact. Then there is a generic extension $\mathrm{V}[G]$ of the ground model V such that the following statements hold:*

(i) The models V and $\mathrm{V}[G]$ have the same bounded subsets of ν.
(ii) Every infinite cardinal μ with $\mu \leq \nu$ or $\mu \geq \kappa$ is preserved in $\mathrm{V}[G]$.
(iii) $\kappa = (\nu^+)^{\mathrm{V}[G]}$.
(iv) If $z \in \mathcal{P}(\nu)^{\mathrm{V}[G]}$, then κ is supercompact in $\mathrm{HOD}_z^{\mathrm{V}[G]}$.

In contrast, results of Shelah show that the $\mathbf{\Sigma}_2$-colouring property always fails at successors of singular strong limit cardinals of uncountable cofinality.

Proposition 8.2. *Let ν be a singular strong limit cardinal of uncountable cofinality. If $z \subseteq \nu$ with $\mathrm{H}(\nu) \subseteq \mathrm{L}[z]$, then no regular cardinal less than or equal to 2^ν has the $\Sigma_2(\nu, z)$-colouring property.*

Proof. A result of Shelah from [34] (see also [9, Section 2]) shows that $\mathcal{P}(\nu) \subseteq \mathrm{HOD}_z$. Pick a regular cardinal $\kappa \leq 2^\nu$. Then $\kappa \leq (2^\nu)^{\mathrm{HOD}_z}$ and therefore HOD_z contains a subset A of κ with $(2^\nu)^{\mathrm{L}[A]} \geq \kappa$. By Proposition 3.9, there is such a subset A of κ with the property that the set $\{A\}$ is $\Sigma_2(\nu, z)$-definable. But then Corollary 3.2 implies that κ does not have the $\Sigma_2(\nu, z)$-colouring property. □

We close this section by showing that the validity of the Σ_1-colouring property at the successor of a singular cardinal has much larger consistency strength than the corresponding statement for successors of regular cardinals.

Lemma 8.3. *If there is a singular cardinal ν such that the cardinal ν^+ has the Σ_1-colouring property, then there is an inner model with a measurable cardinal.*

Proof. Assume that the above conclusion fails. Set $\kappa = \nu^+$ and let K^{DJ} denote the *Dodd–Jensen core model.*

Claim. *There is a subset A of κ with the property that the set $\{A\}$ is $\Sigma_1(\kappa)$-definable and*

$$A \;=\; \{\prec\gamma, \alpha, s_\gamma(\alpha)\succ \mid \gamma < \kappa,\ \alpha < \lambda\}. \tag{4}$$

for some ordinal $\lambda < \kappa$ and some sequence $\langle s_\gamma : \lambda \longrightarrow \gamma \mid \gamma < \kappa\rangle$ of surjections.

Proof of the Claim. By our assumption, the *Covering Theorem* for K (see [10, Theorem 5.17]) implies that $\kappa = (\nu^+)^{\mathrm{K}}$ and this shows that there is a sequence $\langle s_\gamma : \nu \longrightarrow \gamma \mid \gamma < \kappa\rangle$ of surjections that is an element of K.

First, assume that there are no *iterable premice* (see [12, Section 1]). Then results of Dodd and Jensen (see [12, p. 238]) show that $\mathrm{K}^{DJ} = \mathrm{L}$. Let A denote the $<_{\mathrm{L}}$-least subset of κ with the property that there is a $\lambda < \kappa$ and a sequence $\langle s_\gamma : \lambda \longrightarrow \gamma \mid \gamma < \kappa\rangle$ of surjections with (4). Then it is easy to see that the set $\{A\}$ is $\Sigma_1(\kappa)$-definable.

Next, if $\mathrm{K} \neq \mathrm{L}$, then the results of Dodd and Jensen mentioned above show that K^{DJ} is equal to the union of the *lower parts* $lp(M)$ of all iterable premice M. Let $\mathcal{A}$ denote the class of all subsets A of κ with the property that there is an iterable premouse $M = \mathrm{L}_\alpha[F]$ such that $A \in lp(M)$ and A is the $<_{\mathrm{L}[F]}$-minimal subset of κ in M with the property that there is an ordinal $\lambda < \kappa$ and a sequence $\langle s_\gamma : \lambda \longrightarrow \gamma \mid \gamma < \kappa \rangle$ in $lp(M)$ with (4). Then our assumptions imply that $\mathcal{A}$ is non-empty and, since the proof of [30, Lemma 2.3] shows that the class of all iterable premice is $\Sigma_1(\kappa)$-definable, we know that $\mathcal{A}$ is definable in the same way. But then a comparison argument (see [12, Lemma 1.12(7)]) shows that $\mathcal{A}$ consists of a single subset of κ. □

Let A be the subset of κ given by the above claim. Then κ is not a limit cardinal in $\mathrm{L}[A]$ and therefore Corollary 3.2 shows that κ does not have the Σ_1-colouring property. □

9. Definable Homeomorphisms

We present the results that were the initial motivation for the work presented in this paper. Remember that, given an uncountable regular cardinal κ, the *generalized Baire space* of κ consists of the set ${}^\kappa\kappa$ of all functions from κ to κ equipped with the topology whose basic open sets consist of all extensions of functions of the form $s : \alpha \longrightarrow \kappa$ with $\alpha < \kappa$. The *generalized Cantor space* of κ is the subspace of ${}^\kappa\kappa$ given by the set ${}^\kappa 2$ of all binary functions. A classical result of Hung and Negrepontis from [18] then shows that an uncountable regular cardinal κ is weakly compact if and only if the generalized Baire space ${}^\kappa\kappa$ is not homeomorphic to the generalized Cantor space ${}^\kappa 2$ of κ . Motivated by this characterization, Andretta and Motto Ros recently showed that the theory $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}$ proves that the generalized Baire space of ω_1 is not homeomorphic to the generalized Cantor space of ω_1 (see [2, Section 6.1]). By combining this result with work of Woodin on the Π_2-maximality of the $\mathbb{P}_{max}$-extension of $\mathrm{L}(\mathbb{R})$ (see [26, Lemma 2.10 and Theorem 7.3]), one can directly conclude that the existence of infinitely many Woodin cardinals with a measurable cardinal above them all implies that no homeomorphism between the generalized Baire space of ω_1 and the generalized Cantor space of ω_1 is definable by a Σ_1-formula that only uses the cardinal ω_1 and elements of $\mathrm{H}(\omega_1)$ as parameters, because Woodin's results show that the same formula defines a homeomorphism of these spaces in $\mathrm{L}(\mathbb{R})$. The question whether the above conclusion can be derived from weaker large cardinal assumptions was the initial motivation for the work presented in this paper. In combination with Theorem 5.1, the following lemma answers this question affirmatively.

Lemma 9.1. *If κ is an uncountable regular cardinal and z is a set with the property that κ has the $\Sigma_n(z)$-colouring property, then no homeomorphism between ${}^\kappa\kappa$ and ${}^\kappa 2$ is $\Sigma_n(\kappa, z)$-definable.*

Proof. Assume, towards a contradiction, that there is a Σ_n-formula $\varphi(v_0, \ldots, v_3)$ such that there is a homeomorphism $h : {}^\kappa\kappa \longrightarrow {}^\kappa 2$ with the property that for every $x \in {}^\kappa\kappa$, the image $h(x)$ is the unique set y such that $\varphi(\kappa, x, y, z)$ holds. Given $\alpha < \kappa$, let U_α denote the open subset of ${}^\kappa\kappa$ consisting of all functions $x \in {}^\kappa\kappa$ with $x(0) = \alpha$ and let x_α denote the unique element of U_α with $x_\alpha(\beta) = 0$ for all $0 < \beta < \kappa$. If $\alpha < \kappa$, then $h[U_\alpha]$ is an open subset of ${}^\kappa 2$ that contains $h(x_\alpha)$ and hence there is a $\gamma < \kappa$ with the property that $h[U_\alpha]$ contains all extensions of $h(x_\alpha) \restriction \gamma$ in ${}^\kappa 2$. But this shows that for all $\alpha < \kappa$, there is a unique minimal $\gamma_\alpha < \kappa$ with the property that

$$\alpha = \beta \iff h(x_\alpha) \restriction \gamma_\alpha \subseteq h(x_\beta)$$

holds for all $\beta < \kappa$. Then the resulting map

$$\iota : \kappa \longrightarrow {}^{<\kappa}2; \ \alpha \longmapsto h(x_\alpha) \restriction \gamma_\alpha$$

is an injection and our assumptions imply that it is $\Sigma_n(\kappa, z)$-definable. Since all elements of $\mathrm{ran}(\iota)$ are pairwise incompatible, we can use Lemma 3.5 to conclude that κ does not have the $\Sigma_n(z)$-colouring property. □

In combination with Theorem 5.1, the above lemma shows that the existence of a measurable cardinal above a Woodin cardinal implies that no homeomorphism between the generalized Baire space of ω_1 and the generalized Cantor space of ω_1 is definable by a Σ_1-formula with parameters in $\mathrm{H}(\omega_1) \cup \{\omega_1\}$. The next lemma shows that the implication proven above can be reversed in certain models of set theory.

Lemma 9.2. *Let κ be an uncountable regular cardinal and let z be a set with the property that there is a good $\Sigma_n(\kappa,z)$-well-ordering of $\mathrm{H}(\kappa)$ of length κ. If κ does not have the $\Sigma_n(z)$-colouring property, then there is a $\Sigma_n(\kappa,z)$-definable homeomorphism between ${}^\kappa\kappa$ and ${}^\kappa 2$.*

Proof. By Lemma 3.5, our assumptions yield a $\Sigma_n(\kappa,z)$-definable injection $\iota:\kappa\longrightarrow{}^{<\kappa}2$ with the property that for all $x\in{}^\kappa 2$, there is an $\alpha<\kappa$ such that there is no $\beta<\kappa$ with $x\restriction\alpha\subseteq\iota(\beta)$. Set

$$T \ = \ \{t\in{}^{<\kappa}2 \mid \exists\alpha<\kappa \ t\subseteq\iota(\alpha)\}$$

and define ∂T to be the set of all $t\in{}^{<\kappa}2\setminus T$ with the property that $t\restriction\alpha\in T$ holds for all $\alpha<\mathrm{lh}(t)$. Then our assumptions imply that for every $x\in{}^\kappa 2$, there is a unique $t_x\in\partial T$ with $t_x\subseteq x$.

Claim. *The set ∂T has cardinality κ.*

Proof of the Claim. Assume that ∂T has cardinality less than κ. Since ι is an injection, there is an $\alpha<\kappa$ such that there is no $t\in\partial T$ with $\iota(\alpha)\subseteq\partial T$. Pick $x\in{}^\kappa 2$ with $\iota(\alpha)\subseteq x$. Then $t_x\nsubseteq\iota(\alpha)$ and therefore $\iota(\alpha)\subseteq t_x\in\partial T$, a contradiction. □

Note that our assumption on ι imply that the set ∂T is $\Sigma_n(\kappa,z)$-definable. By the above claim, the existence of a good $\Sigma_n(\kappa,z)$-well-ordering of $\mathrm{H}(\kappa)$ of length κ then yields the existence of a $\Sigma_n(\kappa,z)$-definable bijection $b:\kappa\longrightarrow\partial T$. Given $y\in{}^\kappa\kappa$, we can then find a unique element $h(y)$ of ${}^\kappa 2$ with the property that there is a continuous increasing sequence $\langle\beta_\alpha\mid\alpha<\kappa\rangle$ of ordinals less than κ with $\beta_0=0$, $\beta_{\alpha+1}=\beta_\alpha+\mathrm{lh}(b(y(\alpha)))$ and $h(y)(\beta_\alpha+\beta)=b(y(\alpha))(\beta)$ for all $\alpha<\kappa$ and $\beta<\mathrm{lh}(b(y(\alpha)))$.

Claim. *The map $h:{}^\kappa\kappa\longrightarrow{}^\kappa 2$ is a homeomorphism.*

Proof of the Claim. Given $x\in{}^\kappa 2$, there is a unique element $g(x)$ of ${}^\kappa\kappa$ with the property that there exists a sequence $\langle x_\alpha\mid\alpha<\kappa\rangle$ of elements of ${}^\kappa 2$ and a continuous increasing sequence $\langle\beta_\alpha\mid\alpha<\kappa\rangle$ of ordinals less than κ such that the following statements hold:

(i) $x_0=x$ and $\beta_0=0$.
(ii) $\beta_{\alpha+1}=\beta_\alpha+\mathrm{lh}(t_{x_\alpha})$ and $b(g(x)(\alpha))=t_{x_\alpha}$ for all $\alpha<\kappa$.
(iii) $x_\alpha(\beta)=x(\beta_\alpha+\beta)$ for all $\alpha,\beta<\kappa$.

Then it is easy to check that $g:{}^\kappa 2\longrightarrow{}^\kappa\kappa$ and $h:{}^\kappa\kappa\longrightarrow{}^\kappa 2$ are continuous functions with $g\circ h=\mathrm{id}_{{}^\kappa\kappa}$ and $h\circ g=\mathrm{id}_{{}^\kappa 2}$. □

Finally, the above construction ensure that h is $\Sigma_n(\kappa,z)$-definable. □

The above results show that the assumption $\mathrm{V}=\mathrm{L}$ or, more generally, $\mathrm{V}=\mathrm{K}^{DJ}$ implies that an uncountable regular cardinal κ has the $\Sigma_1(z)$-colouring property if and only if there is no $\Sigma_1(\kappa,z)$-definable homeomorphism between ${}^\kappa\kappa$ and ${}^\kappa 2$. Moreover, a combination of the above lemma with [7, Theorem 2] (as in the proof of Proposition 6.1) shows that BPFA implies that for every $z\subseteq\omega_1$ with $\omega_1=\omega_1^{\mathrm{L}[z]}$, there is a $\Sigma_1(\omega_2,z)$-definable homeomorphism between the generalized Baire space of ω_2 and the generalized Cantor space of ω_2. Finally, by combining the construction from the proof of Lemma 9.2 with arguments from the proof of [29, Theorem 5.2], it is possible to show that the existence of a single Woodin cardinal is compatible with the existence of a $\Sigma_1(\omega_1)$-definable homeomorphism between the generalized Baire space of ω_1 and the generalized Cantor space of ω_1.

10. Open questions

We close this paper by presenting some questions left open by the above results. The results of Section 6 show that both PFA^{++} and $\mathsf{MM}^{+\omega}$ do not imply that ω_2 has the Σ_1-colouring property. Since these results rely on the fact that these forcing axioms are compatible with the existence of a well-ordering of the reals that is definable over $\langle\mathrm{H}(\omega_2),\in\rangle$ and it is commonly expected that stronger forcing axioms imply the non-existence of such a well-ordering, it is natural to conjecture that such axioms also rule out the existence of simply definable partitions without large homogeneous sets.

Question 10.1. Do very strong forcing axioms, like MM^{++} or MM^{+++} (defined by Viale in [38]), imply that ω_2 has the Σ_1-colouring property?

In contrast, the above results also leave open the possibility that *Martin's Maximum* is not only compatible with a failure of the Σ_1-colouring property at ω_2, but outright implies such a failure.

Question 10.2. Is MM consistent with the statement that ω_2 has the Σ_1-colouring property?

While the results of Section 7 provide many examples of inaccessible non-weakly compact cardinals with the $\mathbf{\Sigma}_1$-colouring property, they leave open the question whether small inaccessible cardinals can possess this property.

Question 10.3. Is it consistent that the first inaccessible cardinal has the $\mathbf{\Sigma}_1$-colouring property?

Somewhat surprisingly, the above results show that successors of singular strong limit cardinals of uncountable cofinality never have the $\mathbf{\Sigma}_2$-colouring property. This leaves open the following question.

Question 10.4. Is it consistent that the successor of a singular cardinal of uncountable cofinality has the $\mathbf{\Sigma}_1$-colouring property?

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Mathematisches Institut, Rheinische Friedrich-Wilhelms-Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany

E-mail address: `pluecke@math.uni-bonn.de`