Σ^1_1-DEFINABILITY AT UNCOUNTABLE REGULAR CARDINALS

PHILIPP LÜCKE

ABSTRACT. Let κ be an infinite cardinal. A subset of $({}^\kappa\kappa)^n$ is a $\mathbf{\Sigma}^1_1$-subset if it is the projection $p[T]$ of all cofinal branches through a subtree T of $({}^{<\kappa}\kappa)^{n+1}$ of height κ. We define $\mathbf{\Sigma}^1_k$-, $\mathbf{\Pi}^1_k$- and $\mathbf{\Delta}^1_k$-subsets of $({}^\kappa\kappa)^n$ as usual.

Given an uncountable regular cardinal κ with $\kappa = \kappa^{<\kappa}$ and an arbitrary subset A of ${}^\kappa\kappa$, we show that there is a $<\kappa$-closed forcing $\mathbb{P}$ that satisfies the κ^+-chain condition and forces A to be a $\mathbf{\Delta}^1_1$-subset of ${}^\kappa\kappa$ in every $\mathbb{P}$-generic extension of V. We give some applications of this result and the methods used in its proof.

(i) Given any set x, we produce a partial order with the above properties that forces x to be an element of $\mathrm{L}(\mathcal{P}(\kappa))$.

(ii) We show that there is a partial order with the above properties forcing the existence of a well-ordering of ${}^\kappa\kappa$ whose graph is a $\mathbf{\Delta}^1_2$-subset of ${}^\kappa\kappa \times {}^\kappa\kappa$.

(iii) We provide a short proof of a result due to Mekler and Väänänen by using the above forcing to add a tree T of cardinality and height κ such that T has no cofinal branches and every tree from the ground model of cardinality and height κ without a cofinal branch quasi-order embeds into T.

(iv) We will show that generic absoluteness for $\mathbf{\Sigma}^1_3({}^\kappa\kappa)$-formulae (i.e., formulae with parameters which define $\mathbf{\Sigma}^1_3$-subsets of ${}^\kappa\kappa$) under $<\kappa$-closed forcings that satisfy the κ^+-chain condition is inconsistent.

In another direction, we use methods from the proofs of the above results to show that $\mathbf{\Sigma}^1_1$- and $\mathbf{\Delta}^1_1$-subsets have some useful structural properties in certain ZFC-models.

1. INTRODUCTION

Let κ be an uncountable regular cardinal. The set of all functions $f : \kappa \longrightarrow \kappa$ is called *generalized Baire space for* κ. We study the definable subsets of this space and their structural properties. A systematic study of this space was initiated by Alan Mekler and Jouko Väänänen (see [17] and [22]) and was extended by Philipp Schlicht, Samuel Coskey and others. A discussion of some of these results can be found in [8, Chapter IV]. In addition, a number of publications revealed deep connections to infinitary logic and model theory (see, for example, [8], [12], [18] [20] and [21]).

Before we start a more detailed introduction to this paper, we give a brief review of our notation.

- Given a nonempty set X and $A \subseteq X^{n+1}$, we define
$$\exists^x A = \{\langle x_0, \ldots, x_{n-1}\rangle \in X^n \mid (\exists x_n)\ \langle x_0, \ldots, x_n\rangle \in A\}.$$
- Given a cardinal κ and a class C, we let $[C]^{<\kappa}$ denote the class of all sets x of cardinality less than κ with $x \subseteq C$.

The author's research was supported by the Deutsche Forschungsgemeinschaft (Grant SCHI 484/4-1 and SFB 878).

- For all $\lambda \in \mathrm{On}$, we let ${}^{\lambda}X$ denote the set of all functions f with $\mathrm{dom}(f) = \lambda$ and $\mathrm{ran}(f) \subseteq X$. We set ${}^{<\lambda}X = \bigcup_{\alpha<\lambda} {}^{\alpha}X$. If κ is a cardinal, then we let $\kappa^{<\lambda}$ denote the cardinality of ${}^{<\lambda}\kappa$.
- Given a nonempty set X, we call a set T a *tree on* X^n if there is a $\gamma \in \mathrm{On}$ such that $T \subseteq ({}^{<\gamma}X)^n$ and the following statements hold.
  (1) If $\langle s_0, \ldots, s_{n-1}\rangle \in T$, then $\mathrm{lh}(s_0) = \cdots = \mathrm{lh}(s_{n-1})$.
  (2) If $\langle s_0, \ldots, s_{n-1}\rangle \in T$ and $\alpha < \mathrm{lh}(s_0)$, then $\langle s_0 \restriction \alpha, \ldots, s_{n-1} \restriction \alpha\rangle \in T$.
  In the above situation, we call T a *subtree of* $({}^{<\gamma}X)^n$. Given a tuple $t = \langle t_0, \ldots, t_{n-1}\rangle \in T$, we define $\mathrm{lh}(t) = \mathrm{lh}(t_0)$ and call the ordinal $\mathrm{ht}(T) = \mathrm{lub}\{\mathrm{lh}(t) \mid t \in T\}$ the *height of* T.
- We say that a tree T_0 on X is an *end-extension* of a tree T_1 on X if $T_1 = T_0 \cap {}^{<\mathrm{ht}(T_1)}X$ holds.
- Given a tree T on X, a tuple of functions $\langle x_0, \ldots, x_{n-1}\rangle \in \left({}^{\mathrm{ht}(T)}X\right)^n$ is called a *cofinal branch through* T if the tuple $\langle x_0 \restriction \alpha, \ldots, x_{n-1} \restriction \alpha\rangle$ is an element of T for every $\alpha < \mathrm{ht}(T)$. We let $[T]$ denote the set of all cofinal branches through T. If T is a tree on X^{n+1} of height λ, then we define $p[T] = \exists^x [T] \subseteq ({}^{\lambda}X)^n$.

We equip the spaces $({}^{\kappa}\kappa)^n$ with the usual topological structure induced by basic open sets of the form

$$U_{s_0,\ldots,s_{n-1}} = \{\langle x_0, \ldots, x_{n-1}\rangle \in ({}^{\kappa}\kappa)^n \mid s_0 \subseteq x_0, \ldots, s_{n-1} \subseteq x_{n-1}\}$$

with $s_0, \ldots, s_{n-1} \in {}^{<\kappa}\kappa$. Note that closed sets in this topology are of the form $[T]$ for some tree T on κ^n of height κ.

Definition 1.1. Let κ be an infinite cardinal. A subset A of $({}^{\kappa}\kappa)^n$ is a *κ-Borel subset* if it is contained in the smallest algebra of sets on $({}^{\kappa}\kappa)^n$ that contains all open subsets and is closed under unions of size κ.

The following definition directly generalizes the notion of a projective subset of Baire Space to our setting.

Definition 1.2. Let κ be an infinite cardinal.

(1) A subset A of $({}^{\kappa}\kappa)^n$ is a *$\mathbf{\Sigma}^1_1$-subset* if there is a tree T on κ^{n+1} with $A = p[T]$.
(2) A subset A of $({}^{\kappa}\kappa)^n$ is a *$\mathbf{\Pi}^1_k$-subset* if $({}^{\kappa}\kappa)^n \setminus A$ is a $\mathbf{\Sigma}^1_k$-subset.
(3) A subset A of $({}^{\kappa}\kappa)^n$ is a *$\mathbf{\Sigma}^1_{k+1}$-subset* if there is a $\mathbf{\Pi}^1_k$-subset B of $({}^{\kappa}\kappa)^{n+1}$ with $A = \exists^x B$.
(4) A subset A of $({}^{\kappa}\kappa)^n$ is a *$\mathbf{\Delta}^1_k$-subset* if it is both a $\mathbf{\Sigma}^1_k$-subset and a $\mathbf{\Pi}^1_k$-subset.

Fix an uncountable regular κ with $\kappa = \kappa^{<\kappa}$. In Section 2 we will present a folklore result showing that the $\mathbf{\Sigma}^1_1$-subsets are exactly the subsets of $({}^{\kappa}\kappa)^n$ that are definable in the structure $\langle \mathrm{H}(\kappa^+), \in\rangle$ by a Σ_1-formula with parameters. This shows that the $\mathbf{\Sigma}^1_1$-subsets form an interesting and rich class of subsets. Moreover, this result can be used to show that the κ-Borel subsets of ${}^{\kappa}\kappa$ form a proper subclass of the class of $\mathbf{\Delta}^1_1$-subsets (see [8, Theorem 18]).

The initial motivation of this work was to find generalizations of the following coding result due to Leo Harrington to uncountable regular cardinals κ.

Theorem 1.3 ([10, Theorem 1.7])**.** *Assume $\omega_1 = \omega_1^{\mathrm{L}}$. For every subset A of ${}^{\omega}\omega$, there is a partial order $\mathbb{P}$ with the following properties.*

(1) *$\mathbb{P}$ satisfies the countable chain condition.*
(2) *If G is $\mathbb{P}$-generic over V, then A is a $\mathbf{\Pi}^1_2$-subset of ${}^{\omega}\omega$ in $\mathrm{V}[G]$.*

We give a brief overview of related existing results. If "$\mathrm{V} = \mathrm{L}[x]$" holds in the ground model for some $x \subseteq \kappa$, then we can apply Solovay's *almost disjoint coding* forcing (see [14]) to make an arbitrary subset of ${}^\kappa\kappa$ $\mathbf{\Sigma}^1_1$-definable in a forcing extension of $\mathrm{L}[x]$ and in any further forcing extension in which κ remains a cardinal. This follows from the absoluteness properties of this coding and the fact that $({}^\kappa\kappa)^{\mathrm{L}[x]}$ is a $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa$ in all such extensions. Section 4 of this paper contains a detailed outline of the properties of this forcing.

Now, assume that the (GCH) holds at κ. If A is an arbitrary subset of ${}^\kappa\kappa$, then results of Sy-David Friedman show that there is a $<\kappa$-closed partial order that satisfies the κ^+-chain condition and adds a $\mathbf{\Sigma}^1_1$-definition of A. Moreover, this coding is absolute with respect to all further forcing extensions that preserve the regularity of κ and κ^+. This coding technique is called *Canonical Function coding.* A detailed discussion of this technique can be found in [1], [6] and [7].

We will present a coding result which only requires the assumption that the set of bounded subsets of κ has cardinality κ in the ground model. In particular, the hypothesis "$2^\kappa = \kappa^+$" is not needed. Before we state this result, we need to introduce some vocabulary.

Definition 1.4. Let κ be an infinite regular cardinal and Γ be a class of partial orders that contains the trivial partial order. We say that a subset A of ${}^\kappa\kappa$ is Γ-*persistently* $\mathbf{\Sigma}^1_1$ if there is a tree T on $\kappa \times \kappa$ such that $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\check{A} = p[\check{T}]$" holds for every $\mathbb{P}$ in Γ.

We are now ready to state our first main result. See [4, Definition 5.14] for the definition of α-strategically closed partial orders. As usual, we write *σ-strategically closed* instead of $(\omega + 1)$-strategically closed.

Theorem 1.5. *Let κ be a regular uncountable cardinal with $\kappa = \kappa^{<\kappa}$. For every subset A of ${}^\kappa\kappa$, there is a partial order $\mathbb{P}$ that satisfies the following statements.*

(1) *$\mathbb{P}$ is $<\kappa$-closed, satisfies the κ^+-chain condition and has cardinality at most 2^κ.*
(2) *If G is $\mathbb{P}$-generic over V, then A is Γ-persistently $\mathbf{\Sigma}^1_1$ in $\mathrm{V}[G]$, where Γ is the class of all σ-strategically closed partial orders in $\mathrm{V}[G]$ that preserve the regularity of κ.*

By combining the above absoluteness properties with uncountable versions of results from the proof of Theorem 1.3 in [10], we are able to prove our second main result.

Theorem 1.6. *Let κ be a regular uncountable cardinal with $\kappa = \kappa^{<\kappa}$. For every subset A of ${}^\kappa\kappa$, there is a partial order $\mathbb{P}$ that satisfies the following statements.*

(1) *$\mathbb{P}$ is $<\kappa$-closed, satisfies the κ^+-chain condition and has cardinality at most 2^κ.*
(2) *If G is $\mathbb{P}$-generic over V, then A is a $\mathbf{\Delta}^1_1$-subset of ${}^\kappa\kappa$ in $\mathrm{V}[G]$.*

This coding will also have certain absoluteness properties.

Definition 1.7. Let κ be an infinite regular cardinal and Γ be a class of partial orders that contains the trivial partial order. We say that a subset A of ${}^\kappa\kappa$ is Γ-*persistently* $\mathbf{\Delta}^1_1$ if there are trees T_0 and T_1 on $\kappa \times \kappa$ such that T_0 witnesses that A is Γ-persistently $\mathbf{\Sigma}^1_1$ and $\mathbb{1}_{\mathbb{P}} \Vdash$ "$p[\check{T}_1] = {}^{\check{\kappa}}\check{\kappa} \setminus p[\check{T}_0]$" holds for all $\mathbb{P}$ in Γ.

The proof of this result will show that there is a nontrivial class Γ of $<\kappa$-closed partial orders that satisfy the κ^+-chain condition such that the set coded in Theorem 1.6 is actually Γ-persistently $\mathbf{\Delta}^1_1$ in the generic extension. Both the forcing $\mathrm{Add}(\kappa, 1)$ that adds a Cohen-subset of κ and the *almost disjoint coding* forcings at κ are contained in this class Γ and this allows us to analyze certain structural properties of A in $\mathrm{V}[G]$.

In the following, we present some applications of the above results and the methods used in their proofs.

The *Anticoding Theorem*, proven by Itay Neeman and Jindřich Zapletal (see [19]), says that in the presence of large cardinals proper forcings do not code any set of ordinals from the ground model into $\mathrm{L}(\mathbb{R})$ of the forcing extension unless that set is already an element of $\mathrm{L}(\mathbb{R})$ of the ground model. Given an uncountable regular cardinal κ with $\kappa = \kappa^{<\kappa}$, an easy application of the above results shows that it is possible to code new sets of ordinals into $\mathrm{L}(\mathcal{P}(\kappa))$ by forcing with a κ-proper partial order (see [11, Definition 3.4] for a definition of this class of partial orders).

Corollary 1.8. *If κ is a regular uncountable cardinal with $\kappa = \kappa^{<\kappa}$ and X is an arbitrary set, then there is a partial order $\mathbb{P}$ with the following properties.*

(1) *$\mathbb{P}$ is $<\kappa$-closed and satisfies the κ^+-chain condition.*
(2) *If G is $\mathbb{P}$-generic over V, then $\mathbb{1}_{\mathbb{Q}} \Vdash$ "$\check{X} \in \mathrm{L}(\mathcal{P}(\check{\kappa}))$" holds in $\mathrm{V}[G]$ for every σ-strategically closed partial order $\mathbb{Q}$ in $\mathrm{V}[G]$ that preserve the regularity of κ.*

Next, we consider definable well-orders of ${}^\kappa\kappa$. In [7], Sy-David Friedman and Peter Holy construct a class-sized partial order preserving ZFC and large cardinals that forces (GCH) and adds a well-order of ${}^\kappa\kappa$ whose graph is a $\mathbf{\Delta}^1_1$-subset of ${}^\kappa\kappa \times {}^\kappa\kappa$ for every uncountable regular cardinal κ. In another direction, David Asperó and Sy-David Friedman showed in [2] that there is a class-sized partial order with the above preservation properties that forces (GCH) and adds a well-order ${}^\kappa\kappa$ that is definable in the structure $\langle \mathrm{H}(\kappa^+), \in \rangle$ by a formula without parameters for every uncountable regular cardinal κ. A detailed discussion of the above results and the related problem of obtaining lightface well-orders of low quantifier complexity can be found in the first part of [6].

In the following result, we apply Theorem 1.5 to add a definable well-order of ${}^\kappa\kappa$ with a forcing that preserves both cofinalities and the value of 2^κ.

Theorem 1.9. *If κ is a regular uncountable cardinal with $\kappa = \kappa^{<\kappa}$, then there is a partial order $\mathbb{P}$ with the following properties.*

(1) *$\mathbb{P}$ is $<\kappa$-closed, satisfies the κ^+-chain condition and has cardinality 2^κ.*
(2) *If G is $\mathbb{P}$-generic over V, then there is a well-ordering of $({}^\kappa\kappa)^{\mathrm{V}[G]}$ whose graph is a $\mathbf{\Delta}^1_2$-subset of ${}^\kappa\kappa \times {}^\kappa\kappa$ in $\mathrm{V}[G]$.*

Our next application deals with a quasi-ordering of trees that arises naturally in infinitary model theory (see [12] and [23]). Remember that a structure $\langle \mathbb{T}, \lhd_{\mathbb{T}} \rangle$ is a *tree* if $\lhd_{\mathbb{T}}$ is a well-founded strict ordering on $\mathbb{T}$ and the set $\mathrm{prec}_{\mathbb{T}}(t) = \{u \in \mathbb{T} \mid t \leq_{\mathbb{T}} u\}$ is well-ordered by $\lhd_{\mathbb{T}}$ for each $t \in \mathbb{T}$. As usual, we will just write $\mathbb{T}$ instead of $\langle \mathbb{T}, \lhd_{\mathbb{T}} \rangle$. As above, a branch through a tree $\mathbb{T}$ is a linearly ordered subset of $\mathbb{T}$. Given an infinite cardinal κ, we let $\mathcal{T}_\kappa$ denote the class of all trees $\mathbb{T}$ of cardinality at most κ such that every branch through $\mathbb{T}$ has length less than κ.

Let $\mathbb{T}_0$ and $\mathbb{T}_1$ be elements of $\mathcal{T}_\kappa$. We say that $\mathbb{T}_0$ *is order-preserving embeddable into* $\mathbb{T}_1$(abbreviated by $\mathbb{T}_0 \leq \mathbb{T}_1$) if there is a function $f : \mathbb{T}_0 \longrightarrow \mathbb{T}_1$ such that

$$t_0 \lhd_{\mathbb{T}_0} t_1 \longrightarrow f(t_0) \lhd_{\mathbb{T}_1} f(t_1)$$

holds for all $t_0, t_1 \in \mathbb{T}_0$. Note that f need not be injective.

There is a natural correspondence between elements of $\mathcal{T}_\omega$ and countable ordinals and the above ordering of trees is equal to the ordering of the ordinals under this correspondence. We may therefore think of elements of $\mathcal{T}_\kappa$ as analogs of ordinals. We can combine Theorem 1.5 with the *Boundedness Lemma for* ${}^\kappa\kappa$ to get an easy and short proof of the following statement that was proved in [17, Proof of Theorem 15] in the case "$\kappa = \omega_1$".

Theorem 1.10. *If κ is a regular uncountable cardinal with $\kappa = \kappa^{<\kappa}$, then there is a partial order $\mathbb{P}$ with the following properties.*

(1) *$\mathbb{P}$ is $<\kappa$-closed, satisfies the κ^+-chain condition and has cardinality at most 2^κ.*
(2) *If G is $\mathbb{P}$-generic over V, then there is a $\mathbb{T}_G \in \mathcal{T}_\kappa^{\mathrm{V}[G]}$ such that $\mathbb{T} \leq \mathbb{T}_G$ holds for every tree $\mathbb{T} \in \mathcal{T}_\kappa^{\mathrm{V}}$.*

Next, we consider generalizations of notions of projective absoluteness to our uncountable setting. Given an uncountable regular cardinal κ with $\kappa = \kappa^{<\kappa}$, the constructions carried out in the proof of Theorem 1.9 show that we can define a $\mathbf{\Sigma}^1_3$-subset of ${}^\kappa\kappa$ that is empty in every $\mathrm{Add}(\kappa, \kappa^+)$-generic extension of the ground model and nonempty in a generic extension of the ground model by a certain $<\kappa$-closed forcing that satisfies the κ^+-chain condition. This shows that *generic absoluteness for $\mathbf{\Sigma}^1_3({}^\kappa\kappa)$-formulae under forcings with the above properties is inconsistent with the axioms of* ZFC for such cardinals κ.

Theorem 1.11. *Let κ be an uncountable regular cardinal with $\kappa = \kappa^{<\kappa}$ and $a \subseteq \kappa$ such that ${}^{<\kappa}\kappa \in \mathrm{L}[a]$. Then there is a tree T on κ^3 contained in $\mathrm{L}[a]$ and a partial order $\mathbb{P}$ such that the following statements hold.*

(1) *$\mathbb{P}$ is $<\kappa$-closed and satisfies the κ^+-chain condition.*
(2) $\mathbb{1}_{\mathbb{P}} \Vdash$ "$(\exists x \in {}^{\check\kappa}\check\kappa)(\forall y \in {}^{\check\kappa}\check\kappa)\ \langle x, y\rangle \in p[\check T]$ ".
(3) $\mathbb{1}_{\mathrm{Add}(\kappa,\kappa^+)} \Vdash$ "$(\forall x \in {}^{\check\kappa}\check\kappa)(\exists y \in {}^{\check\kappa}\check\kappa)\ \langle x, y\rangle \notin p[\check T]$ ".

Again, the above result was known in the case where (GCH) holds at κ. The *Canonical Function coding* mentioned above can be applied to construct such trees and extensions under this assumption.

In the last section, we will construct ZFC-models in which generic absoluteness for $\mathbf{\Sigma}^1_2$-subsets of ${}^\kappa\kappa$ under certain classes of $<\kappa$-closed forcings holds for a regular uncountable cardinal κ with $\kappa = \kappa^{<\kappa}$. Together with methods developed in the proofs of the above results, this will allow us to show that various statements about the lengths of $\mathbf{\Sigma}^1_1$-definable well-orders of subsets of ${}^\kappa\kappa$ (whose consistency can be established with the help of the above coding results) are independent from the axioms of ZFC.

Acknowledgements. The results presented in this paper form a part of the author's Ph.D. thesis supervised by Ralf Schindler. The author would like to thank him for his support and many valuable comments. In addition, the author is indebted to Sy-David Friedman and Philipp Schlicht for helpful discussions of the subject of this paper. Finally, the author would like to thank the anonymous referee for numerous useful suggestions and corrections.

2. Some preliminary basic results

Given an uncountable regular cardinal κ, it is a well-known that a subset of ${}^\kappa\kappa$ is $\mathbf{\Sigma}^1_1$ if and only if it is definable in the structure $\langle \mathrm{H}(\kappa^+), \in\rangle$ by a Σ_1-formula with parameters. In this section, we will give a proof of this folklore result that emphasises the absoluteness properties of this correspondence.

Before we start, we fix some more notation. We let $\prec\cdot,\cdot\succ : \mathrm{On}\times\mathrm{On} \longrightarrow \mathrm{On}$ denote Gödel's Pairing function. Given an ordinal λ closed under Gödel-Pairing, $f \in {}^\lambda X$ for some nonempty set X and $\alpha < \lambda$, we define $(f)_\alpha$ to be the unique function $g \in {}^\lambda X$ with $g(\beta) = f(\prec\alpha,\beta\succ)$ for all $\beta < \lambda$.

Using Gödel-Pairing to code κ-many branches into one branch, it is easy to prove the following proposition.

Proposition 2.1. *Let κ be an infinite cardinal.*

(1) *If $\langle T_\alpha \mid \alpha < \kappa\rangle$ is a sequence of trees on κ^{n+1}, then there are trees T_U and T_I on κ^{n+1} such that*
$$p[T_U] = \bigcup_{\alpha<\kappa} p[T_\alpha] \textit{ and } p[T_I] = \bigcap_{\alpha<\kappa} p[T_\alpha]$$
hold in every transitive ZFC*-model that contains* V*.*

(2) *If T is a tree on κ^{n+2}, then there is a tree T_* on κ^{n+1} such that $p[T_*] = \exists^x p[T]$ holds in every transitive* ZFC*-model that contains* V*.* □

Given a limit ordinal λ closed under Gödel-Pairing and $x \in {}^\lambda 2$, we define $\in_x$ to be the unique binary relation on λ such that
$$\alpha \in_x \beta \iff x(\prec\alpha,\beta\succ) = 1$$
holds for all $\alpha,\beta < \lambda$.

Proposition 2.2. *Let κ be an uncountable regular cardinal. There is a tree T on $\kappa\times\kappa$ such that*
$$p[T] = \{x \in {}^\kappa 2 \mid \langle\kappa,\in_x\rangle \textit{ is well-founded and extensional }\} \tag{1}$$
holds in every transitive ZFC*-model that contains* V *and has the same ${}^{<\kappa}\kappa$ as* V*.*

Proof. Given $\lambda < \kappa$ closed under Gödel-Pairing, we define T^λ to be the set of all pairs $\langle s,t\rangle \in {}^\lambda 2 \times {}^\lambda\kappa$ such that $\langle\lambda,\in_s\rangle$ is well-founded and, if $\alpha,\beta,\gamma < \lambda$ with $\alpha \neq \beta$ and $t(\prec\alpha,\beta\succ) = \gamma$, then $s(\prec\gamma,\alpha\succ) \neq s(\prec\gamma,\beta\succ)$. We define T to be the tree on $\kappa\times\kappa$ consisting of all $\langle s,t\rangle$ with $\mathrm{lh}(s) = \mathrm{lh}(t)$ and $\langle s\upharpoonright\lambda, t\upharpoonright\lambda\rangle \in T^\lambda$ for all $\lambda \leq \mathrm{lh}(s)$ closed under Gödel-Pairing. □

Proposition 2.3. *Let κ be an infinite cardinal, $\varphi(v_0,\ldots,v_{n-1})$ be a formula in the language of set theory and $\alpha_0,\ldots,\alpha_{n-1} < \kappa$. There is a tree T on $\kappa\times\kappa$ such that*
$$p[T] = \{x \in {}^\kappa 2 \mid \langle\kappa,\in_x\rangle \models \varphi(\alpha_0,\ldots,\alpha_{n-1})\} \tag{2}$$
holds in every transitive ZFC*-model that contains* V *and has the same ${}^{<\kappa}\kappa$ as* V*.*

Proof. We can assume that $\varphi(v_0,\ldots,v_{n-1})$ is in prenex normal form. We construct the corresponding trees inductively. If φ is atomic (or the negation of an atomic formula), then T is simply the tree of all $\langle s,t\rangle \in {}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$ with $\mathrm{lh}(s) = \mathrm{lh}(t)$ and either $\mathrm{lh}(s) \leq \prec\alpha_0,\alpha_1\succ$ or $s(\prec\alpha_0,\alpha_1\succ) = 1$ (or $s(\prec\alpha_0,\alpha_1\succ) = 0$ in the case of a negated atomic formula).

If $\varphi(v_0,\ldots,v_{n-1}) \equiv (\exists x)\ \varphi_0(v_0,\ldots,v_{n-1},x)$ and $\alpha < \kappa$, then we can use the induction hypothesis to find a tree T_α on $\kappa \times \kappa$ such that

$$p[T_\alpha] = \{x \in {}^\kappa 2 \mid \langle \kappa, \in_x \rangle \models \varphi_0(\alpha_0,\ldots,\alpha_{n-1},\alpha)\}$$

holds in every transitive ZFC-model that contains V and has the same ${}^{<\kappa}\kappa$ as V. By Proposition 2.1, there is a tree T on $\kappa\times\kappa$ with the property that $p[T] = \bigcup_{\alpha<\kappa} p[T_\alpha]$ holds upwards-absolutely. This implies that T satisfies (2) in every transitive ZFC-model that contains V and has the same ${}^{<\kappa}\kappa$ as V.

The trees in the universal quantifier case, the disjunction case and the conjunction case are constructed in the same fashion using Proposition 2.1. □

Note that the sets mentioned in (1) and (2) are actually κ-Borel subsets of ${}^\kappa\kappa$. In particular, if κ has uncountable cofinality, then the set of codes for well-founded relations on κ is closed in ${}^\kappa\kappa$.

Lemma 2.4. *Let κ be an uncountable regular cardinal. Given a Σ_1-formula $\varphi \equiv \varphi(u_0,\ldots,u_{n-1},v_0,\ldots,v_m)$ and $p_0,\ldots,p_m \in \mathrm{H}(\kappa^+)$, there is a tree T on κ^{n+1} such that*

$$(3)\quad p[T] = \{\langle x_0,\ldots,x_{n-1}\rangle \in ({}^\kappa\kappa)^n \mid \langle \mathrm{H}(\kappa^+),\in\rangle \models \varphi(x_0,\ldots,x_{n-1},p_0,\ldots,p_m)\}$$

holds in every transitive ZFC*-model that contains* V *and has the same ${}^{<\kappa}\kappa$ as* V.

Proof. Fix bijections $b_j : \kappa \longrightarrow \mathrm{tc}(\{p_j\}\cup\kappa)$ for all $j \leq m$. Let M be a transitive ZFC-model containing V with the same ${}^{<\kappa}\kappa$ as V and $x_0,\ldots,x_{n-1} \in ({}^\kappa\kappa)^M$. Now, $\langle \mathrm{H}(\kappa^+)^M,\in\rangle$ is a model of $\varphi(x_0,\ldots,x_{n-1},p_0,\ldots,p_m)$ if and only if there is a transitive $N \in \mathrm{H}(\kappa^+)^M$ with $\kappa,x_0,\ldots,x_{n-1},p_0,\ldots,p_m \in N$ and $\langle N,\in\rangle$ is a model of this statement. If $\varphi(\vec{u},\vec{v}) \equiv (\exists x)\varphi_0(\vec{u},x,\vec{v})$, then the above statement is equivalent to the existence of $x \in ({}^\kappa 2)^M$ and $y,z_0,\ldots,z_m \in ({}^\kappa\kappa)^M$ with the following properties.

1. (1) $\langle\kappa,\in_x\rangle$ is well-founded and extensional and
$$\langle\kappa,\in_x\rangle \models \varphi_0(0,\ldots,n,\omega,\ldots,\omega+m).$$
2. (2) $\langle\kappa,\in_x\rangle \models$ "$\omega^3 \in \mathrm{On}$", $\langle\kappa,\in_x\rangle \models$ "$\omega^2+j = \mathrm{tc}(\{\omega+j\}\cup\omega^3)$" for all $j \leq m$ and $\langle\kappa,\in_x\rangle \models$ "$i : \omega^3 \longrightarrow \omega^3$" for all $i \leq n$.
3. (3) For all $\alpha,\beta < \kappa$, $\langle\kappa,\in_x\rangle \models$ "$\alpha\dot{\in}\beta \wedge \beta\dot{\in}\omega^3$" if and only if $\alpha = y(\gamma)$ and $\beta = y(\delta)$ for some $\gamma < \delta < \kappa$.
4. (4) For all $\alpha,\beta < \kappa$ and $i < n$, $\langle\kappa,\in_x\rangle \models$ "$i(\alpha) = \beta$" if and only if $\alpha = y(\gamma)$ and $\beta = (y\circ x_i)(\gamma)$ for some $\gamma < \kappa$.
5. (5) For all $\alpha,\beta < \kappa$ and $j \leq m$, $\langle\kappa,\in_x\rangle \models$ "$\alpha\dot{\in}\beta \wedge \beta\dot{\in}(\omega^2+j)$" if and only if $\alpha = (z_j\circ b_j)(\gamma)$ and $\beta = (z_j\circ b_j)(\delta)$ for some $\gamma,\delta < \kappa$ with $b_j(\gamma) \in b_j(\delta)$.

Using Proposition 2.2 and 2.3, there is a tree T_0 on κ^{m+n+3} with the property that, for all M as above, $\langle x_0,\ldots,x_{n-1},x,y,z_0,\ldots,z_m\rangle \in [T]^M$ if and only if $x,y,\vec{z}$ witness that $\langle \mathrm{H}(\kappa^+)^M,\in\rangle \models \varphi(\vec{x},\vec{p})$ holds. By Proposition 2.1, this completes the proof of the lemma. □

Let κ be an infinite cardinal with $\kappa = \kappa^{<\kappa}$. Given $n < \omega$, there is a Σ_1-formula $\varphi(u_0,\ldots,u_{n-1},v_0,v_1)$ such that for every tree T on κ^{n+1} the equality (3) holds with $m = 2$, $p_0 = \kappa$ and $p_1 = T$ in every transitive ZFC-model that contains V. This shows that $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$ correspond to $\mathbf{\Sigma}_1(\mathrm{H}(\kappa^+))$-subsets in a way that is upwards-absolute between transitive ZFC-models with the same ${}^{<\kappa}\kappa$. We will often use this folklore fact to keep constructions in our proofs simple.

There is a similar correspondence for κ-Borel subsets: a subset A of ${}^\kappa\kappa$ is κ-Borel if and only if there is a transitive set M of cardinality κ, a formula $\varphi \equiv \varphi(v_0, \ldots, v_{n-1})$ in the language of set theory expanded by an unary relation symbol and parameters $z_0, \ldots, z_{m-1} \in M$ such that $\kappa \in M$, $\langle M, \in\rangle \models \mathrm{ZF}^-$ and

$$x \in A \iff \langle M, \in, x\rangle \models \varphi(z_0, \ldots, z_{n-1})$$

holds for all $x \in {}^\kappa\kappa$.

3. Generic tree coding

This section contains the proof of Theorem 1.5. For the rest of this paper, we fix a **regular uncountable cardinal** κ with $\kappa = \kappa^{<\kappa}$ and an **enumeration** $\langle s_\alpha \mid \alpha < \kappa\rangle$ of ${}^{<\kappa}\kappa$ with $\mathrm{lh}(s_\alpha) \leq \alpha$ for all $\alpha < \kappa$ and $\{\alpha < \kappa \mid s = s_\alpha\}$ unbounded in κ for all $s \in {}^{<\kappa}\kappa$.

Our coding forcing will be a modification of the standard forcing that adds a Kurepa tree (see [13, §3]). The main idea behind this modification is that it is possible to code information about the elements of a subset A of ${}^\kappa\kappa$ into the cofinal branches of the generic tree.

Definition 3.1. Given a nonempty subset A of ${}^\kappa\kappa$, we define $\mathbb{P}(A)$ to be the partial order consisting of conditions $p = \langle T_p, f_p, h_p\rangle$ with the following properties.

(1) T_p is a subtree of ${}^{<\kappa}2$ that satisfies the following statements.
  (a) T_p has cardinality less than κ.
  (b) If $t \in T_p$ with $\mathrm{lh}(t) + 1 < \mathrm{ht}(T_p)$, then t has two immediate successors in T_p.
(2) $f_p : A \xrightarrow{part} [T_p]$ is a partial function such that $\mathrm{dom}(f_p)$ is a nonempty set of cardinality less than κ.
(3) $h_p : A \xrightarrow{part} \kappa$ is a partial function with the following properties.
  (a) $\mathrm{dom}(h_p) = \mathrm{dom}(f_p)$.
  (b) For all $x \in \mathrm{dom}(h_p)$ and $\alpha, \beta < \mathrm{ht}(T_p)$ with $\alpha = \prec h_p(x), \beta\succ$, we have

$$s_\beta \subseteq x \iff f_p(x)(\alpha) = 1.$$

We define $p \leq_{\mathbb{P}(A)} q$ to hold if the following statements are satisfied.

(a) T_p is an end-extension of T_q.
(b) For all $x \in \mathrm{dom}(f_q)$, $x \in \mathrm{dom}(f_p)$ and $f_q(x)$ is an initial segment of $f_p(x)$.
(c) $h_q = h_p \restriction \mathrm{dom}(h_q)$.

Lemma 3.2. *$\mathbb{P}(A)$ is $<\kappa$-closed, satisfies the κ^+-chain condition and has cardinality at most 2^κ.*

Proof. If $\lambda \in \mathrm{Lim} \cap \kappa$ and $\langle p_\mu \mid \mu < \lambda\rangle$ is a strictly $\leq_{\mathbb{P}(A)}$-descending sequence in $\mathbb{P}(A)$, then we define $T = \bigcup_{\mu<\lambda} T_{p_\mu}$, $h = \bigcup_{\mu<\lambda} h_\mu$ and

$$f(x) = \bigcup\{f_{p_\mu}(x) \mid \mu < \lambda, x \in \mathrm{dom}(f_{p_\mu})\}$$

for all $x \in \mathrm{dom}(h)$. It is easy to see that $p = \langle T, f, h\rangle \in \mathbb{P}(A)$ and $p \leq_{\mathbb{P}(A)} p_\mu$ holds for all $\mu < \lambda$.

Next, assume that $\langle p_\mu \mid \mu < \kappa^+\rangle$ enumerates an antichain in $\mathbb{P}(A)$. By our assumptions, we can assume $T_{p_\mu} = T_{p_\rho}$ for all $\mu, \rho < \kappa^+$. A Δ-system argument allows us to assume the existence of an $r \subseteq A$ with $r = \mathrm{dom}(f_{p_\mu}) \cap \mathrm{dom}(f_{p_\rho})$,

$f_{p_\mu} \upharpoonright r = f_{p_\rho} \upharpoonright r$ and $h_{p_\mu} \upharpoonright r = h_{p_\rho} \upharpoonright r$ for all $\mu < \rho < \kappa^+$. But this shows that $\langle T_{p_0}, f_{p_0} \cup f_{p_1}, h_{p_0} \cup h_{p_1}\rangle$ is a common extension of p_0 and p_1, a contradiction.

Finally, the assumption $\kappa = \kappa^{<\kappa}$ implies that there are only κ-many subtrees of ${}^{<\kappa}2$ of height less than κ and 2^κ-many partial functions with the above properties. □

The next lemma will allow us to show that various subsets of $\mathbb{P}(A)$ are dense.

Lemma 3.3. *Fix a condition p in $\mathbb{P}(A)$ and a sequence $\langle c_x \in {}^\kappa 2 \mid x \in \mathrm{dom}(f_p)\rangle$. There exists a $\leq_{\mathbb{P}(A)}$-descending sequence $\langle p_\mu \in \mathbb{P}(A) \mid \mathrm{ht}(T_p) \leq \mu < \kappa\rangle$ such that $p = p_{\mathrm{ht}(T_p)}$ and the following statements hold for all $\mathrm{ht}(T_p) \leq \mu < \kappa$.*

1. *$\mathrm{dom}(f_{p_\mu}) = \mathrm{dom}(f_p)$ and $\mathrm{ht}(T_{p_\mu}) = \mu$.*
2. *If $x \in \mathrm{dom}(f_p)$ and $\mu \neq \prec h_p(x), \beta\succ$ for all $\beta < \kappa$, then*
$$f_{p_{\mu+1}}(x)(\mu) = c_x(\mu).$$
3. *If $\mu \in \mathrm{Lim}$, then $\mathrm{ran}(f_{p_\mu}) = T_{p_{\mu+1}} \cap {}^\mu 2$.*

Proof. We construct the sequences inductively. If $\mu \in \mathrm{Lim}$, then we define $T_{p_\mu} = \bigcup\{T_{p_{\bar\mu}} \mid \mathrm{ht}(T_p) \leq \bar\mu < \mu\}$. Given $x \in \mathrm{dom}(f_p)$, we define

$$f_{p_\mu}(x) = \bigcup\{f_{p_{\bar\mu}}(x) \mid \mathrm{ht}(T_p) \leq \bar\mu < \mu\}.$$

If $\mu = \bar\mu + 1$ with $\bar\mu \notin \mathrm{Lim}$, then $T_{p_{\bar\mu}}$ has a maximal level and there is only one suitable tree T_{p_μ} of height μ end-extending it. In particular, $f_{p_{\bar\mu}}(x) \in T_{p_\mu}$ for all $x \in \mathrm{dom}(f_p)$. For all $x \in \mathrm{dom}(f_p)$, we define $f_{p_\mu}(x)$ to be the unique element t of ${}^\mu 2$ with $f_{p_{\bar\mu}}(x) \subseteq t$ and

$$t(\bar\mu) = \begin{cases} 1, & \text{if } \bar\mu = \prec h_p(x), \beta\succ \text{ and } s_\beta \subseteq x, \\ 0, & \text{if } \bar\mu = \prec h_p(x), \beta\succ \text{ and } s_\beta \nsubseteq x, \\ c_x(\bar\mu), & \text{otherwise.} \end{cases}$$

Finally, if $\mu = \bar\mu + 1$ with $\bar\mu \in \mathrm{Lim}$, then we set $T_{p_\mu} = T_{p_{\bar\mu}} \cup \mathrm{ran}(f_{p_{\bar\mu}})$ and define f_{p_μ} as in the first successor case. □

Corollary 3.4. *The following sets are dense subsets of $\mathbb{P}(A)$.*

1. *$C_\mu = \{p \in \mathbb{P}(A) \mid \mathrm{ht}(T_p) > \mu\}$ for all $\mu < \kappa$.*
2. *$D_x = \{p \in \mathbb{P}(A) \mid x \in \mathrm{dom}(f_p)\}$ for all $x \in A$.*
3. *$E_{x,y} = \{p \in \mathbb{P}(A) \mid x, y \in \mathrm{dom}(f_p), f_p(x) \neq f_p(y)\}$ for all $x, y \in A$.*
4. *$F_z = \{p \in \mathbb{P}(A) \mid \mathrm{ht}(T_p) = \mu + 1,\ z \upharpoonright \mu \notin T_p\}$ for all $z \in {}^\kappa 2$.*

Proof. (i) This statement follows directly from Lemma 3.3.

(ii) Given $p \in \mathbb{P}(A)$ with $x \notin \mathrm{dom}(f_p)$ and $b \in [T_p] \neq \emptyset$, we define

$$q = \langle T_p, f_p \cup \{\langle x, b\rangle\}, h_p \cup \{\langle x, \mathrm{ht}(T_p)\rangle\}\rangle.$$

Then $q \in D_x$ and $q \leq_{\mathbb{P}(A)} p$.

(iii) Given $p \in \mathbb{P}(A)$, we can apply the above result to find $q \leq_{\mathbb{P}(A)} p$ with $x, y \in \mathrm{dom}(f_q)$. There is $\mathrm{ht}(T_q) \leq \mu < \kappa$ with $\prec h_q(x), \beta_0\succ \neq \mu \neq \prec h_q(y), \beta_1\succ$ for all $\beta_0, \beta_1 < \kappa$ and we can use Lemma 3.3 to find $q^* \leq_{\mathbb{P}(A)} q$ with $\mathrm{ht}(T_{q^*}) = \mu + 1$ and $f_{q^*}(x)(\mu) \neq f_{q^*}(y)(\mu)$.

(iv) Fix $p \in \mathbb{P}(A)$ and $\mathrm{ht}(T_p) \leq \mu < \kappa$ with $\mu \neq \prec h_p(x), \beta\succ$ for all $x \in \mathrm{dom}(f_p)$ and $\beta < \kappa$. Using Lemma 3.3, we can find $q \leq_{\mathbb{P}(A)} p$ with $\mathrm{ht}(T_q) = \mu + 1$, $\mathrm{dom}(f_q) = \mathrm{dom}(f_p)$ and $f_q(x)(\mu) = 1 - z(\mu)$ for all $x \in \mathrm{dom}(f_p)$.

In particular, $z \restriction (\mu+1) \notin \mathrm{ran}(f_q)$. Another application of the above lemma gives us conditions $s \leq_{\mathbb{P}(A)} r \leq_{\mathbb{P}(A)} q$ with $\mathrm{ht}(T_s) = \mathrm{ht}(T_r)+1 = \mathrm{ht}(T_q)+\omega+1$, $\mathrm{dom}(f_s) = \mathrm{dom}(f_p)$ and $T_s \cap {}^{\mathrm{ht}(T_r)}2 = \mathrm{ran}(f_r)$. Since $z \restriction \mathrm{ht}(T_r) \neq f_r(x)$ for all $x \in \mathrm{dom}(f_p)$, we have $z \restriction \mathrm{ht}(T_r) \notin T_s$. □

Corollary 3.5. *Let G be $\mathbb{P}(A)$-generic over* V. *The following statements hold true in* $\mathrm{V}[G]$.

(1) $T_G = \bigcup_{p \in G} T_p$ *is subtree of* ${}^{<\kappa}2$ *of height* κ *with* $[T_G] \cap \mathrm{V} = \emptyset$.
(2) *If we define* $F_G(x) = \bigcup\{f_p(x) \mid p \in G, x \in \mathrm{dom}(f_p)\}$ *for all* $x \in A$, *then* $F_G : A \longrightarrow [T_G]$ *is an injection.*
(3) *Let* $H_G = \bigcup_{p \in G} h_g$. *Then* $H_G : A \longrightarrow \kappa$ *and*

$$s_\beta \subseteq x \iff F_G(x)(\prec H_G(x), \beta \succ) = 1 \tag{4}$$

for all $x \in A$ *and* $\beta < \kappa$. □

We now show how the branches of T_G correspond to elements of A in an absolute and bijective way.

Lemma 3.6. *Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}(A)$-name such that*

$$\mathbb{1}_{\mathbb{P}(A)} \Vdash \text{“}\dot{\mathbb{Q}} \textit{ is a } \sigma\textit{-strategically closed partial order and forcing with } \dot{\mathbb{Q}} \textit{ preserves the regularity of } \check{\kappa}\text{”}. \tag{5}$$

If $G_0 * G_1$ is $(\mathbb{P}(A)*\dot{\mathbb{Q}})$-generic over V, *then $F_{G_0} : A \longrightarrow [T_{G_0}]^{\mathrm{V}[G_0][G_1]}$ is surjective.*

Proof. Fix names $\dot{F}, \dot{T} \in \mathrm{V}^{\mathbb{P}(A)*\dot{\mathbb{Q}}}$ such that $\dot{F}^{H_0 * H_1} = F_{H_0}$ and $\dot{T}^{H_0*H_1} = T_{H_0}$ holds whenever $H_0 * H_1$ is $(\mathbb{P}(A)*\dot{\mathbb{Q}})$-generic over V. Assume, toward a contradiction, that there is a name $\tau \in \mathrm{V}^{\mathbb{P}(A)*\dot{\mathbb{Q}}}$ and a condition $\langle p, \dot{q}\rangle$ in $\mathbb{P}(A)*\dot{\mathbb{Q}}$ with

$$\langle p, \dot{q}\rangle \Vdash \text{“}\tau \in [\dot{T}] \wedge \tau \notin \check{\mathrm{V}} \wedge \tau \notin \mathrm{ran}(\dot{F})\text{”}.$$

For each $r \leq_{\mathbb{P}(A)*\dot{\mathbb{Q}}} \langle p, \dot{q}\rangle$, we define a partial function $t_r : \kappa \xrightarrow{part} 2$ in V by setting

$$t_r = \bigcup\{s \in {}^{<\kappa}2 \mid r \Vdash \text{“}\check{s} \subseteq \tau\text{”}\}.$$

We have $t_r \in {}^{<\kappa}2$ for all $r \leq_{\mathbb{P}(A)*\dot{\mathbb{Q}}} \langle p, \dot{q}\rangle$, because $r \Vdash \text{“}\tau \notin \check{V}\text{”}$. Moreover, since $\langle p, \dot{q}\rangle \Vdash \text{“}(\forall \alpha < \check{\kappa})\ \tau \restriction \alpha \in \check{\mathrm{V}}\text{”}$, the set $\{r \leq_{\mathbb{P}(A)*\dot{\mathbb{Q}}} \langle p, \dot{q}\rangle \mid \alpha \subseteq \mathrm{dom}(t_r)\}$ is dense below $\langle p, \dot{q}\rangle$ for all $\alpha < \kappa$.

Let $\langle p', \dot{q}'\rangle \leq_{\mathbb{P}(A)*\dot{\mathbb{Q}}} \langle p, \dot{q}\rangle$ and $d = \mathrm{dom}(f_{p'})$. Since

$$\langle p, \dot{q}\rangle \Vdash \text{“}\textit{The cardinality of } \check{d} \textit{ is less than } \mathrm{cof}(\check{\kappa}) \textit{ and } \tau \neq \dot{F}(x) \textit{ for all } x \in \check{d}\text{”},$$

there is an $r \leq_{\mathbb{P}(A)*\dot{\mathbb{Q}}} \langle p', \dot{q}'\rangle$ and an $\alpha < \kappa$ such that

$$r \Vdash \text{“}(\forall x \in \check{d})(\exists \beta < \check{\alpha})\ \tau(\beta) \neq \dot{F}(x)(\beta)\text{”}.$$

Then there is a condition $r_* = \langle p'', \dot{q}''\rangle \leq_{\mathbb{P}(A)*\dot{\mathbb{Q}}} r$ such that $\alpha \subseteq \mathrm{dom}(t_{r_*})$ and $\mathrm{ht}(T_{p''}) \geq \alpha$. This implies that for all $x \in \mathrm{dom}(f_{p'})$ there is a $\beta \in \mathrm{dom}(t_{r_*})$ such that $f_{p''}(x)(\beta) \neq t_{r_*}(\beta)$.

Let $\dot{\sigma}$ be a $\mathbb{P}(A)$-name with

$$\mathbb{1}_{\mathbb{P}(A)} \Vdash \text{“}\dot{\sigma} \textit{ is a winning strategy for Player } \text{Even in } \mathcal{G}_{\omega+1}(\dot{\mathbb{Q}})\text{”}.$$

Given $\langle p_0, \dot{q}_1\rangle \leq_{\mathbb{P}(A)*\dot{\mathbb{Q}}} \langle p, \dot{q}\rangle$, the above remarks allow us to construct a strictly $\leq_{\mathbb{P}(A)*\dot{\mathbb{Q}}}$-descending sequence $\langle\langle p_n, \dot{q}_{2n+1}\rangle \mid n < \omega\rangle$ of conditions in $\mathbb{P}(A)*\dot{\mathbb{Q}}$ and a sequence $\langle \dot{q}_{2n} \in \mathrm{V}^{\mathbb{P}(A)} \mid n < \omega\rangle$ of names such that the following statements hold for all $n < \omega$.

(1) $\dot{q}_0 = \dot{\mathbb{1}}_{\dot{\mathbb{Q}}}$, $\mathbb{1}_{\mathbb{P}(A)} \Vdash$ "$\dot{q}_{2n} \in \dot{\mathbb{Q}}$", $p_n \Vdash$ "$\dot{q}_{2n+1} \leq_{\dot{\mathbb{Q}}} \dot{q}_{2n}$" and

$$p_n \Vdash \text{“}\dot{q}_{2n+2} = \dot{\sigma}(\dot{q}_0, \ldots, \dot{q}_{2n+1})\text{”}.$$

(2) $\mathrm{ht}(T_{p_n}) \subseteq \mathrm{dom}(t_{\langle p_{n+1}, \dot{q}_{2n+3}\rangle})$ and $\mathrm{dom}(t_{\langle p_n, \dot{q}_{2n+1}\rangle}) \subsetneq \mathrm{ht}(T_{p_{n+1}})$.

(3) If $x \in \mathrm{dom}(f_{p_n})$, then there is an $\alpha \in \mathrm{dom}(t_{\langle p_{n+1}, \dot{q}_{2n+3}\rangle})$ with

$$f_{p_{n+1}}(x)(\alpha) \neq t_{\langle p_{n+1}, \dot{q}_{2n+3}\rangle}(\alpha).$$

By the proof of Lemma 3.2, the sequence $\langle p_n \mid n < \omega\rangle$ has a greatest lower bound p_ω in $\mathbb{P}(A)$. Note that $T_{p_\omega} = \bigcup_{n<\omega} T_{p_n}$ and $\mathrm{dom}(f_{p_\omega}) = \bigcup_{n<\omega} \mathrm{dom}(f_{p_n})$ hold. If $\dot{R} \in \mathrm{V}^{\mathbb{P}(A)}$ is the canonical name for the sequence $\langle \dot{q}_n \mid n < \omega\rangle$, then

$$p_\omega \Vdash \text{“}\dot{R} \textit{ is a run of the game } \mathcal{G}_\omega(\dot{\mathbb{Q}}) \textit{ in which } \text{Even played according to } \dot{\sigma}\text{”}.$$

Hence there is a name $\dot{q}_\omega \in \mathrm{V}^{\mathbb{P}(A)}$ with $\mathbb{1}_{\mathbb{P}(A)} \Vdash$ "$\dot{q}_\omega \in \dot{\mathbb{Q}}$" and $p_\omega \Vdash$ "$\dot{q}_\omega \leq_{\dot{\mathbb{Q}}} \dot{q}_n$" for all $n < \omega$. This implies $\langle p_\omega, \dot{q}_\omega\rangle \leq_{\mathbb{P}(A)*\dot{Q}} \langle p_n, \dot{q}_{2n+1}\rangle$ for all $n < \omega$. We define $t = t_{\langle p_\omega, \dot{q}_\omega\rangle} \restriction \mathrm{ht}(T_{p_\omega}) \in [T_{p_\omega}]$. Since $\langle p_\omega, \dot{q}_\omega\rangle \Vdash$ "$\check{t} \subseteq \tau \wedge \tau \in [\dot{T}]$" holds, we can conclude $\langle p_\omega, \dot{q}_\omega\rangle \Vdash$ "$\check{t} \in \dot{T}$".

By our construction, we have $\mathrm{ht}(T_{p_\omega}) \in \mathrm{Lim}$ and $t \notin \mathrm{ran}(f_{p_\omega})$. We can apply Lemma 3.3 to find a condition $p^* \leq_{\mathbb{P}(A)} p_\omega$ with $\mathrm{ht}(T_{p^*}) = \mathrm{ht}(T_{p_\omega}) + 1$ and $t \notin T_{p^*}$. This obviously implies $\langle p^*, \dot{q}_\omega\rangle \Vdash$ "$\check{t} \notin \dot{T}$", a contradiction. □

Corollary 3.7. *Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}(A)$-name such that (5) holds. If $G_0 * G_1$ is $(\mathbb{P}(A)*\dot{\mathbb{Q}})$-generic over* V*, then the following statements are equivalent for all $y \in ({}^\kappa\kappa)^{\mathrm{V}[G_0][G_1]}$.*

(1) *$y \in A$.*

(2) *There is $z \in [T_{G_0}]^{\mathrm{V}[G_0][G_1]}$ and $\gamma < \kappa$ such that*

$$s_\beta \subseteq y \iff z(\prec\gamma, \beta\succ) = 1 \tag{6}$$

holds for all $\beta < \kappa$

Proof. If $y \in A$, then the equivalence (6) holds with $z = F_{G_0}(y)$ and $\gamma = H_{G_0}(y)$ by Corollary 3.5.

In the other direction, let $z \in [T_{G_0}]^{\mathrm{V}[G_0][G_1]}$ and $\gamma < \kappa$ witness that (6) holds for $y \in ({}^\kappa\kappa)^{\mathrm{V}[G_0][G_1]}$. By Lemma 3.6, we have $z = F_{G_0}(x) \in \mathrm{V}[G_0]$ for some $x \in A$. Pick $p \in G_0$ with $x \in \mathrm{dom}(f_p)$.

Assume, toward a contradiction, that $\gamma \neq h_p(x) = H_{G_0}(x)$. By Lemma 3.3, this implies that the set

$$D_s = \{q \leq_{\mathbb{P}(A)} p \mid \mathrm{ht}(T_q) = \mu + 1,\ \mu = \prec\gamma, \beta\succ,\ f_q(x)(\mu) = 0,\ s_\beta = s\}$$

is dense below p for all $s \in {}^{<\kappa}\kappa$ and there is a $q \in G_0 \cap D_{y\restriction 1}$ with $q \leq_{\mathbb{P}(A)} p$. Then there is a $\beta < \kappa$ with $\mathrm{ht}(T_q) = \prec\gamma, \beta\succ + 1$, $z(\prec\gamma, \beta\succ) = 0$ and $s_\beta = y \restriction 1 \subseteq y$, contradicting (6). This shows $\gamma = H_{G_0}(x)$ and we can conclude that

$$s_\beta \subseteq y \iff z(\prec\gamma, \beta\succ) = 1 \iff F_{G_0}(x)(\prec H_{G_0}(x), \beta\succ) = 1 \iff s_\beta \subseteq x$$

holds for all $\beta < \kappa$. This proves $y = x \in A$. □

We are now ready to prove our first main result.

Proof of Theorem 1.5. Let G_0 be $\mathbb{P}(A)$-generic over V. In $V[G_0]$, define T to be the set consisting of pairs $\langle t, u\rangle$ such that $t \in {}^{<\kappa}\kappa$, $u \in {}^{<\kappa}\kappa$ and there is $\gamma < \kappa$ and $v \in T_{G_0}$ with $\mathrm{lh}(z) = \mathrm{lh}(u) = \mathrm{lh}(v)$, $u(\alpha) = \prec\gamma, v(\alpha)\succ$ for all $\alpha < \mathrm{lh}(s)$ and

$$s_\beta \subseteq t \iff v(\prec\gamma, \beta\succ) = 1$$

for all $\beta < \mathrm{lh}(s)$ with $\prec\gamma, \beta\succ < \mathrm{lh}(s)$. It is easy to check that T is a tree.

Let $\mathbb{Q}$ be a σ-strategically closed partial order in $V[G_0]$ and G_1 be $\mathbb{Q}$-generic over $V[G_0]$. There is a name $\dot{\mathbb{Q}} \in V^{\mathbb{P}(A)}$ such that $\mathbb{Q} = \dot{\mathbb{Q}}^{G_0}$ and (5) holds in V.

If $\langle x, y\rangle \in [T]^{V[G_0][G_1]}$, then there is $z \in [T_{G_0}]^{V[G_0][G_1]}$ and $\gamma < \kappa$ with $y(\beta) = \prec\gamma, z(\beta)\succ$ and

$$s_\beta \subseteq x \iff z(\prec\gamma, \beta\succ) = 1$$

for all $\beta < \kappa$. By Corollary 3.7, this implies $x \in A$.

Conversely, if $x \in A$ and $y \in ({}^{\kappa}\kappa)^{V[G_0]}$ with $y(\alpha) = \prec H_{G_0}(x), F_{G_0}(x)(\alpha)\succ$, then $\langle x, y\rangle \in [T]^{V[G_0][G_1]}$ by our assumptions on s and Corollary 3.7. □

We close this section by showing that Theorem 1.5 directly implies the statement of Corollary 1.8. Given functions $x, y \in {}^{\kappa}\kappa$, we let $\prec x, y\succ$ denote the unique function $z \in {}^{\kappa}\kappa$ with $x = (z)_0$, $y = (z)_1$ and $(z)_\alpha = \mathrm{id}_\kappa$ for all $1 < \alpha < \kappa$.

Proof of Corollary 1.8. Let ν be the cardinality of $\mathrm{tc}(\{X\})$ and let $\dot{e} \in V^{\mathrm{Add}(\kappa,\nu^+)}$ be a name for an injection of $\mathrm{tc}(\{X\})$ into ${}^{\kappa}\kappa \setminus \{\mathrm{id}_\kappa\}$. Let $\dot{A} \in V^{\mathrm{Add}(\kappa,\nu^+)}$ be a name such that

$$\dot{A}^G = \{\prec\dot{e}^G(b), \dot{e}^G(c)\succ \mid b, c \in \mathrm{tc}(\{X\}),\ b \in c\} \ \cup\ \{\prec\mathrm{id}_\kappa, \dot{e}^G(b)\succ \mid b \in X\}$$

holds whenever G is $\mathrm{Add}(\kappa, \nu^+)$-generic over V. Pick a name $\dot{\mathbb{P}} \in V^{\mathrm{Add}(\kappa,\nu^+)}$ with $\mathbb{1}_{\mathrm{Add}(\kappa,\nu^+)} \Vdash$ "$\dot{\mathbb{P}} = \mathbb{P}(\dot{A})$". The partial order $\mathrm{Add}(\kappa, \nu^+) * \dot{\mathbb{P}}$ is $<\kappa$-closed and satisfies the κ^+-chain condition.

Let $G_0 * G_1$ be $(\mathrm{Add}(\kappa, \nu^+) * \dot{\mathbb{P}})$-generic over V, $\mathbb{Q}$ be a σ-strategically closed partial order in $V[G_0][G_1]$ that preserve the regularity of κ and H be $\mathbb{Q}$-generic over $V[G_0][G_1]$. By Theorem 1.5, $\dot{A}^{G_0}$ is a $\mathbf{\Sigma}^1_1$-subset of ${}^{\kappa}\kappa$ in $V[G_0][G_1][H]$. This shows that both $\mathrm{ran}(\dot{e}^{G_0})$ and the relation

$$E = \{\langle\dot{e}^{G_0}(b), \dot{e}^{G_1}(c)\rangle \mid b, c \in \mathrm{tc}(\{X\}),\ b \in c\}$$

are elements of $L(\mathcal{P}(\kappa))$ in $V[G_0][G_1][H]$. Since this model can compute the transitive collapse of the well-founded and extensional relation $\langle\mathrm{ran}(\dot{b}^{G_0}), E\rangle$ and this function is equal to the inverse of $\dot{e}^{G_0}$, we can conclude that $\mathrm{tc}(\{X\})$ is an element of $L(\mathcal{P}(\kappa))$ in $V[G_0][G_1][H]$. Finally, we have

$$X = \{b \in \mathrm{tc}(\{X\}) \mid \prec\mathrm{id}_\kappa, \dot{e}^{G_0}(b)\succ \in \dot{A}^{G_0}\}$$

and we can conclude that X is also an element of $L(\mathcal{P}(\kappa))$ in $V[G_0][G_1][H]$. □

4. Almost disjoint coding

In [10, Section 1], Leo Harrington uses the method of *almost disjoint coding* forcing invented by Robert Solovay (see [14]) to prove Theorem 1.3. Working towards a proof of Theorem 1.6, we generalize this approach to uncountable cardinalities. Note that all results of this section are also true if κ is countable.

Definition 4.1. Given $A \subseteq {}^{\kappa}\kappa$, we define $\mathbb{Q}(A)$ to be the partial order consisting of conditions $p = \langle t_p, a_p\rangle$ with $t_p \in {}^{<\kappa}2$ and $a_p \in [A]^{<\kappa}$. The ordering $p \leq_{\mathbb{Q}(A)} q$ is defined by the following clauses.

(1) $t_q \subseteq t_p$ and $a_q \subseteq a_p$.
(2) $(\forall x \in a_q)(\forall \alpha \in \mathrm{dom}(t_p) \setminus \mathrm{dom}(t_q))$ $[s_\alpha \subseteq x \longrightarrow t_p(\alpha) = 0]$.

It is easy to check that this is in fact a partial order. In addition, it is easy to see that two conditions p and q are compatible if and only if t_p and t_q are compatible as elements of ${}^{<\kappa}2$ and $\langle t_p \cup t_q, a_p \cup a_q \rangle \leq_{\mathbb{Q}(A)} p, q$.

Lemma 4.2. *$\mathbb{Q}(A)$ is $<\kappa$-closed, satisfies the κ^+-chain condition and has cardinality at most 2^κ.*

Proof. If $\mu < \kappa$, $\langle p_\alpha \mid \alpha < \mu \rangle$ is a $\leq_{\mathbb{Q}(A)}$-descending sequence, $t = \bigcup_{\alpha<\mu} t_{p_\alpha}$ and $a = \bigcup_{\alpha<\mu} a_{p_\alpha}$, then $\langle t, a \rangle \in \mathbb{Q}(A)$ and $\langle t, a \rangle \leq_{\mathbb{Q}(A)} p_\alpha$ for all $\alpha < \mu$. It is easy to see that any two conditions in $\mathbb{Q}(A)$ with the same first coordinate are compatible and this shows that any antichain in $\mathbb{Q}(A)$ has cardinality at most $\kappa^{<\kappa} = \kappa$. The cardinality statement follows directly from our assumptions on κ. □

Proposition 4.3. *The following sets are dense subsets of $\mathbb{Q}(A)$.*

(1) $C_\mu = \{p \in \mathbb{Q}(A) \mid \mu \in \mathrm{dom}(t_p)\}$ *for all* $\mu < \kappa$.
(2) $D_x = \{p \in \mathbb{Q}(A) \mid x \in a_p\}$ *for all* $x \in A$.
(3) $E_{\alpha,y} = \{p \in \mathbb{Q}(A) \mid (\exists \beta \in \mathrm{dom}(t_p) \setminus \alpha)\ [t_p(\beta) = 1 \wedge s_\beta \subseteq y]\}$ *for all* $\alpha < \kappa$ *and* $y \in {}^\kappa\kappa \setminus A$.

Proof. (i) Given $\alpha < \kappa$ and $p \in \mathbb{Q}(A)$ with $\alpha \notin \mathrm{dom}(t_p)$, we define

$$t(\beta) = \begin{cases} t_p(\beta), & \text{if } \beta \in \mathrm{dom}(t_p), \\ 0, & \text{if } \beta \in (\alpha+1) \setminus \mathrm{dom}(s_p). \end{cases}$$

Obviously, $\langle t, a_p \rangle \leq_{\mathbb{Q}(A)} p$ and $\langle t, a_p \rangle \in C_\alpha$.

(ii) For all $p \in \mathbb{Q}(A)$, $p^* = \langle t_p, a_p \cup \{x\} \rangle \leq_{\mathbb{Q}(A)} p$ and $p^* \in D_x$.

(iii) Given $p \in \mathbb{Q}(A)$, there is an $\alpha < \beta \in \kappa \setminus \mathrm{dom}(s_p)$ with $x \restriction \beta \neq y \restriction \beta$ for all $x \in a_p$. We can find $\beta \leq \gamma < \kappa$ with $s_\gamma = y \restriction \beta$ and define $t : \gamma + 1 \longrightarrow 2$ by

$$t(\delta) = \begin{cases} t_p(\delta), & \text{if } \delta \in \mathrm{dom}(s_p), \\ 0, & \text{if } \delta \in \gamma \setminus \mathrm{dom}(t_p). \\ 1, & \text{if } \delta = \gamma. \end{cases}$$

Then $\langle t, a_p \rangle \leq_{\mathbb{Q}(A)} p$ and $\langle t, a_p \rangle \in E_{\alpha,y}$. □

The following theorem summarizes the properties of $\mathbb{Q}(A)$.

Theorem 4.4. *Let G be $\mathbb{Q}(A)$-generic over V. If we define $t_G = \bigcup\{t_p \mid p \in G\}$, then $t_G \in {}^\kappa 2$ and*

$$x \in A \Longleftrightarrow (\exists \beta < \kappa)(\forall \beta \leq \alpha < \kappa)\ [s_\alpha \subseteq x \rightarrow t_G(\alpha) = 0] \tag{7}$$

for all $x \in ({}^\kappa\kappa)^{\mathrm{V}}$. Moreover,

(8)
$$G = \{p \in \mathbb{Q}(A) \mid t_p \subseteq t_G \wedge (\forall \alpha \in \kappa \setminus \mathrm{dom}(t_p))(\forall x \in a_p)\ \ [s_\alpha \subseteq x \rightarrow t_G(\alpha) = 0]\}.$$

Proof. By Proposition 4.3, t_G is a function with domain κ and for every $x \in A$ there is a $p \in G$ with $x \in a_p$.

Assume, toward a contradiction, that $t_G(\alpha) = 1$ and $s_\alpha \subseteq x$ holds for some $\alpha \in \kappa \setminus \mathrm{dom}(t_p)$. There is a $q \in G$ with $q \leq_{\mathbb{Q}(A)} p$ and $\alpha \in \mathrm{dom}(t_q)$. But this means $0 = t_q(\alpha) = t_G(\alpha)$, a contradiction. Given $<\in {}^\kappa\kappa \setminus A$ and $\beta < \kappa$, there is $p \in G \cap E_{\beta,y}$ and this shows that there is an $\beta < \alpha < \kappa$ with $t_G(\alpha) = 1$ and $s_\alpha \subseteq y$.

Given $p \in G$, the above argument shows that p is also an element of the right set. Next, assume $p \in \mathbb{Q}(A)$ is a member of the set on the right. There is a $q \in G$ with $a_p \subseteq a_q$ and $t_p \subseteq t_q$. If $\alpha \in \mathrm{dom}(t_q) \setminus \mathrm{dom}(t_p)$ and $x \in a_p$ with $s_\alpha \subseteq x$, then $t_q(\alpha) = t_G(\alpha) = 0$. This shows $q \leq_{\mathbb{Q}(A)} p$ and $p \in G$. □

We close this section by introducing two forcing-theoretical properties and investigating their relevance to $\mathbb{Q}(A)$.

Definition 4.5. A partial order $\mathbb{P}$ is a *q-lattice ("quasi-lower-semi-lattice")* if the $\mathbb{P}$-minimum $p_0 \wedge_{\mathbb{P}} p_1$ exists for all compatible conditions $p_0, p_1 \in \mathbb{P}$. Let $\mathbb{Q}$ be a suborder of $\mathbb{P}$ and a q-lattice itself. We call $\mathbb{Q}$ a *sublattice of* $\mathbb{P}$ if $q_0 \wedge_{\mathbb{P}} q_1 = q_0 \wedge_{\mathbb{Q}} q_1$ holds for all $q_0, q_1 \in \mathbb{Q}$, which are compatible in $\mathbb{Q}$.

The partial order $\mathrm{Add}(\kappa, 1)$ is clearly a q-lattice and the remark following the definition of $\mathbb{Q}(A)$ directly implies that $\mathbb{Q}(A)$ is also a q-lattice with

$$p \wedge_{\mathbb{Q}(A)} q = \langle t_p \cup t_q, a_p \cup a_q \rangle$$

for all compatible $p, q \in \mathbb{Q}(A)$. Moreover, if $B \subseteq A$, then $\mathbb{Q}(B)$ is a sublattice of $\mathbb{Q}(A)$, every antichain in $\mathbb{Q}(B)$ is an antichain in $\mathbb{Q}(A)$ and every $\mathbb{Q}(B)$-nice name is a $\mathbb{Q}(A)$-nice name.

Definition 4.6. Let $\mathbb{P}$ be a partial order. We call $\dot{\mathbb{Q}} \in \mathrm{V}^{\mathbb{P}}$ a $\mathbb{P}$-*innocuous forcing* if there is a q-lattice $\mathbb{Q}_0$ with $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{\mathbb{Q}}$ *is a sublattice of* $\check{\mathbb{Q}}_0$".

We give a simple example of $\mathbb{P}$-innocuous forcings that will be important later.

Proposition 4.7. *If* $\mathbb{P}$ *is a* $<\kappa$*-closed forcing and* $\dot{\mathbb{Q}} \in \mathrm{V}^{\mathbb{P}}$ *with*

$$\mathbb{1}_{\mathbb{P}} \Vdash (\exists B) \ \left[B \subseteq \check{A} \wedge \dot{\mathbb{Q}} = \mathbb{Q}(B) \right],$$

then $\dot{\mathbb{Q}}$ *is a* $\mathbb{P}$*-innocuous forcing.*

Proof. Set $\mathbb{Q}_0 = \mathbb{Q}(A)$. We show $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{\mathbb{Q}}$ *is a sublattice of* $\check{\mathbb{Q}}_0$". Let G be $\mathbb{P}$-generic over V. We have $\mathbb{Q}_0 = \mathbb{Q}(A)^{\mathrm{V}[G]}$, because $\mathbb{P}$ is $<\kappa$-closed. An application of the above remarks in $\mathrm{V}[G]$ shows that $\dot{\mathbb{Q}}^G$ is a sublattice of $\mathbb{Q}_0$ in $\mathrm{V}[G]$. □

5. Innocent forcings

In this section, we complete the proof of Theorem 1.6. As mentioned in the Introduction, the $\mathbf{\Delta}^1_1$-coding we construct will have certain absoluteness properties. We are now ready to introduce the corresponding class of partial orders.

Definition 5.1. Let M be an inner model, ν be a cardinal, $\mathbb{P}$ be a partial order contained in M and G be $\mathbb{P}$-generic over M. We define $\Gamma_M(\mathbb{P}, G, \nu)$ to be the class of all $<\nu$-closed partial orders $\mathbb{Q}$ that satisfy the ν^+-chain condition and have the property that there is a $\mathbb{P}$-name $\dot{\mathbb{Q}}$ in M with $\mathbb{Q} = \dot{\mathbb{Q}}^G$ and

$$\langle M, \in \rangle \models \text{"}\dot{\mathbb{Q}} \textit{ is a } \mathbb{P}\textit{-innocuous forcing"}.$$

If $\mathbb{P}$ is a partial order and G is $\mathbb{P}$-generic over V, then results of Richard Laver (see [16, Theorem 3]) show that V is a class in $\mathrm{V}[G]$. In particular, $\Gamma_{\mathrm{V}}(\mathbb{P}, G, \nu)$ is a class in $\mathrm{V}[G]$ for every cardinal ν.

In the following, we continue to modify coding results from [10] to our context to prove the following *absoluteness version* of Theorem 1.6.

Theorem 5.2. *Let κ be a regular uncountable cardinal with $\kappa = \kappa^{<\kappa}$. For every subset A of ${}^\kappa\kappa$, there is a partial order $\mathbb{P}$ with the following properties.*

1. *$\mathbb{P}$ is $<\kappa$-closed, satisfies the κ^+-chain condition and has cardinality at most 2^κ.*
2. *If G is $\mathbb{P}$-generic over V, then A is a $\Gamma_{\mathrm{V}}(\mathbb{P}, G, \kappa)$-persistently $\mathbf{\Delta}^1_1$ in $\mathrm{V}[G]$.*

Following [10], we start by introducing another notion of forcing.

Definition 5.3. Given $A \subseteq {}^\kappa\kappa$, we define $\mathbb{Q}^+(A) = \bigoplus_{\gamma<\kappa^+} \mathbb{Q}(A)$ to be the κ^+-product forcing of $\mathbb{Q}(A)$ with $<\kappa$-support.

Lemma 5.4. *$\mathbb{Q}^+(A)$ is a $<\kappa$-closed q-lattice that satisfies the κ^+-chain condition and has cardinality at most 2^κ.*

Proof. Since $\mathbb{Q}^+(A)$ is the product with $<\kappa$-support and $\mathbb{Q}(A)$ is $<\kappa$-closed, it follows directly that $\mathbb{Q}^+(A)$ is also $<\kappa$-closed.

Given two compatible conditions $\vec{q}_0 = \left(q^0_\gamma\right)_{\gamma<\kappa^+}$ and $\vec{q}_1 = \left(q^1_\gamma\right)_{\gamma<\kappa^+}$, it is easy to check that q^0_γ and q^1_γ are compatible for all $\gamma < \kappa^+$ and $\left(q^0_\gamma \wedge_{\mathbb{Q}(A)} q^1_\gamma\right)_{\gamma<\kappa^+}$ is the $\mathbb{Q}^+(A)$-minimum of $\vec{q}_0$ and $\vec{q}_1$.

Assume, toward a contradiction, that $\langle \vec{q}_\delta \mid \delta < \kappa^+ \rangle$ enumerates an anti-chain in $\mathbb{Q}^+(A)$ with $\vec{q}_\delta = \left(q^\delta_\gamma\right)_{\gamma<\kappa^+}$ for each $\delta < \kappa^+$. By the Δ-System Lemma, we may assume that there is an $r \subseteq \kappa^+$ of cardinality less than κ such that $\mathrm{supp}(\vec{q}_\delta) \cap \mathrm{supp}(\vec{q}_{\bar\delta}) = r$ holds for all $\delta < \bar\delta < \kappa^+$. The set $\{\langle t_{q^\delta_\gamma} \in {}^{<\kappa}\kappa \mid \gamma \in r\rangle \mid \delta < \kappa^+\}$ is a subset of ${}^r({}^{<\kappa}\kappa)$ and this set has cardinality κ by our assumptions. Hence there are $\delta < \bar\delta < \kappa^+$ with $t_{q^\delta_\gamma} = t_{q^{\bar\delta}_\gamma}$ for all $\gamma \in r$. But this shows that $\vec{q}_\delta$ and $\vec{q}_{\bar\delta}$ are compatible in $\mathbb{Q}^+(A)$, a contradiction.

By our assumptions, the set $S = \{\mathrm{supp}(\vec{q}) \mid \vec{q} \in \mathbb{Q}^+(A)\}$ has cardinality κ^+ and for each $s \in S$ there are at most 2^κ-many $\vec{q} \in \mathbb{Q}^+(A)$ with $\mathrm{supp}(\vec{q}) = s$. □

Let $G = \bigoplus_{\gamma<\kappa^+} G_\alpha$ be $\mathbb{Q}^+(A)$-generic over V and $x \in ({}^\kappa\kappa)^{\mathrm{V}[G]}$. Since $\mathbb{Q}^+(A)$ satisfies the κ^+-chain condition in V, there is an $\delta < \kappa^+$ with $x \in \mathrm{V}[\langle G_\beta \mid \gamma < \delta\rangle]$. Now, Theorem 4.4 shows that

$$x \in A \iff (\forall \gamma < \kappa^+)(\exists \beta < \kappa)(\forall \beta \leq \alpha < \kappa)\ [s_\alpha \subseteq x \longrightarrow t_{G_\gamma}(\alpha) = 0] \tag{9}$$

holds in $\mathrm{V}[G]$. This shows that a $\mathbf{\Sigma}^1_1$-definition of the set $\{t_{G_\gamma} \in {}^\kappa\kappa \mid \gamma < \kappa^+\}$ would yield a $\mathbf{\Pi}^1_1$-definition of A in $\mathrm{V}[G]$. In order to make the set of all t_{G_γ}'s $\mathbf{\Sigma}^1_1$-definable, we need to show that the equivalence (9) also holds in certain forcing extensions of $\mathrm{V}[G]$. We introduce a class of forcings with this property.

Definition 5.5. Let $\mathbb{P}$ be a q-lattice. We call $\dot{\mathbb{Q}} \in \mathrm{V}^{\mathbb{P}}$ a $\mathbb{P}$-*innocent forcing* if $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{\mathbb{Q}}$ is a partial order" and there is a dense subset $D \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ such that

$$p_0 \wedge_{\mathbb{P}} p_1 \Vdash \text{“}\dot{q}_0 \textit{ and } \dot{q}_1 \textit{ are compatible in } \dot{\mathbb{Q}}\text{”}$$

holds for all compatible $\langle p_0, \dot{q}_0\rangle, \langle p_1, \dot{q}_1\rangle \in D$.

Lemma 5.6. *Let $\dot{\mathbb{P}}$ be an $\mathbb{Q}^+(A)$-innocent forcing with*

$$\mathbb{1}_{\mathbb{Q}^+(A)} \Vdash \text{“}\dot{\mathbb{P}} \textit{ is } <\check{\kappa}\textit{-closed and satisfies the } \check{\kappa}^+\textit{-chain condition”}.$$

If $G_0 * G_1$ is $(\mathbb{Q}^+(A) * \dot{\mathbb{P}})$-generic over V with $G_0 = \bigoplus_{\gamma<\kappa^+} \bar{G}_\gamma$, then

$$x \in A \iff (\forall \gamma < \kappa^+)(\exists \beta < \kappa)(\forall \beta \leq \alpha < \kappa)\ [s_\alpha \subseteq x \longrightarrow t_{\bar{G}_\gamma}(\alpha) = 0]$$

holds in $\mathrm{V}[G_0][G_1]$ for all $x \in ({}^\kappa\kappa)^{\mathrm{V}[G_0][G_1]}$.

Proof. Let D be a dense subset of $\mathbb{Q}^+(A) * \dot{\mathbb{P}}$ witnessing that $\dot{\mathbb{P}}$ is a $\mathbb{Q}^+(A)$-innocent forcing. Let $\dot{\eta}_0$ be a $\mathbb{Q}^+(A)$-name with with the property that, whenever G is $\mathbb{Q}^+(A)$-generic over V and $G = \bigoplus_{\gamma<\kappa^*} \bar{G}_\gamma$, then

$$\dot{\eta}_0^G : \kappa^+ \longrightarrow ({}^\kappa 2)^{\mathrm{V}[G]};\ \gamma \longmapsto t_{\bar{G}_\gamma}$$

Let $\dot{\eta}$ denote the canonical $(\mathbb{Q}^+(A) * \dot{\mathbb{P}})$-name corresponding to $\dot{\eta}_0$.

Assume, towards a contradiction, that there is a name $\dot{x} \in \mathrm{V}^{\mathbb{Q}^+(A)*\dot{\mathbb{P}}}$ and a condition $r_0 \in \mathbb{Q}^+(A) * \dot{\mathbb{P}}$ such that

$$r_0 \Vdash \text{“}\dot{x} \in ({}^{\check{\kappa}}\check{\kappa} \setminus \check{A})\ \wedge\ (\forall \gamma < \check{\kappa}^+)(\exists \beta < \check{\kappa})(\forall \beta \leq \alpha < \check{\kappa})\ [\check{s}_\alpha \subseteq \dot{x} \longrightarrow \dot{\eta}(\gamma)(\alpha) = 0]\text{”}$$

holds. Given $\alpha < \kappa$, we pick a maximal antichain $\mathcal{A}_\alpha$ in $\{r \in D \mid r \Vdash \text{“}\check{s}_{\check{\alpha}} \subseteq \dot{x}\text{”}\}$ and define $\mathcal{A} = \bigcup\{\mathcal{A}_\alpha \mid \alpha < \kappa\}$. Our assumptions imply that $\mathbb{Q}^+(A) * \dot{\mathbb{P}}$ satisfies the κ^+-chain condition and therefore $\mathcal{A}$ has cardinality κ. This shows that there is a $\gamma_* < \kappa^+$ with the property that, whenever $\langle \vec{q}, \dot{p}\rangle \in \mathcal{A}$ and $\vec{q} = (q_\gamma)_{\gamma<\kappa^+}$, then $q_{\gamma_*} = \mathbb{1}_{\mathbb{Q}(A)}$.

We can find an $r_1 \in D$ and $\beta_* < \kappa$ with $r_1 \leq_{\mathbb{Q}^+(A)*\dot{\mathbb{P}}} r_0$ and

$$r_1 \Vdash \text{“}(\forall \check{\beta}_* \leq \alpha < \check{\kappa})\ [\check{s}_\alpha \subseteq \dot{x} \longrightarrow \eta(\check{\gamma}_*)(\alpha) = 0]\text{”}.$$

Let $r_1 = \langle \vec{q}_1, \dot{p}_1\rangle$, $\vec{q}_1 = \left(q^1_\gamma\right)_{\gamma<\kappa^+}$ and $q^1_{\gamma_*} = \langle t_1, a_1\rangle$.

Now, let $G_0 * G_1$ be $(\mathbb{Q}^+(A) * \dot{\mathbb{P}})$-generic over V with $r_1 \in G_0 * G_1$ and set $x = \dot{x}^{G_0 * G_1}$. Since this partial order is $<\kappa$-closed and every initial segment of x is an element of V, we can find an $\alpha_* < \kappa$ with $\beta_* < \alpha_*$, $\mathrm{dom}(t_1) < \alpha_*$, $s_{\alpha_*} \subseteq x$ and $s_{\alpha_*} \nsubseteq y$ for all $y \in a_1$.

Our construction ensures that there is an $r_2 \in \mathcal{A}_{\alpha_*} \cap G$. Let $r_2 = \langle \vec{q}_2, \dot{p}_2\rangle$ and $\vec{q}_2 = \left(q^2_\gamma\right)_{\gamma<\kappa^+}$. The conditions r_1 and r_2 are compatible and elements of D. Hence, we can find an $r = \langle \vec{q}, \dot{p}\rangle \leq_{\mathbb{Q}^+(A)*\dot{\mathbb{P}}} r_1, r_2$ with $\vec{q} = (q_\gamma)_{\gamma<\kappa}$ and $q_\gamma = q^1_\gamma \wedge_{\mathbb{Q}(A)} q^2_\gamma$ for all $\gamma < \kappa^+$. In particular, $q_{\gamma_*} = q^1_{\gamma_*} = \langle t_1, a_1\rangle$.

We define $t_* \in {}^{<\kappa}\kappa$ by setting

$$t_*(\delta) = \begin{cases} t_1(\delta), & \text{if } \delta \in \mathrm{dom}(t_1), \\ 0, & \text{if } \delta \in \alpha_* \setminus \mathrm{dom}(t_1), \\ 1, & \text{if } \delta = \alpha_*. \end{cases}$$

By the choice of α_*, we have $\langle t_*, a_1\rangle \leq_{\mathbb{Q}(A)} \langle t_1, a_1\rangle = q_{\gamma^*}$. If we define

$$q^*_\gamma = \begin{cases} q_\gamma, & \text{if } \gamma \neq \gamma_*, \\ \langle t_*, a_1\rangle, & \text{if } \gamma = \gamma^*, \end{cases}$$

then $r_* = \langle \left(q^*_\gamma\right)_{\gamma<\kappa^+}, \dot{p}\rangle \leq r$. Let H be $(\mathbb{Q}^+(A) * \dot{\mathbb{P}})$-generic over V with $H = H_0 * H_1$, $H_0 = \bigoplus_{\gamma<\kappa^+} \bar{H}_\gamma$ and $r_* \in H$. The above construction yields $r_1 \in H$, $s_{\alpha_*} \subseteq \dot{x}^H$ and $t_{\bar{H}_{\gamma_*}}(\alpha_*) = 1$, a contradiction. □

Let $\mathbb{P}$ be a partial order, $\dot{\mathbb{Q}} \in \mathrm{V}^{\mathbb{P}}$ with $\mathbb{1}_{\mathbb{P}} \Vdash \text{“}\dot{\mathbb{Q}}$ *is a partial order* ” and G be $\mathbb{P}$-generic over V. Using $\mathbb{P}$, $\dot{\mathbb{Q}}$ and G as parameters, we can recursively define a class function

$$\mathrm{t}_G : \mathrm{V}^{\mathbb{P}*\dot{\mathbb{Q}}} \longrightarrow \mathrm{V}[G]^{\dot{\mathbb{Q}}^G}$$

in $\mathrm{V}[G]$ that satisfies

$$\mathrm{t}_G(\sigma) = \{\langle \mathrm{t}_G(\tau), \dot{q}^G\rangle \mid \langle \tau, \langle p, \dot{q}\rangle\rangle \in \sigma, p \in G\}$$

for all $\sigma \in \mathrm{V}^{\mathbb{P}*\dot{\mathbb{Q}}}$. If H is $\dot{\mathbb{Q}}^G$-generic over $\mathrm{V}[G]$, then an easy induction shows that $\sigma^{G*H} = \mathrm{t}_G(\sigma)^H$ holds for all $\sigma \in \mathrm{V}^{\mathbb{P}*\dot{\mathbb{Q}}}$. Given $\sigma \in \mathrm{V}^{\mathbb{P}*\dot{\mathbb{Q}}}$, we let $\mathrm{T}(\sigma)$ denote the class of all $\tau \in \mathrm{V}^{\mathbb{P}}$ such that $\tau^G = \mathrm{t}_G(\sigma)$ whenever G is $\mathbb{P}$-generic over V.

Next, suppose $\dot{\mathbb{R}} \in \mathrm{V}^{\mathbb{P}*\dot{\mathbb{Q}}}$ with $\mathbb{1}_{\mathbb{P}*\dot{\mathbb{Q}}} \Vdash$ "$\dot{\mathbb{R}}$ *is a partial order* " and $\dot{\mathbb{S}} \in \mathrm{V}^{\mathbb{P}}$. We write $\dot{\mathbb{S}} = \dot{\mathbb{Q}} *_{\mathbb{P}} \dot{\mathbb{R}}$ if there is a $\sigma \in \mathrm{T}(\dot{\mathbb{R}})$ with $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{\mathbb{S}} = \dot{\mathbb{Q}} * \sigma$".

By the above remarks, there is a map

$$\iota : (\mathbb{P} * \dot{\mathbb{Q}}) * \dot{\mathbb{R}} \longrightarrow \mathbb{P} * \dot{\mathbb{S}}$$

such that for every $\langle\langle p, \dot{q}\rangle, \dot{r}\rangle \in (\mathbb{P} * \dot{\mathbb{Q}}) * \dot{\mathbb{R}}$ there is an $\dot{s} \in \mathrm{V}^{\mathbb{P}}$ and $\rho \in \mathrm{T}(\dot{r})$ with $\iota(\langle p, \dot{q}\rangle, \dot{r}) = \langle p, \dot{s}\rangle$ and $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{s} = \langle \dot{q}, \rho\rangle$".

Lemma 5.7. *The map ι is a dense embedding.* □

Lemma 5.8. *Let $\mathbb{P}$ be a q-lattice, $\dot{\mathbb{Q}} \in \mathrm{V}^{\mathbb{P}}$ be a $\mathbb{P}$-innocuous forcing and $\dot{\mathbb{R}} \in \mathrm{V}^{\mathbb{P}*\dot{\mathbb{Q}}}$ be a $(\mathbb{P}*\dot{\mathbb{Q}})$-innocuous forcing. If $\dot{\mathbb{S}} \in \mathrm{V}^{\mathbb{P}}$ satisfies $\dot{\mathbb{S}} = \dot{\mathbb{Q}} *_{\mathbb{P}} \dot{\mathbb{R}}$, then $\dot{\mathbb{S}}$ is a $\mathbb{P}$-innocent forcing.*

Proof. Let $\mathbb{Q}_0$ witness that $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-innocuous forcing and $\mathbb{R}_0$ witness that $\dot{\mathbb{R}}$ is a $(\mathbb{P} * \dot{\mathbb{Q}})$-innocuous forcing. We define D_0 to be the set

$$\{\langle\langle p, \dot{q}\rangle, \dot{r}\rangle \in (\mathbb{P} * \dot{\mathbb{Q}}) * \dot{\mathbb{R}} \mid (\exists q \in \mathbb{Q}_0)(\exists r \in \mathbb{R}_0)\ [p \Vdash \text{“}\dot{q} = \check{q}\text{”} \wedge \langle p, \dot{q}\rangle \Vdash \text{“}\dot{r} = \check{r}\text{”}]\}.$$

Pick $\langle\langle p_0, \dot{q}_0\rangle, \dot{r}\rangle \in (\mathbb{P} * \dot{\mathbb{Q}}) * \dot{\mathbb{R}}$. There is a $\langle p_1, \dot{q}\rangle \leq_{\mathbb{P}*\dot{\mathbb{Q}}} \langle p_0, \dot{q}_0\rangle$ and $r \in \mathbb{R}_0$ with $\langle p_1, \dot{q}\rangle \Vdash$ "$\dot{r} = \check{r}$". In addition, there is a $p \leq_{\mathbb{P}} p_1$ and a $q \in \mathbb{Q}_0$ with $p \Vdash$ "$\dot{q} = \check{q}$". This means $\langle\langle p, \dot{q}\rangle, \dot{r}\rangle \in D_0$ and $\langle\langle p, \dot{q}\rangle, \dot{r}\rangle \leq_{(\mathbb{P}*\dot{\mathbb{Q}})*\dot{\mathbb{R}}} \langle\langle p_0, \dot{q}_0\rangle, \dot{r}\rangle$.

Let $\langle\langle p_0, \dot{q}_0\rangle, \dot{r}_0\rangle, \langle\langle p_1, \dot{q}_1\rangle, \dot{r}_1\rangle, \langle\langle p_2, \dot{q}_2\rangle, \dot{r}_2\rangle \in (\mathbb{P} * \dot{\mathbb{Q}}) * \dot{\mathbb{R}}$ with

$$\langle\langle p_0, \dot{q}_0\rangle, \dot{r}_0\rangle \leq_{(\mathbb{P}*\dot{\mathbb{Q}})*\dot{\mathbb{R}}} \langle\langle p_1, \dot{q}_1\rangle, \dot{r}_1\rangle, \langle\langle p_2, \dot{q}_2\rangle, \dot{r}_2\rangle$$

and $\langle\langle p_1, \dot{q}_1\rangle, \dot{r}_1\rangle, \langle\langle p_2, \dot{q}_2\rangle, \dot{r}_2\rangle \in D_0$. Fix conditions $q_1, q_2 \in \mathbb{Q}_0$ and $r_1, r_2 \in \mathbb{R}_0$ with $p_i \Vdash$ "$\dot{q}_i = \check{q}_i$" and $\langle p_i, \dot{q}_i\rangle \Vdash$ "$\dot{r}_i = \check{r}_i$". Clearly, $p_1 \wedge_{\mathbb{P}} p_2$ exists and there is a $p \in \mathbb{P}$ and $q \in \mathbb{Q}_0$ such that $p \leq_{\mathbb{P}} p_0$ and $p \Vdash$ "$\check{q} = \dot{q}_0 \leq_{\dot{\mathbb{Q}}} \check{q}_1, \check{q}_2$". But this shows that $q \leq_{\mathbb{Q}_0} q_1, q_2$ and $q_1 \wedge_{\mathbb{Q}_0} q_2$ exists. If we pick $\dot{q} \in \mathrm{V}^{\mathbb{P}}$ with $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{q} \in \dot{\mathbb{Q}}$" and $(p_1 \wedge_{\mathbb{P}} p_2) \Vdash$ "$\dot{q} = \check{q}_1 \wedge_{\check{\mathbb{Q}}_0} \check{q}_2$", then $\langle p_1 \wedge_{\mathbb{P}} p_2, \dot{q}\rangle \leq_{\mathbb{P}*\dot{\mathbb{Q}}} \langle p_1, \dot{q}_1\rangle, \langle p_2, \dot{q}_2\rangle$.

In the same way, we can show that $r_1 \wedge_{\mathbb{R}_0} r_2$ exists and there is an $\dot{r} \in \mathrm{V}^{\mathbb{R}}$ with $\mathbb{1}_{\mathbb{P}*\dot{\mathbb{Q}}} \Vdash$ "$\dot{r} \in \dot{\mathbb{R}}$" and $\langle p_1 \wedge_{\mathbb{P}} p_2, \dot{q}\rangle \Vdash$ "$\dot{r} = \check{r}_1 \wedge_{\check{\mathbb{R}}_0} \check{r}_2$". This means

$$\langle\langle p_1 \wedge_{\mathbb{P}} p_2, \dot{q}\rangle, \dot{r}\rangle \leq_{(\mathbb{P}*\dot{\mathbb{Q}})*\dot{\mathbb{R}}} \langle\langle p_1, \dot{q}_1\rangle, \dot{r}_1\rangle, \langle\langle p_2, \dot{q}_2\rangle, \dot{r}_2\rangle.$$

Define $D \subseteq \mathbb{P} * \dot{\mathbb{S}}$ to be the image of D_0 under ι. By the above Lemma, D is a dense subset of $D \subseteq \mathbb{P} * \dot{\mathbb{S}}$. Given two compatible conditions $d_0, d_1 \in D$ with $d_i = \iota(\langle p_i, \dot{q}_i\rangle, \dot{r}_i) = \langle p_i, \dot{s}_i\rangle$, we have shown that there are $\dot{q} \in \mathrm{V}^{\mathbb{P}}$ and $\dot{r} \in \mathrm{V}^{\mathbb{P}*\dot{\mathbb{Q}}}$ with

$$\langle\langle p_1 \wedge_{\mathbb{P}} p_2, \dot{q}\rangle, \dot{r}\rangle \leq_{(\mathbb{P}*\dot{\mathbb{Q}})*\dot{\mathbb{R}}} \langle\langle p_1, \dot{q}_1\rangle, \dot{r}_1\rangle, \langle\langle p_2, \dot{q}_2\rangle, \dot{r}_2\rangle.$$

This gives us an $\dot{s} \in \mathrm{V}^{\mathbb{P}}$ with

$$\langle p_1 \wedge_{\mathbb{P}} p_2, \dot{s}\rangle = \iota(\langle p_1 \wedge_{\mathbb{P}} p_2, \dot{q}\rangle, \dot{r}) \leq_{\mathbb{P}*\dot{\mathbb{S}}} d_0, d_1.$$

□

The techniques developed above allow us to prove the *absoluteness version* of our second main result.

Proof of Theorem 5.2. For the remainder of the proof, we fix a subset B of ${}^\kappa\kappa$ of cardinality κ^+ such that

$$(x)_\alpha = (y)_\beta \iff [x = y \wedge \alpha = \beta]$$

holds for all $x, y \in B$ and $\alpha, \beta < \kappa$. In addition, we fix an injective enumeration $\langle b_\gamma \mid \gamma < \kappa^+ \rangle$ of B.

By Theorem 1.5, there is a $<\kappa$-closed forcing $\mathbb{P}_0$ of cardinality at most 2^κ satisfying the κ^+-chain condition with the property that, whenever G_0 is $\mathbb{P}_0$-generic over V and $\mathbb{Q} \in \mathrm{V}[G_0]$ is a $<\kappa$-closed partial order, then both A and B are $\mathbf{\Sigma}^1_1$-subsets of $({}^\kappa\kappa)^{\mathrm{V}[G_0][G_1]}$ in every $\mathbb{Q}$-generic extension $\mathrm{V}[G_0][G_1]$ of $\mathrm{V}[G_0]$.

If G_0 is $\mathbb{P}_0$-generic over V, then $\mathbb{Q}(A)^{\mathrm{V}} = \mathbb{Q}(A)^{\mathrm{V}[G_0]}$ and $\mathbb{Q}^+(A)^{\mathrm{V}} = \mathbb{Q}^+(A)^{\mathrm{V}[G_0]}$. This shows that $\mathbb{P}_0 \times \mathbb{Q}^+(A)$ is a $<\kappa$-closed forcing that satisfies the κ^+-chain condition in V. In addition, there are names $\dot{C}, \dot{\mathbb{R}} \in \mathrm{V}^{\mathbb{P}_0 \times \mathbb{Q}^+(A)}$ with the property that, whenever $G_0 \times G_1$ is $(\mathbb{P}_0 \times \mathbb{Q}^+(A))$-generic over V with $G_1 = \bigoplus_{\gamma<\kappa^+} \bar{G}_\gamma$, then

$$\dot{C}^{G_0 \times G_1} = \{(b_\gamma)_{\prec\bar{\alpha},\alpha\succ} \in {}^\kappa\kappa \mid \alpha, \bar{\alpha} < \kappa, \gamma < \kappa^+, \bar{\alpha} = t_{\bar{G}_\gamma}(\alpha)\}$$

and $\dot{\mathbb{R}}^{G_0 \times G_1} = \mathbb{Q}(\dot{C}^{G_0 \times G_1})$ in $\mathrm{V}[G_0][G_1]$. Notice $\dot{C}^{G_0 \times G_1} \subseteq ({}^\kappa\kappa)^{\mathrm{V}}$ and $\dot{\mathbb{R}}^{G_0 \times G_1}$ is a sublattice of $\mathbb{Q}({}^\kappa\kappa)^{\mathrm{V}}$ in $\mathrm{V}[G_0][G_1]$. We define

$$\mathbb{P} = (\mathbb{P}_0 \times \mathbb{Q}^+(A)) * \dot{\mathbb{R}}.$$

This partial order is $<\kappa$-closed and satisfies the κ^+-chain condition.

In V, we define

$$D_0 = \{\langle p, \vec{q}, r\rangle \in \mathbb{P}_0 \times \mathbb{Q}^+(A) \times \mathbb{Q}({}^\kappa\kappa) \mid \langle p, \vec{q}\,\rangle \Vdash \text{“}\check{r} \in \dot{\mathbb{R}}\text{”}\}.$$

For each $\vec{d} = \langle p, \vec{q}, r\rangle \in D_0$, there is an $s_{\vec{d}} \in (\mathbb{P}_0 \times \mathbb{Q}^+(A)) * \dot{\mathbb{R}}$ with $s_{\vec{d}} = \langle\langle p, \vec{q}\rangle, \dot{r}\rangle$ and $\langle p, \vec{q}\rangle \Vdash$ “$\dot{r} = \check{r}$”. Clearly, there is a subset D of $\mathbb{P}$ that is closed under descending $\leq_{\mathbb{P}}$-sequences of length less than κ, has cardinality at most 2^κ and contains the dense subset $\{s_{\vec{d}} \mid \vec{d} \in D_0\}$. The partial order $\langle D, \leq_{\mathbb{P}}\restriction (D \times D)\rangle$ satisfies the κ^+-chain condition and is forcing-equivalent to $\mathbb{P}$. We continue to work with $\mathbb{P}$.

Let $G = (G_0 \times G_1) * G_2$ be $\mathbb{P}$-generic over V. There are trees $T_0, T_B \in \mathrm{V}[G_0]$ on $\kappa \times \kappa$ such that $A = p[T_0]^{\mathrm{V}[G_0][\bar{G}]}$ and $B = p[T_B]^{\mathrm{V}[G_0][\bar{G}]}$ hold in every generic extension $\mathrm{V}[G_0][\bar{G}]$ of $\mathrm{V}[G_0]$ by a $<\kappa$-closed forcing.

The results of Section 2 show that there is a tree $T_S \in \mathrm{V}[G]$ on $\kappa \times \kappa$ such that $p[T_S]$ is the set of all $x \in {}^\kappa\kappa$ with

$$(\exists y \in p[T_B])(\forall \alpha, \bar{\alpha} < \kappa)\ [x(\alpha) = \bar{\alpha} \longleftrightarrow (\exists \beta < \kappa)(\forall \beta \leq \bar{\beta} < \kappa) \\ [s_{\bar{\beta}} \subseteq (y)_{\prec\bar{\alpha},\alpha\succ} \longrightarrow t_{G_2}(\bar{\beta}) = 0]] \tag{10}$$

in every transitive ZFC-model that contains $\mathrm{V}[G]$ and has the same ${}^{<\kappa}\kappa$ as $\mathrm{V}[G]$.

Fix a $\gamma < \kappa^+$. The definition of $\dot{C}$ and the equivalence (7) imply that the function $b_\gamma \in B = p[T_B]^{\mathrm{V}[G]}$ witnesses that $t_{\bar{G}_\gamma} \in p[T_S]^{\mathrm{V}[G]}$ holds.

By the results of Section 2, there is a tree $T_1 \in \mathrm{V}[G]$ on $\kappa \times \kappa$ such that

$$p[T_1] = \{x \in {}^\kappa\kappa \mid (\exists y \in p[T_S])(\forall \beta < \kappa)(\exists \beta \leq \alpha < \kappa)\ [s_\alpha \subseteq x \wedge y(\alpha) = 1]\} \tag{11}$$

holds in every transitive ZFC-model that contains $\mathrm{V}[G]$ and has the same ${}^{<\kappa}\kappa$ as $\mathrm{V}[G]$.

Let $\mathbb{S}$ be an element of $\Gamma_{\mathrm{V}}(\mathbb{P}, G, \kappa)$. We work in $\mathrm{V}[G_0]$. Since $\dot{\mathbb{R}}^{G_0 \times G_1}$ is a sublattice of $\mathbb{Q}({}^\kappa\kappa)^{\mathrm{V}[G_0]}$, we can find a $\mathbb{Q}^+(A)$-innocuous forcing $\dot{\mathbb{R}}_0 \in \mathrm{V}[G_0]^{\mathbb{Q}^+(A)}$ with $\dot{\mathbb{R}}_0^{G_1} = \dot{\mathbb{R}}^{G_0 \times G_1}$. By our assumptions, there is a $(\mathbb{Q}^+(A) * \dot{\mathbb{R}}_0)$-innocuous forcing

$\dot{\mathbb{S}} \in \mathrm{V}[G_0]^{\mathbb{Q}^+(A)*\dot{\mathbb{R}}_0}$ with $\mathbb{S} = \dot{\mathbb{S}}^{G_1 * G_2}$. Pick $\dot{\mathbb{T}} \in \mathrm{V}[G_0]^{\mathbb{Q}^+(A)}$ with $\dot{\mathbb{T}} = \dot{\mathbb{R}}_0 *_{\mathbb{Q}^+(A)} \dot{\mathbb{S}}$. This means

$$\mathbb{1}_{\mathbb{Q}^+(A)} \Vdash \text{“}\dot{\mathbb{T}} \textit{ is } {<}\check{\kappa}\textit{-closed and satisfies the } \check{\kappa}^+\textit{-chain condition”}$$

and $\dot{\mathbb{T}}$ is a $\mathbb{Q}^+(A)$-innocent forcing by Lemma 5.8.

Let H be $\mathbb{S}$-generic over $\mathrm{V}[G]$. We have $A = p[T_0]^{\mathrm{V}[G][H]}$, $B = p[T_B]^{\mathrm{V}[G][H]}$ and $\{t_{\bar{G}_\gamma} \mid \gamma < \kappa^+\} \subseteq p[T_S]^{\mathrm{V}[G][H]}$.

Suppose $x \in p[T_S]^{\mathrm{V}[G][H]}$. Then x satisfies (10) in $\mathrm{V}[G][H]$ and there is a $\gamma < \kappa^+$ with

$$\begin{aligned} x(\alpha) = \bar{\alpha} &\iff (\exists \bar{\beta} < \kappa)(\forall \bar{\beta} \leq \delta < \kappa)\ [s_\delta \subseteq (b_\gamma)_{\prec \bar{\alpha}, \alpha \succ} \to t_{G_2}(\delta) = 0] \\ &\iff (b_\gamma)_{\prec \bar{\alpha}, \alpha \succ} \in \dot{C}^{G_0 \times G_1} \end{aligned}$$

for all $\alpha, \bar{\alpha} < \kappa$. We can conclude $x = t_{\bar{G}_\gamma}$ and $p[T_S]^{\mathrm{V}[G][H]} = \{t_{\bar{G}_\gamma} \mid \gamma < \kappa^+\}$.

There is a $\bar{H} \in \mathrm{V}[G][H]$ that is $\dot{\mathbb{T}}^{G_1}$-generic over $\mathrm{V}[G_0][G_1]$ with $\mathrm{V}[G][H] = \mathrm{V}[G_0][G_1][\bar{H}]$. The above remarks and Lemma 5.6 show that $x \in ({}^\kappa\kappa)^{\mathrm{V}[G][H]}$ is an element of A if and only if

$$(\forall \gamma < \kappa^+)(\exists \beta < \kappa)(\forall \beta \leq \alpha < \kappa)\ [s_\alpha \subseteq x \longrightarrow t_{\bar{G}_\gamma}(\alpha) = 0].$$

By the above computations, $x \in ({}^\kappa\kappa)^{\mathrm{V}[G][H]}$ is *not* an element of A if and only if

$$(\exists y \in p[T_S])(\forall \beta < \kappa)(\exists \beta \leq \alpha < \kappa)\ [s_\alpha \subseteq x \wedge y(\alpha) = 1]$$

holds in $\mathrm{V}[G][H]$. Since the equality (11) still holds in $\mathrm{V}[G][H]$, we can conclude

$$p[T_1]^{\mathrm{V}[G][H]} = ({}^\kappa\kappa)^{\mathrm{V}[G][H]} \setminus A.$$

□

6. Definable well-orders of ${}^\kappa\kappa$

This section is devoted to the proof of the following result that directly implies the statement of Theorem 1.9.

Theorem 6.1. *Let $\lhd$ be a well-ordering of ${}^\kappa\kappa$,*

$$A = \{\prec x, y \succ \mid x, y \in {}^\kappa\kappa \textit{ with either } x = y \textit{ or } x \lhd y\}$$

and G be $\mathbb{P}(A)$-generic over V.

1. *There is a well-ordering of $({}^\kappa\kappa)^{\mathrm{V}[G]}$ whose graph is a $\mathbf{\Delta}^1_2$-subset of ${}^\kappa\kappa$ in* $\mathrm{V}[G]$.
2. *The set $({}^\kappa\kappa)^{\mathrm{V}[G]}$ is Γ_κ-persistently $\mathbf{\Sigma}^1_1$ in $\mathrm{V}[G]$, where Γ_κ is the class of all ${<}\kappa$-closed partial orders in* $\mathrm{V}[G]$.

The idea behind the proof of this statement is to use $\lhd$ in the $\mathbb{P}(A)$-generic extension to define a well-ordering $\lhd^*$ of $\mathrm{H}(\kappa^+)^{\mathrm{V}}$ in $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ and well-order $({}^\kappa\kappa)^{\mathrm{V}[G]}$ by identifying functions in ${}^\kappa\kappa$ with the $\lhd^*$-least nice name in $\mathrm{H}(\kappa^+)^{\mathrm{V}}$ representing this function. We introduce some vocabulary needed in the following arguments.

Definition 6.2. Let Γ be a class of partial orders that contains the trivial partial order. We say that a set X is Γ-*persistently* $\mathbf{\Sigma}_1(\mathrm{H}(\kappa^+))$ if there is a Σ_1-formula $\varphi \equiv \varphi(u, v_0, \ldots, v_{n-1})$ and parameters $y_0, \ldots, y_{n-1} \in \mathrm{H}(\kappa^+)$ such that

$$X = \{x \in \mathrm{H}(\kappa^+)^{\mathrm{V}[G]} \mid \langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in \rangle \models \varphi(x, y_0, \ldots, y_{n-1})\}$$

holds whenever $\mathbb{Q}$ is a partial order in Γ and G is $\mathbb{Q}$-generic over V.

Proposition 6.3. *Let A be a subset of ${}^\kappa\kappa$ and G be $\mathbb{P}(A)$-generic over* V*. If Γ_κ denotes the class of all $<\kappa$-closed partial orders in* V$[G]$*, then the sets A, $\mathbb{P}(A)^{\mathrm{V}}$, G, $\mathbb{P}(A)^{\mathrm{V}} \setminus G$ and the relation*

$$\perp_{\mathbb{P}(A)^{\mathrm{V}}} = \{\langle p,q\rangle \in \mathbb{P}(A)^{\mathrm{V}} \times \mathbb{P}(A)^{\mathrm{V}} \mid p \textit{ and } q \textit{ are incompatible in } \mathbb{P}(A)^{\mathrm{V}}\}$$

are Γ_κ-persistently $\mathbf{\Sigma}_1(\mathrm{H}(\kappa^+)^{\mathrm{V}[G]})$.

Proof. We work in $\mathrm{V}[G]$. Theorem 1.5 directly implies that A is Γ_κ-persistently $\mathbf{\Sigma}_1(\mathrm{H}(\kappa^+)^{\mathrm{V}[G]})$. If $\mathrm{V}[G][H]$ is a generic extension of $\mathrm{V}[G]$ by a forcing in Γ_κ, then $\mathbb{P}(A)^{\mathrm{V}} = \mathbb{P}(A) = \mathbb{P}(A)^{\mathrm{V}[G][H]} \subseteq \mathrm{H}(\kappa^+)$ and the absoluteness of the definition of A implies that $\mathbb{P}(A)$ is Γ_κ-persistently $\mathbf{\Sigma}_1(\mathrm{H}(\kappa^+)^{\mathrm{V}[G]})$.

A pair $\langle p,q\rangle$ of conditions in $\mathbb{P}(A)$ is an incompatible in $\mathbb{P}(A)$ if and only if one of the following statements holds true.

1. T_p is not an end-extension of T_q or T_q is not an end-extension of T_p.
2. T_p is an end-extension of T_q and there is an $x \in \mathrm{dom}(f_p) \cap \mathrm{dom}(f_q)$ with either $f_q(x) \nsubseteq f_p(x)$ or $h_p(x) \neq h_q(x)$.
3. Same as (2), but with the roles of p and q exchanged.
4. T_p is an end-extension of T_q and there is an $x \in \mathrm{dom}(f_q) \setminus \mathrm{dom}(f_p)$ such that for all $z \in [T_p]$ with $f_q(x) \subseteq z$ there is $\beta < \kappa$ with $\prec h_q(x), \beta\succ < \mathrm{ht}(T_p)$ and either $s_\beta \subseteq x$ and $z(\prec h_q(x),\beta\succ) = 0$ or $s_\beta \nsubseteq x$ and $z(\prec h_q(x),\beta\succ) = 1$.
5. Same as (4), but with the roles of p and q exchanged.

Since all of those statements are absolute between $\mathrm{V}[G]$ and generic extensions of $\mathrm{V}[G]$ by forcings in Γ_κ, we can conclude that $\perp_{\mathbb{P}(A)^{\mathrm{V}}}$ is Γ_κ-persistent $\mathbf{\Sigma}_1(\mathrm{H}(\kappa^+))$.

Given $y \in A$ and $\gamma < \kappa$, the proof of Corollary 3.7 shows that $H_G(x) = \gamma$ holds if and only if there is a $z \in [T_G]$ such that (6) holds for all $\beta < \kappa$. By Lemma 3.6, $[T_G] = [T_G]^{\mathrm{V}[G][H]}$ holds whenever $\mathrm{V}[G][H]$ is a generic extension of $\mathrm{V}[G]$ by a forcing in Γ_κ. This shows that the graph of H_G is Γ_κ-persistently $\mathbf{\Sigma}_1(\mathrm{H}(\kappa^+))$. In combination with (4), this implies that the graph of F_G is Γ_κ-persistently $\mathbf{\Sigma}_1(\mathrm{H}(\kappa^+))$.

The filter G consists of all conditions p in $\mathbb{P}(A)$ such that T_G is an end extension of T_p and, if $x \in \mathrm{dom}(f_p)$, then $f_p(x) = F_G(x) \restriction \mathrm{ht}(T_p)$ and $h_p(x) = H_G(x)$. In combination with the above computations, this allows us to conclude that G is Γ_κ-persistently $\mathbf{\Sigma}_1(\mathrm{H}(\kappa^+))$.

Finally, a condition p in $\mathbb{P}(A)$ is not an element of G if there is a q in G that is incompatible with p. Using the above computations, $\mathbb{P}(A)^{\mathrm{V}} \setminus G$ is Γ_κ-persistently $\mathbf{\Sigma}_1(\mathrm{H}(\kappa^+))$. □

Proof of Theorem 6.1. (i) Work in $\mathrm{V}[G]$ and let Γ_κ denote the class of all $<\kappa$-closed partial orders in $\mathrm{V}[G]$. We have

$$x \in \mathrm{V} \iff \langle \mathrm{H}(\kappa^+), \in\rangle \models (\exists z \in A)(\forall \alpha < \kappa)\ x(\alpha) = z(\prec 0, \alpha\succ)$$

for all $z \in {}^\kappa\kappa$. This shows that $({}^\kappa\kappa)^{\mathrm{V}}$ is Γ_κ-persistently $\mathbf{\Sigma}^1_1(\mathrm{H}(\kappa^+))$.

Define $\psi \equiv \psi(u,v,w)$ to be the Σ_1-formula

$$\begin{aligned}(12)\quad & (\exists f : w \longrightarrow \mathrm{tc}(\{u\} \cup w) \textit{ bijection})(\forall \alpha, \beta < w) \\ & [(v(\prec 0, \prec\alpha,\beta\succ\succ) = 1 \leftrightarrow f(\alpha) \in f(\beta)) \wedge (v(\prec 1, \alpha\succ) = 1 \leftrightarrow f(\alpha) \in u)].\end{aligned}$$

Let $\mathrm{V}[G][H]$ be a generic extension of $\mathrm{V}[G]$ by a forcing in Γ_κ. Given a function $x \in ({}^\kappa 2)^{\mathrm{V}[G][H]}$, we let e_x denote the relation on κ defined by

$$\alpha\ e_x\ \beta \iff x(\prec 0, \prec\alpha,\beta\succ\succ) = 1.$$

If $\langle\kappa, e_x\rangle$ is well-founded and extensional, then we let t_x denote image of the corresponding collapsing map c_x and $a_x = \{c_x(\alpha) \mid x(\prec 1, \alpha\succ) = 1\}$.

Given $a \in \mathrm{H}(\kappa^+)^{\mathrm{V}[G][H]}$, there is an $x \in ({}^\kappa 2)^{\mathrm{V}[G][H]}$ such that $\psi(a, x, \kappa)$ holds in $\mathrm{H}(\kappa^+)^{\mathrm{V}[G][H]}$. Moreover, if $\psi(a, x, \kappa)$ holds in $\mathrm{H}(\kappa^+)^{\mathrm{V}[G][H]}$, then $\langle\kappa, e_x\rangle$ is well-founded and extensional, $a = a_x$, $\mathrm{tc}(\{a\} \cup \kappa) = t_x$ and c_x is the unique bijection witnessing that $\psi(a, x, \kappa)$ holds. In particular, if $a, b \in \mathrm{H}(\kappa^+)^{\mathrm{V}[G][H]}$ and $x \in ({}^\kappa 2)^{\mathrm{V}[G][H]}$ such that both $\psi(a, x, \kappa)$ and $\psi(b, x, \kappa)$ hold in $\mathrm{H}(\kappa^+)^{\mathrm{V}[G][H]}$, then $a = b$. Finally, these computations show that a is an element of $\mathrm{H}(\kappa^+)^{\mathrm{V}}$ if and only if $\psi(a, x, \kappa)$ holds in $\mathrm{H}(\kappa^+)^{\mathrm{V}[G][H]}$ for some $x \in ({}^\kappa 2)^{\mathrm{V}}$. We can conclude that $\mathrm{H}(\kappa^+)^{\mathrm{V}}$ is Γ_κ-persistently $\mathbf{\Sigma}_1(\mathrm{H}(\kappa^+))$.

Let N denote the set of all functions $n : \kappa \times \kappa \longrightarrow \mathbb{P}(A)$ in V with the property that the set $A^n_\alpha = \{n(\alpha, \beta) \in \mathbb{P}(A) \mid \beta < \kappa\}$ is an anti-chain in $\mathbb{P}(A)$ for all $\alpha < \kappa$. By Proposition 6.3 and the above computations, N is Γ_κ-persistently $\mathbf{\Sigma}_1(\mathrm{H}(\kappa^+))$.

By the results of Section 2, there is a tree T on κ^3 with the property that, whenever $\mathrm{V}[G][H]$ is a generic extension of $\mathrm{V}[G]$ by a forcing in Γ_κ, then $p[T]^{\mathrm{V}[G][H]}$ is equal to the set of all $\langle x, y\rangle \in ({}^\kappa\kappa)^{\mathrm{V}[G][H]} \times ({}^\kappa 2)^{\mathrm{V}}$ such that

$$\begin{aligned} \psi(n, y, \kappa) \wedge (\forall \alpha, \beta < \kappa)\ [(x(\alpha) = \beta \rightarrow (\exists \gamma < \kappa)\ n(\prec\alpha, \beta\succ, \gamma) \in G) \\ \wedge\ (x(\alpha) \neq \beta \rightarrow (\forall \gamma < \kappa)\ n(\prec\alpha, \beta\succ, \gamma) \notin G)]. \end{aligned} \tag{13}$$

holds in $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G][H]}, \in\rangle$ for some $n \in N$. For every $x \in {}^\kappa\kappa$ there is a $y \in ({}^\kappa 2)^{\mathrm{V}}$ with $\langle x, y\rangle \in p[T]$, because there is an $n \in N$ such that

$$\tau_n^G = \{\prec\alpha, \beta\succ \mid \alpha, \beta < \kappa,\ x(\alpha) = \beta\},$$

where τ_n is the $\mathbb{P}(A)$-nice name $\bigcup_{\alpha<\kappa} \{\check{\alpha}\} \times A^n_\alpha$. Moreover, if $\langle x_0, y\rangle, \langle x_1, y\rangle \in p[T]$, then $x_0 = x_1$.

Now, define a relation $\lhd^*$ on ${}^\kappa\kappa$ by setting

$$\begin{aligned} x_0 \lhd^* x_1 \iff (\exists z_0, z_1 \in ({}^\kappa 2)^{\mathrm{V}})\ [\langle x_0, z_0\rangle, \langle x_1, z_1\rangle \in p[T] \wedge z_0 \lhd z_1 \\ \wedge\ (\forall \bar{z}_0, \bar{z}_1 \in ({}^\kappa 2)^{\mathrm{V}})[(\bar{z}_0 \lhd z_0 \wedge \bar{z}_1 \lhd z_1) \rightarrow (\langle x_0, \bar{z}_0\rangle \notin p[T] \vee \langle x_1, \bar{z}_1\rangle \notin p[T])]]. \end{aligned}$$

By the above constructions and the results of Section 2, the graph of this relation is a $\mathbf{\Sigma}^1_2$-subset of ${}^\kappa\kappa \times {}^\kappa\kappa$. It is easy to check that this relation is linear, strict and total. In particular, its graph is a $\mathbf{\Delta}^1_2$-subset of ${}^\kappa\kappa \times {}^\kappa\kappa$. Assume, toward a contradiction, that there is a strictly $\lhd^*$-descending sequence of elements in ${}^\kappa\kappa$ of length ω. The definition gives us a strictly $\lhd$-descending sequence of elements in $({}^\kappa 2)^{\mathrm{V}}$ of the same length. Since $\mathbb{P}(A)$ is σ-closed, this sequence is an element of V, a contradiction.

(ii) By Proposition 2.1, there is a tree T_* on $\kappa \times \kappa$ such that $p[T_*]^{\mathrm{V}[G][H]} = \exists^x(p[T]^{\mathrm{V}[G][H]})$ holds whenever $\mathrm{V}[G][H]$ is a generic extension of $\mathrm{V}[G]$ by a forcing in Γ_κ. Let $\mathrm{V}[G][H]$ be such an extension and x be an element of $p[T_*]^{\mathrm{V}[G][H]}$. There is a $y \in ({}^\kappa\kappa)^{\mathrm{V}[G][H]}$ with $\langle x, y\rangle \in p[T]^{\mathrm{V}[G][H]}$. By the construction of T, y is an element of $({}^\kappa 2)^{\mathrm{V}}$ and there is an $n \in N$ witnessing that (13) holds in $\mathrm{H}(\kappa^+)^{\mathrm{V}[G][H]}$. In V, we can construct the $\mathbb{P}(A)$-nice name τ_n and, since $\tau_n^G \in \mathrm{V}[G]$, we can conclude $x \in \mathrm{V}[G]$. This shows that $p[T_*]^{\mathrm{V}[G][H]} \subseteq ({}^\kappa\kappa)^{\mathrm{V}[G]}$ and the above computations already show $({}^\kappa\kappa)^{\mathrm{V}[G]} = \exists^x(p[T]^{\mathrm{V}[G]}) = p[T_*]^{\mathrm{V}[G]} \subseteq p[T_*]^{\mathrm{V}[G][H]}$. $\square$

7. The perfect subset property and absoluteness

We generalize the perfect subset property of subsets of Baire space to subsets of arbitrary function spaces ${}^\kappa\kappa$ and establish a connection between this property and generic absoluteness.

Definition 7.1. Let λ be a limit ordinal.

We say that a map $\iota : {}^{<\lambda}2 \longrightarrow ({}^{<\lambda}\lambda)^n$ is a *continuous order-embedding* if the following statements hold for all $s_0, s_1 \in {}^{<\lambda}2$ with $\iota(s_i) = \langle t^i_0, \ldots, t^i_{n-1}\rangle$.

1. If $s_0 \subsetneq s_1$, then $t^0_k \subsetneq t^1_k$ for all $k < n$.
2. If s_0 and s_1 are incompatible in ${}^{<\lambda}2$, then there is a $k < n$ such that t^0_k and t^1_k are incompatible in ${}^{<\lambda}\lambda$.
3. If $\mathrm{lh}(s_0) \in \mathrm{Lim} \cap \lambda$ and $k < n$, then
$$t^0_k = \bigcup \{u^\alpha_k \mid (\exists \alpha < \mathrm{lh}(s_0))\ \iota(s_0 \upharpoonright \alpha) = \langle u^\alpha_0, \ldots, u^\alpha_{n-1}\rangle\}.$$

Definition 7.2. Let λ be a limit ordinal and A be a subset of ${}^\lambda\lambda$. We say that A *contains a perfect subset* if there is a continuous order-embedding $\iota : {}^{<\lambda}2 \longrightarrow {}^{<\lambda}\lambda$ such that $[T_\iota] \subseteq A$, where T_ι is the tree
$$T_\iota = \{t \in {}^{<\lambda}\lambda \mid (\exists s \in {}^{<\lambda}2)\ t \subseteq \iota(s)\}.$$
on λ.

Let $\mathcal{C}$ be a class of subsets of ${}^\kappa\kappa$. We say that subsets in $\mathcal{C}$ have the *perfect subset property* if every subset in $\mathcal{C}$ of cardinality bigger than κ contains a perfect subset. We present existing results related to the above definitions following [8, Chapter IV].

- We call a tree T on κ a *weak κ-Kurepa tree* if $\mathrm{ht}(T) = \kappa$, $[T]$ has cardinality at least κ^+ and there are stationary many $\alpha < \kappa$ such that $T \cap {}^\alpha\kappa$ has the same cardinality as α. The idea of using Kurepa trees to construct closed subsets without the perfect subset property goes back to [17, Section 5].

  Let $\iota : {}^{<\kappa}2 \longrightarrow {}^{<\kappa}\kappa$ be a continuous order-embedding and T be a tree on κ of height κ with $[T_\iota] \subseteq [T]$. First, assume that there is an $\alpha < \kappa$ such that $\iota"{}^\alpha 2 \nsubseteq {}^{<\beta}\kappa$ for all $\beta < \kappa$. Let α be minimal with this property. By the regularity of κ, there is a $\beta < \kappa$ with $\iota"{}^{<\alpha}2 \subseteq {}^{<\beta}\kappa$. The set $C = \{s \in {}^\alpha 2 \mid \mathrm{lh}(\iota(s)) \geq \beta\}$ has cardinality κ and $\iota(s) \upharpoonright \beta \in T$ for all $s \in C$. We can conclude that $T \cap {}^\beta\kappa$ has cardinality at least κ in this case. Now, assume that for every $\alpha < \kappa$ there is a $\beta < \kappa$ with $\iota"{}^\alpha 2 \subseteq T \cap {}^{<\beta}\kappa$. Then the set $\{\alpha < \kappa \mid \iota"{}^\alpha 2 \subseteq T \cap {}^\alpha\kappa\}$ is closed and unbounded in κ. In both cases, T is not a weak κ-Kurepa tree.

  The existence of weak κ-Kurepa trees therefore provides examples for the failure of the perfect subset property for closed subsets of ${}^\kappa\kappa$. In particular, if "V = L" holds, then the perfect subset property for closed sets fails for all uncountable regular cardinals (see [8, Section IV.2]).
- If all closed subsets of ${}^\kappa\kappa$ have the perfect subset property, then there are no κ-Kurepa trees and κ^+ is inaccessible in L by an argument of Robert Solovay (see [13, Section 4]).
- Let $\nu > \kappa$ be an inaccessible cardinal and G be $\mathrm{Col}(\kappa, {<}\nu)$-generic over V. An argument of Philipp Schlicht shows that $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$ in V$[G]$ have the perfect subset property. We will provide a proof of this statement in Section 9 (Proposition 9.9).

- Large cardinal properties of κ do not imply the perfect subset property for closed subsets of ${}^\kappa\kappa$. If κ is a supercompact cardinal, then there is a partial order that preserves the supercompactness of κ and adds a weak κ-Kurepa tree (see [8, Section IV.2]).

To further investigate the perfect subset property for $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$, we need a well-known result saying that ZFC proves *generic absoluteness for $\mathbf{\Sigma}^1_1$-subsets under $<\kappa$-closed forcings.*

Proposition 7.3. *Let T be a tree on κ^n of height κ and $\mathbb{P}$ be a $<\kappa$-closed partial order. If there is a $p \in \mathbb{P}$ with $p \Vdash$ "$[\check{T}] \neq \emptyset$", then $[T] \neq \emptyset$.*

Proof. Let $p \Vdash$ "$\langle \tau_0, \dots, \tau_{n-1} \rangle \in [\check{T}]$" for some names $\tau_0, \dots, \tau_{n-1} \in \mathrm{V}^{\mathbb{P}}$. Given $\alpha < \kappa$, the set of conditions $q \in \mathbb{P}$ with

$$(\exists \langle t_0, \dots, t_{n-1} \rangle \in T) \ \left[\mathrm{lh}(t_0) \geq \alpha \wedge q \Vdash \text{“}\check{t}_0 \subseteq \tau_0 \wedge \dots \wedge \check{t}_{n-1} \subseteq \tau_{n-1}\text{”}\right]$$

is dense below p. Since $\mathbb{P}$ is $<\kappa$-closed, we can define a $\leq_{\mathbb{P}}$-descending sequence $\langle p_\alpha \in \mathbb{P} \mid \alpha < \kappa \rangle$ and an ascending sequence $\langle \langle t_0^\alpha, \dots, t_{n-1}^\alpha \rangle \in T \mid \alpha < \kappa \rangle$ in V such that $p_0 = p$, $\mathrm{lh}(t_0^\alpha) \geq \alpha$ and $p_\alpha \Vdash$ "$\check{t}_i^\alpha \subseteq \tau_i$" holds for all $\alpha < \kappa$ and $i < n$. But this construction implies that the tuple $\langle \bigcup_{\alpha<\kappa} t_0^\alpha, \dots, \bigcup_{\alpha<\kappa} t_{n-1}^\alpha \rangle$ is an element of $[T]$ in V. □

We look at a stronger version of the perfect subset property for $\mathbf{\Sigma}^1_1$-subsets.

Definition 7.4. Let T be a tree on κ^{n+1}. A *$\exists^x$-perfect ("projection-perfect") embedding into T* is a continuous order-embedding $\iota : {}^{<\kappa}2 \longrightarrow ({}^{<\kappa}\kappa)^{n+1}$ with the following properties.

1. $\mathrm{ran}(\iota) \subseteq T$.
2. If $s_0, s_1 \in {}^{<\kappa}2$ are incompatible sequences with $\iota(s_i) = \langle t_0^i, \dots, t_n^i \rangle$, then there is a $k < n$ such that the sequences t_k^0 and t_k^1 are incompatible in ${}^{<\kappa}\kappa$.

The idea behind the above definition is that a $\exists^x$-perfect embedding into T witnesses that the projection $p[T]$ has a perfect subset.

Proposition 7.5. *Let T be a tree on $\kappa \times \kappa$ and ι be a $\exists^x$-perfect embedding into T. If we define $\bar{\iota} : {}^{<\kappa}2 \longrightarrow {}^{<\kappa}\kappa$ to be the continuous order-embedding such that $\bar{\iota}(s) = t_0$ for all $s \in {}^{<\kappa}2$ with $\iota(s) = \langle t_0, t_1 \rangle$, then $\bar{\iota}$ witnesses that $p[T]$ has a perfect subset in every transitive* ZFC-*model containing* V. □

The following lemma establishes a connection between the existence of $\exists^x$-perfect embeddings and absoluteness properties of $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$.

Lemma 7.6. *The following statements are equivalent for every tree T on $\kappa \times \kappa$ of height κ.*

1. *There is a $\exists^x$-perfect embedding into T.*
2. *If $\mathbb{P}$ is $<\kappa$-closed partial order, then $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\mathcal{P}(\check{\kappa}) \nsubseteq \check{\mathrm{V}} \rightarrow p[\check{T}] \nsubseteq \check{\mathrm{V}}$".*
3. $\mathbb{1}_{\mathrm{Add}(\kappa,1)} \Vdash$ *"$p[\check{T}] \nsubseteq \check{\mathrm{V}}$".*
4. *There is a $<\kappa$-closed partial order $\mathbb{P}$ with $\mathbb{1}_{\mathbb{P}} \Vdash$ "$p[\check{T}] \nsubseteq \check{\mathrm{V}}$".*

Proof. Assume (i) holds, ι is a $\exists^x$-perfect embedding into T and $\mathbb{P}$ is a $<\kappa$-closed partial order that adds a new subset of κ. If we define

$$S = \{\langle t_0 \restriction \alpha, t_1 \restriction \alpha \rangle \in T \mid \langle t_0, t_1 \rangle \in \mathrm{ran}(\iota), \ \alpha \leq \mathrm{lh}(t_0)\},$$

then S is a subtree of T of height κ.

Let G be $\mathbb{P}$-generic over V, $x_0 \in ({}^{\kappa}2)^{\mathrm{V}[G]} \setminus \mathrm{V}$ and define

$$y = \bigcup\{t_0 \mid (\exists \alpha < \kappa)\ \iota(x_0 \restriction \alpha) = \langle t_0, t_1\rangle\},$$

Clearly, $y \in p[S]^{\mathrm{V}[G]} \subseteq p[T]^{\mathrm{V}[G]}$. Assume, toward a contradiction, that $y \in \mathrm{V}$ holds. Then the tree $S_y = \{t \in {}^{<\kappa}\kappa \mid \langle y \restriction \mathrm{lh}(t), t\rangle \in S\}$ is an element of V and $[S_y]^{\mathrm{V}[G]} \neq \emptyset$. By Proposition 7.3, there is a $z \in [S_y]^{\mathrm{V}}$ and this means $\langle y, z\rangle \in [S]^{\mathrm{V}}$. But this means that there is an $x_1 \in ({}^{\kappa}2)^{\mathrm{V}}$ with $y = \bigcup\{t_0 \mid (\exists \alpha < \kappa)\ \iota(x_1 \restriction \alpha) = \langle t_0, t_1\rangle\}$. Given $\alpha < \kappa$ with $x_0(\alpha) \neq x_1(\alpha)$ and $\iota(x_i \restriction (\alpha+1)) = \langle t_0^i, t_1^i\rangle$, we have t_0^0 and t_0^1 incompatible and $t_0^0, t_0^1 \subseteq y$, a contradiction.

Now, assume (iv) holds. Fix $\tau_0, \tau_1 \in \mathrm{V}^{\mathbb{P}}$ with $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\tau_0 \notin \mathrm{V} \wedge \langle \tau_0, \tau_1\rangle \in [\check{T}]$". We inductively construct order-embeddings $i : {}^{<\kappa}2 \longrightarrow \mathbb{P}$ and $\iota : {}^{<\kappa}2 \longrightarrow T$ with the following properties.

(1) ι is continuous.
(2) If $s \in {}^{<\kappa}2$ and $\iota(s) = \langle t_0, t_1\rangle$, then $i(s) \Vdash$ "$\check{t}_0 \subseteq \tau_0 \wedge \check{t}_1 \subseteq \tau_1$".
(3) If $s_0, s_1 \in {}^{<\kappa}2$ are incompatible, then $\iota(s_0), \iota(s_1) \in T$ are incompatible.

Assume that $i \restriction {}^{<\alpha}2$ and $\iota \restriction {}^{<\alpha}2$ are already constructed for some $\alpha < \kappa$. If $\alpha \in \mathrm{Lim}$ and $s \in {}^{\alpha}2$, then there is a condition $i(s) \in \mathbb{P}$ with $p \leq_{\mathbb{P}} i(s \restriction \bar{\alpha})$ for all $\bar{\alpha} < \alpha$. Define $\langle t_0, t_1\rangle \in {}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$ by setting

$$t_i = \bigcup\{\bar{t}_i \mid (\exists \bar{\alpha} < \alpha)\ \iota(s \restriction \alpha) = \langle \bar{t}_0, \bar{t}_1\rangle\}.$$

By construction, $i(s) \Vdash$ "$\check{t}_i \subseteq \tau_i$" and this means $\langle t_0, t_1\rangle \in T$. Moreover, given incompatible $s_0, s_1 \in {}^{\alpha}2$, there is an $\bar{\alpha} < \alpha$ such that $s_0 \restriction \bar{\alpha}$ and $s_1 \restriction \bar{\alpha}$ are incompatible and our assumptions imply that $\iota(s_0)$ and $\iota(s_1)$ are also incompatible.

If $\alpha = \bar{\alpha}+1$ and $s \in {}^{\bar{\alpha}}2$, then there are conditions $q_0, q_1 \leq_{\mathbb{P}} i(s)$ and $\beta, \gamma_0, \gamma_1 < \kappa$ with $\beta \geq \mathrm{lh}(\iota(s))$, $q_i \Vdash$ "$\tau_0(\check{\beta}) = \check{\gamma}_i$" and $\gamma_0 \neq \gamma_1$, because we have $i(s) \Vdash$ "$\tau_0 \notin \mathrm{V}$". Given $i < 2$, we can find $i(s^\frown\langle i\rangle) \in \mathbb{P}$ and $\iota(s^\frown\langle i\rangle) = \langle t_0^i, t_1^i\rangle \in T$ with $i(s^\frown\langle i\rangle) \leq q_i$, $\mathrm{lh}(t_0^i) = \beta + 1$ and $i(s^\frown\langle i\rangle) \Vdash$ "$\check{t}_0^i \subseteq \tau_0 \wedge \check{t}_1^i \subseteq \tau_1$". It is easy to check that this partial embedding also satisfies the above properties. □

In the following, we investigate the correlation between the existence of a perfect subset of $\mathbf{\Delta}^1_1$-subsets of the form $p[T_0]$ and the existence of $\exists^x$-perfect embeddings into T_0. We need another notion of absoluteness.

Definition 7.7. Let Γ be a class of partial orders. We say that a subset A of ${}^{\kappa}\kappa$ is *weakly* Γ*-persistently* $\mathbf{\Delta}^1_1$ if there are trees T_0 and T_1 on $\kappa \times \kappa$ such that $p[T_0] = A$, $p[T_1] = {}^{\kappa}\kappa \setminus A$ and $\mathbb{1}_{\mathbb{P}} \Vdash$ "$p[\check{T}_1] = {}^{\check{\kappa}}\check{\kappa} \setminus p[\check{T}_0]$" holds for all partial orders $\mathbb{P}$ in Γ.

Proposition 7.8. *Let $\mathbb{P}$ be a $<\kappa$-closed partial order that adds a new subset of κ, A be a subset of ${}^{\kappa}\kappa$ and T_0, T_1 be trees on $\kappa \times \kappa$ witnessing that A is weakly $\mathbb{P}$-persistently $\mathbf{\Delta}^1_1$. Then A has a perfect subset if and only if $\mathbb{1}_{\mathbb{P}} \nVdash$ "$\check{A} = p[\check{T}_0]$".*

Proof. Pick $p \in \mathbb{P}$ with $p \Vdash$ "$\check{A} \neq p[\check{T}_0]$". Assume, towards a contradiction, that there is a $q \leq_{\mathbb{P}} p$ with $q \Vdash$ "$p[\check{T}_0] \subseteq \check{\mathrm{V}}$". Let G be $\mathbb{P}$-generic over V with $q \in G$ and pick $y \in p[T_0]^{\mathrm{V}[G]} \setminus A \subseteq \mathrm{V}$. Define $T_y = \{t \in {}^{<\kappa}\kappa \mid \langle y \restriction \mathrm{lh}(t), t\rangle \in T_0\} \in \mathrm{V}$. Then $[T_y]^{\mathrm{V}[G]} \neq \emptyset$ and this means $[T_y]^{\mathrm{V}} \neq \emptyset$ by Proposition 7.3. But this implies $y \in p[T_0]^{\mathrm{V}} = A$, a contradiction. Therefore $p \Vdash$ "$p[\check{T}_0] \nsubseteq \check{\mathrm{V}}$" and A has a perfect subset by Lemma 7.6.

In the other direction, let $\iota : {}^{<\kappa}2 \longrightarrow {}^{<\kappa}\kappa$ witnesses that A has a perfect subset and assume, toward a contradiction, that $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\check{A} = p[\check{T}_0]$" holds. Let G be $\mathbb{P}$-generic over V. By construction, $[T_\iota]^{\mathrm{V}[G]} \nsubseteq \mathrm{V}$, $p[T_0]^{\mathrm{V}[G]} = A \subseteq \mathrm{V}$ and $p[T_1]^{\mathrm{V}[G]} =$

$({}^\kappa\kappa)^{\mathrm{V}[G]} \setminus p[T_0]^{\mathrm{V}[G]} = ({}^\kappa\kappa)^{\mathrm{V}[G]} \setminus A$. If we define $T = \{\langle t_0, t_1\rangle \in T_1 \mid t_0 \in T_\iota\} \in \mathrm{V}$, then $[T]^{\mathrm{V}[G]} \neq \emptyset$ and therefore $[T]^{\mathrm{V}} \neq \emptyset$. But this shows that $\emptyset \neq [T_\iota]^{\mathrm{V}} \cap p[T_1]^{\mathrm{V}} \subseteq p[T_0]^{\mathrm{V}} \cap p[T_1]^{\mathrm{V}} = \emptyset$, a contradiction. □

Corollary 7.9. *Let A be a subset of ${}^\kappa\kappa$ that is weakly Γ-persistently $\mathbf{\Delta}^1_1$, where Γ is a class of $<\kappa$-closed partial orders that contains both the trivial partial order and a partial order that adds a new subset of κ. Then A contains no perfect subset if and only if it is Γ-persistently $\mathbf{\Delta}^1_1$.* □

In particular, if A is a subset of ${}^\kappa\kappa$ and G is $\mathbb{P}(A)$-generic over V, then A contains no perfect subset in $\mathrm{V}[G]$.

In combination with Theorem 6.1, the above results allow us to show that generic absoluteness for $\mathbf{\Sigma}^1_3$-subsets of ${}^\kappa\kappa$ under $<\kappa$-closed forcings that satisfy the κ^+-chain condition is inconsistent.

Proof of Theorem 1.11. We fix an $a \in \mathcal{P}(\kappa)$ with ${}^{<\kappa}\kappa \in \mathrm{L}[a]$ and bijections

$$f : \kappa \longrightarrow \{\langle t_0, t_1\rangle \in {}^{<\kappa}\kappa \times {}^{<\kappa}\kappa \mid \mathrm{lh}(t_0) = \mathrm{lh}(t_1)\}$$

and $g : \kappa \longrightarrow {}^{<\kappa}2$ contained in $\mathrm{L}[a]$. Given $x \in {}^\kappa\kappa$, we define $\iota_x = f \circ x \circ g^{-1}$ and $T_x = \{f(\alpha) \mid x(\alpha) = 1\}$. By the results of Section 2, there is a tree $T \in \mathrm{L}[a]$ on κ^3 such that

$$p[T] = \{\langle x, y\rangle \in {}^\kappa\kappa \times {}^\kappa\kappa \mid \text{“}T_x \textit{ is a tree on } \kappa \times \kappa\text{”} \ \wedge \ y \in p[T_x] \ \wedge \ \text{“}\iota_y \textit{ is not a } \exists^x\textit{-perfect embedding into } T_x\text{”}\}$$

holds in every transitive ZFC-model that contains $\mathrm{L}[a]$ and has the same ${}^{<\kappa}\kappa$ as $\mathrm{L}[a]$. This implies that in any ZFC-model with the above properties

$$(\exists x \in {}^\kappa\kappa)(\forall y \in {}^\kappa\kappa)\ \langle x, y\rangle \in p[T] \tag{14}$$

is equivalent to the existence of a tree T_* on $\kappa \times \kappa$ such that “$p[T_*] = {}^\kappa\kappa$” holds and there is no $\exists^x$-perfect embedding into T_*.

We show that $\mathbb{1}_{\mathrm{Add}(\kappa,\kappa^+)} \Vdash$ “$(\forall x \in {}^{\check{\kappa}}\check{\kappa})(\exists y \in {}^{\check{\kappa}}\check{\kappa})\ \langle x, y\rangle \notin p[\check{T}]$” holds in V. Assume, toward a contradiction, that G is $\mathrm{Add}(\kappa,\kappa^+)$-generic over V and $T_* \in \mathrm{V}[G]$ witnesses that (14) holds in $\mathrm{V}[G]$. Since $T_* = T_x$ for some $x \in ({}^\kappa 2)^{\mathrm{V}[G]}$, there is an $\alpha < \kappa^+$ with $T_* \in \mathrm{V}[G \cap \mathrm{Add}(\kappa,\alpha)]$ and $\mathrm{V}[G]$ is an $\mathrm{Add}(\kappa,\kappa)^+$-generic extension of $\mathrm{V}[G \cap \mathrm{Add}(\kappa,\alpha)]$ with $({}^\kappa\kappa)^{\mathrm{V}[G\cap\mathrm{Add}(\kappa,\alpha)]} \subsetneq ({}^\kappa\kappa)^{\mathrm{V}[G]} = p[T_*]^{\mathrm{V}[G]}$. This means

$$\langle \mathrm{V}[G \cap \mathrm{Add}(\kappa,\alpha)], \in\rangle \models \left[(\exists p \in \mathrm{Add}(\kappa,\kappa^+))\ p \Vdash \text{“}p[\check{T}_*] \not\subseteq \check{\mathrm{V}}\text{”}\right]$$

and there is a $\exists^x$-perfect embedding into T_* in $\mathrm{V}[G \cap \mathrm{Add}(\kappa,\alpha)]$ by Lemma 7.6. But this map is also a $\exists^x$-perfect embedding into T_* in $\mathrm{V}[G]$, a contradiction.

In the other direction, define $A \subseteq {}^\kappa\kappa$ as in Theorem 6.1 and let G be $\mathbb{P}(A)$-generic over V. By the second part of the Theorem, there is a tree T_* on $\kappa \times \kappa$ in $\mathrm{V}[G]$ such that $p[T_*]^{\mathrm{V}[G][H]} = ({}^\kappa\kappa)^{\mathrm{V}[G]}$ holds whenever $\mathrm{V}[G][H]$ is a generic extension of $\mathrm{V}[G]$ by a $<\kappa$-closed forcing in $\mathrm{V}[G]$. This obviously implies that “$p[T_*] = {}^\kappa\kappa$” holds in $\mathrm{V}[G]$ and we can apply Lemma 7.6 to show that there are no $\exists^x$-perfect embeddings into T_* in $\mathrm{V}[G]$. We can conclude that

$$\mathbb{1}_{\mathbb{P}(A)} \Vdash \text{“}(\exists x \in {}^{\check{\kappa}}\check{\kappa})(\forall y \in {}^{\check{\kappa}}\check{\kappa})\ \langle x, y\rangle \in p[\check{T}]\text{”}$$

holds in V. □

In the remainder of this section, we generalize the notion of $\mathbf{\Sigma}^1_2$-absoluteness to our uncountable context and investigate its structural implications.

Definition 7.10. Let Γ be a class of partial orders. We say that *generic absoluteness holds for $\mathbf{\Sigma}^1_2$-subsets of ${}^\kappa\kappa$ under forcings in* Γ if the implication

$$p \Vdash \text{“}(\exists x_0,\ldots,x_n \in {}^{\check\kappa}\check\kappa)(\forall y_1,\ldots,y_m \in {}^{\check\kappa}\check\kappa)\langle x_0,\ldots,x_n,y_0,\ldots,y_m\rangle \notin [\check T]\text{”}$$
$$\longrightarrow (\exists x_0,\ldots,x_n \in {}^\kappa\kappa)(\forall y_0,\ldots,y_m \in {}^\kappa\kappa)\langle x_0,\ldots,x_n,y_0,\ldots,y_m\rangle \notin [T]$$

holds true for every partial order $\mathbb{P}$ in Γ, every condition $p \in \mathbb{P}$ and every tree T on κ^{m+n+2}.

In Section 9, we will show that the consistency of generic absoluteness for $\mathbf{\Sigma}^1_2$-subsets of ${}^\kappa\kappa$ under forcing with $<\kappa$-closed partial orders can be established from a relatively mild large cardinal assumption (Lemma 9.7). We will also show that such generic absoluteness for Cohen forcing $\mathrm{Add}(\kappa,1)$ holds in every $\mathrm{Add}(\kappa,\kappa^+)$-generic extension of the ground model (Corollary 9.3).

The referee pointed out that it is possible to establish the consistency of $\mathbf{\Sigma}^1_2$-absoluteness under certain classes of $<\kappa$-closed partial orders without the use of large cardinals. Let Γ be a class of $<\kappa$-closed partial orders such that elements of Γ satisfy the κ^+-chain condition and Γ is closed under forcing iterations with $<\kappa$-support in the ground model and every generic extension by a forcing in Γ. If “$2^\kappa = \kappa^+$” holds in the ground model, then there is a forcing iteration $\langle\langle\vec{\mathbb{P}}_{<\alpha} \mid \alpha \leq \kappa^+\rangle, \langle\dot{\mathbb{P}}_\alpha \mid \alpha < \kappa^+\rangle\rangle$ of partial orders in Γ with $<\kappa$-support and a sequence $\langle \dot t_\alpha \in \mathrm{V}^{\vec{\mathbb{P}}_{<\alpha}} \mid \alpha < \kappa^+\rangle$ of names such that the following statements hold whenever $\alpha < \kappa^+$ with $\alpha = \prec\!\prec\beta,\gamma\succ,\delta\succ$, G is $\vec{\mathbb{P}}_{<\alpha}$-generic over V and $\bar G$ is the corresponding filter in $\vec{\mathbb{P}}_{<\beta}$.

(1) $\dot t^{\bar G}_\beta$ is an enumeration of all subtrees of ${}^{<\kappa}\kappa$ in $\mathrm{V}[\bar G]$ of length κ^+.
(2) If $\dot t^{\bar G}_\beta(\gamma) = T$ and $(\exists \mathbb{Q} \in \Gamma)\ \mathbb{1}_{\mathbb{Q}} \Vdash$ “$p[\check T] \neq {}^{\check\kappa}\check\kappa$” holds in $\mathrm{V}[G]$, then $\mathbb{1}_{\dot{\mathbb{P}}^G_\alpha} \Vdash$ “$p[\check T] \neq {}^{\check\kappa}\check\kappa$” holds in $\mathrm{V}[G]$.

If G is $\vec{\mathbb{P}}_{<\kappa^+}$-generic over V, then generic absoluteness for $\mathbf{\Sigma}^1_2$-subsets of ${}^\kappa\kappa$ under forcings in Γ holds in $\mathrm{V}[G]$.

Proposition 7.11. *Let Γ be a class of $<\kappa$-closed partial order that contains the trivial partial order and assume that generic absoluteness holds for $\mathbf{\Sigma}^1_2$-subsets of ${}^\kappa\kappa$ under forcings in Γ. Then every $\mathbf{\Delta}^1_1$-subset of ${}^\kappa\kappa$ is weakly Γ-persistently $\mathbf{\Delta}^1_1$.*

Proof. Let T_0 and T_1 witness that $p[T_0]$ is a $\mathbf{\Delta}^1_1$-subset of ${}^\kappa\kappa$. By Proposition 2.1, there is a tree T such that “$p[T] = p[T_0] \cup p[T_1]$” holds in V and every generic extension of V by a forcing in Γ.

Assume, toward a contradiction, that $\mathbb{1}_{\mathbb{P}} \not\Vdash$ “${}^{\check\kappa}\check\kappa = p[\check T_0] \cup p[\check T_1]$” holds for some $\mathbb{P} \in \Gamma$. Then there is a $p \in \mathbb{P}$ with

$$p \Vdash \text{“}(\exists x \in {}^{\check\kappa}\check\kappa)(\forall y \in {}^{\check\kappa}\check\kappa)\ \langle x,y\rangle \notin [\check T]\text{”}.$$

By $\mathbf{\Sigma}^1_2$-absoluteness, there is an $x \in {}^\kappa\kappa$ with $x \notin p[T] = {}^\kappa\kappa$, a contradiction.

In the same way, we can use Proposition 7.3 to see that $\mathbb{1}_{\mathbb{P}} \Vdash$ “$p[\check T_0] \cap p[\check T_1] = \emptyset$” holds for every partial order $\mathbb{P}$ in Γ. □

Proposition 7.12. *Assume that generic absoluteness holds for $\mathbf{\Sigma}^1_2$-subsets of ${}^\kappa\kappa$ under* $\mathrm{Add}(\kappa,1)$. *If T is a tree on $\kappa\times\kappa$ of height κ, then $p[T]$ contains a perfect subset if and only if $\mathbb{1}_{\mathrm{Add}(\kappa,1)} \Vdash$ “$p[\check T] \nsubseteq \check{\mathrm{V}}$”.*

Proof. Let $\iota : {}^{<\kappa}2 \longrightarrow {}^{<\kappa}\kappa$ witness that $p[T]$ has a perfect subset and assume, toward a contradiction, that there is a $p \in \mathrm{Add}(\kappa,1)$ with $p \Vdash$ “$p[\check T] \subseteq \check{\mathrm{V}}$”. By the results of Section 2, there is a tree T_* on $\kappa \times \kappa$ such that $p[T_*] = p[T] \cup ({}^\kappa\kappa \setminus [T_\iota])$ holds in V and every $\mathrm{Add}(\kappa, 1)$-generic extension of V. Since $p \Vdash$ "$[\check{T}_\iota] \nsubseteq \check{\mathrm{V}}$", we get $p \Vdash$ "$(\exists x \in {}^{\check\kappa}\check\kappa)(\forall y \in {}^{\check\kappa}\check\kappa)\ \langle x, y\rangle \notin [\check{T}_*]$" and absoluteness gives us an $x \in {}^\kappa\kappa$ with $x \notin p[T_*] = {}^\kappa\kappa$, a contradiction. □

We apply the above results to prove statements about the length of definable well-orders on subsets of ${}^\kappa\kappa$ in the presence of $\mathbf{\Sigma}^1_2$-absoluteness.

Definition 7.13. A $\mathbf{\Sigma}^1_1$*-well-ordering of a subset of* ${}^\kappa\kappa$ is a $\mathbf{\Sigma}^1_1$-subset R of ${}^\kappa\kappa \times {}^\kappa\kappa$ such that $\langle \mathrm{dom}(R), R\rangle$ is a well-ordering, where

$$\mathrm{dom}(R) = \{x \in {}^\kappa\kappa \mid (\exists y)\ [R(x, y) \vee R(y, x)]\}.$$

Clearly, every $\mathbf{\Sigma}^1_1$-well-ordering of a subset of ${}^\kappa\kappa$ has order type less than $(2^\kappa)^+$ and for every $\gamma < \kappa^+$ there is such a well-ordering with order type γ. Moreover, Theorem 1.5 shows it is consistent to have a $\mathbf{\Sigma}^1_1$-well-ordering of a subset of ${}^\kappa\kappa$ of order-type greater than 2^κ.

Proposition 7.14. *Let Γ be a class of $<\kappa$-closed partial orders and assume that generic absoluteness holds for $\mathbf{\Sigma}^1_2$-subsets of ${}^\kappa\kappa$ under forcings in Γ. If T is a tree on κ^3 of such that $p[T]$ is a $\mathbf{\Sigma}^1_1$-well-ordering of a subset of ${}^\kappa\kappa$ and $\mathbb{P} \in \Gamma$, then*

$$\mathbb{1}_{\mathbb{P}} \Vdash \text{“}p[\check{T}] \text{ is a } \mathbf{\Sigma}^1_1\text{-well-ordering of a subset of } {}^{\check\kappa}\check\kappa\text{ ”}.$$

Proof. We prove that $p[T]$ is a linear and well-founded relation in every generic extension by a forcing in Γ; the other properties of a well-ordering can be deduced in the same manner.

By the results of Section 2, there is a tree T_w in $\kappa \times \kappa$ such that

$$p[T_w] = \{x \in {}^\kappa\kappa \mid (\forall n < \omega)\ \langle (x)_{n+1}, (x)_n\rangle \in p[T]\}$$

holds in V and every generic extension of V by a forcing in Γ. By our assumptions, $p[T_w] = \emptyset$ and Proposition 7.3 shows that $\mathbb{1}_{\mathbb{P}} \vDash$ "$p[\check{T}_w] = \emptyset$" holds for all $\mathbb{P}$ in Γ. This shows that $p[T]^{\mathrm{V}[G]}$ is a well-founded relation in every $\mathbb{P}$-generic extension $\mathrm{V}[G]$ of V with $\mathbb{P} \in \Gamma$.

As above, there is a tree T_l on κ^7 such that $p[T_l]$ is equal to the set

$$\begin{aligned}\{\langle x, x_0, x_1, y, y_0, y_1\rangle \in ({}^\kappa\kappa)^6 \mid \langle x, y\rangle \in p[T] \vee \langle y, x\rangle \in p[T]] \\ \vee [\langle x, x_0, x_1\rangle \notin [T] \wedge \langle x_0, x, x_1\rangle \notin [T]] \vee [\langle y, y_0, y_1\rangle \notin [T] \wedge \langle y_0, y, y_1\rangle \notin [T]]\}\end{aligned}$$

in V and every generic extension of V by a forcing in Γ. Assume, toward a contradiction, that there is a $\mathbb{P}$ in Γ and a $\mathbb{P}$-generic extension $\mathrm{V}[G]$ of V such that $p[T]^{\mathrm{V}[G]}$ is not a linear order on its domain. Then there is a $p \in \mathbb{P}$ with

$$p \vDash \text{“}(\exists x, x_0, x_1, y, y_0, y_1 \in {}^{\check\kappa}\check\kappa)(\forall z \in {}^{\check\kappa}\check\kappa)\ \langle x, x_0, x_1, y, y_0, y_1, z\rangle \notin [T_l]\text{”}$$

and, by $\mathbf{\Sigma}^1_2$-absoluteness, $p[T]$ is not linear on its domain in V, a contradiction. □

The proof of the following lemma uses an idea of Philipp Schlicht to show that $\mathbf{\Sigma}^1_2$-absoluteness implies that $\mathbf{\Sigma}^1_1$-well-orders have *small* domains.

Lemma 7.15. *Assume that generic absoluteness holds for $\mathbf{\Sigma}^1_2$-subsets of ${}^\kappa\kappa$ under* $\mathrm{Add}(\kappa, 1)$*. If T is a tree on κ^3 such that $p[T]$ is a $\mathbf{\Sigma}^1_1$-well-ordering of a subset of ${}^\kappa\kappa$, then $\exists^x p[T]$ contains no perfect subset.*

Proof. There is a tree T_* on $\kappa \times \kappa$ such that $p[T_*] = \mathrm{dom}(p[T])$ holds in V and every $\mathrm{Add}(\kappa,1)$-generic extension of V. Assume, toward a contradiction, that $p[T_*]$ contains a perfect subset and let G be $\mathrm{Add}(\kappa,1)$-generic over V. We will construct sequences $\langle G_n \in \mathrm{V}[G] \mid n < \omega\rangle$ and $\langle x_n \in ({}^\kappa\kappa)^{\mathrm{V}[G]} \mid n < \omega\rangle$ such that the following statements hold true for all $n < \omega$.

(1) There is a $\bar{G} \in \mathrm{V}[G]$ such that $(G_n \times \bar{G})$ is $(\mathrm{Add}(\kappa,1) \times \mathrm{Add}(\kappa,1))$-generic over $\mathrm{V}[G_0,\ldots,G_{n-1}]$ and $\mathrm{V}[G] = \mathrm{V}[G_0,\ldots,G_{n-1}][G_n][\bar{G}]$.

(2) We have $x_n \in \mathrm{V}[G_0,\ldots,G_n]$, $\langle x_{n+1}, x_n\rangle \in p[T]^{\mathrm{V}[G_0,\ldots,G_{n+1}]}$ and

$$\langle \mathrm{V}[G_0,\ldots,G_n],\in\rangle \models \left[\mathbb{1}_{\mathrm{Add}(\kappa,1)} \Vdash (\exists x) \ \left[x \notin \check{\mathrm{V}} \wedge \langle x, \check{x}_n\rangle \in p[\check{T}]\right]\right]. \tag{15}$$

There are $H_0, H_1 \in \mathrm{V}[G]$ such that $H_0 \times H_1$ is $(\mathrm{Add}(\kappa,1) \times \mathrm{Add}(\kappa,1))$-generic over V with $\mathrm{V}[G] = \mathrm{V}[H_0][H_1]$. By our assumptions and Proposition 7.12, there are $y_0, y_1 \in \mathrm{V}[G]$ with $y_i \in p[T_*]^{\mathrm{V}[H_i]} \setminus \mathrm{V}$. Since $\mathrm{V}[H_0] \cap \mathrm{V}[H_1] = \mathrm{V}$, we have $y_0 \neq y_1$ and there is an $i_* < 2$ with $\langle y_{1-i_*}, y_{i_*}\rangle \in p[T]^{\mathrm{V}[G]}$. Define $x_0 = y_{i_*}$ and $G_0 = H_{i_*}$. The homogeneity of $\mathrm{Add}(\kappa,1)$ in $\mathrm{V}[G_0]$ and $y_{1-i_*} \in \mathrm{V}[G] \setminus \mathrm{V}[G_0]$ imply (15).

Now assume $G_0,\ldots,G_n$ and $x_0,\ldots,x_n$ with the above properties are already constructed. Hence there are $H_0, H_1 \in \mathrm{V}[G]$ such that $(H_0 \times H_1)$ is $(\mathrm{Add}(\kappa,1) \times \mathrm{Add}(\kappa,1))$-generic over $\mathrm{V}[G_0,\ldots,G_n]$ and $\mathrm{V}[G] = \mathrm{V}[G_0,\ldots,G_n][H_0][H_1]$. By (15), there are $y_0, y_1 \in \mathrm{V}[G]$ with $y_i \in \mathrm{V}[G_0,\ldots,G_n,H_i] \setminus \mathrm{V}[G_0,\ldots,G_n]$ and $\langle y_i, x_n\rangle \in p[T]^{\mathrm{V}[G_0,\ldots,G_n,H_i]}$. Again, there is an $i_* < 2$ with $\langle y_{1-i_*}, y_{i_*}\rangle \in p[T]^{\mathrm{V}[G]}$ and we can define $G_{n+1} = H_{i_*}$ and $x_{n+1} = y_{i_*}$. As above, (15) holds true.

Our construction shows $\langle x_{n+1}, x_n\rangle \in p[T]^{\mathrm{V}[G]}$ for all $n < \omega$. But $p[T]^{\mathrm{V}[G]}$ is a $\mathbf{\Sigma}^1_1$-well-ordering of a subset of ${}^\kappa\kappa$ in $\mathrm{V}[G]$ by Proposition 7.14, a contradiction. □

Corollary 7.16. *Assume that generic absoluteness holds for $\mathbf{\Sigma}^1_2$-subsets of ${}^\kappa\kappa$ under* $\mathrm{Add}(\kappa,1)$. *Then there is no well-ordering of ${}^\kappa\kappa$ whose graph is a $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa \times {}^\kappa\kappa$.* □

8. Embeddings of trees

In this short section, we present an easy proof of Theorem 1.10 with the help of our first main result. Let $\mathcal{TO}_\kappa$ denote the class of all $x \in {}^\kappa\kappa$ such that $\mathbb{T}_x = \langle \kappa, \in_x\rangle$ is a tree that is an element of $\mathcal{T}_\kappa$.

Let $\bar{T}$ be the set of all pairs $\langle s,t\rangle$ in ${}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$ such that $\mathrm{lh}(s) = \mathrm{lh}(t) = \gamma + 1$ for some $\gamma < \kappa$ and either $\langle \lambda, \in_{s\restriction\lambda}\rangle$ is not a tree for some $\lambda \leq \gamma$ closed under Gödel-Pairing or t is injective and

$$(\forall \alpha < \beta \leq \gamma) \ \left[\prec t(\alpha), t(\beta)\succ \leq \gamma \rightarrow s(\prec t(\alpha), t(\beta)\succ) = 1\right].$$

We define T to be the tree $\{\langle s \restriction \alpha, t \restriction \beta\rangle \mid \langle s,t\rangle \in \bar{T}, \ \alpha \leq \mathrm{lh}(s)\}$ on $\kappa \times \kappa$. It is easy to check that $\mathcal{TO}_\kappa = {}^\kappa\kappa \setminus p[T]$ holds in V and every generic extension of V by a $<\kappa$-closed forcing.

Given $y \in {}^\kappa\kappa$, we define $T(y)$ to be the tree $\{t \in {}^{<\kappa}\kappa \mid \langle y \restriction \mathrm{lh}(t), t\rangle \in T\}$ on κ. If $y \in \mathcal{TO}_\kappa$ and $\alpha < \kappa$, then $\langle \{\alpha\} \cup \mathrm{prec}_{\mathbb{T}_y}(\alpha), \in_y\rangle$ is a well-order of successor length and we let $t_y(\alpha) \in {}^{<\kappa}\kappa$ denote the corresponding uncollapsing map. Our construction yields $t_y(\alpha) \in T(y)$ and the map $[\alpha \mapsto t_y(\alpha)]$ shows that $\mathbb{T}_y$ is order-preserving embeddable into $\langle T(y), \subsetneq\rangle$.

The following result was proved in [17] in the case "$\kappa = \omega_1$", but the proof given there directly generalizes to higher cardinalities. It is the uncountable version of the classic *Boundedness Lemma*.

Lemma 8.1 (Boundedness Lemma for ${}^\kappa\kappa$, [17, Corollary 13])**.** *If A is a $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa$ with $A \subseteq \mathcal{TO}_\kappa$, then there is a tree $\mathbb{T}$ in $\mathcal{T}_\kappa$ such that $\mathbb{T}_y \leq \mathbb{T}$ holds for every $y \in A$.*

Proof. Let S be a tree on $\kappa \times \kappa$ with $A = p[S]$ and T be the tree on $\kappa \times \kappa$ defined above. Define S_* to be the tree on κ^3 consisting of triples $\langle s, t, u\rangle$ with $\langle s, t\rangle \in T$ and $\langle s, u\rangle \in S$. Assume towards a contradiction, that there is a $\langle x, y, z\rangle \in [S_*]$. Then $x \in p[S] \cap p[T] = A \cap ({}^\kappa\kappa \setminus \mathcal{TO}_\kappa) = \emptyset$, a contradiction. If $y \in A$ with $\langle y, z\rangle \in [S]$ and $t \in T(y)$, then $\langle y \restriction \mathrm{lh}(t), t, z \restriction \mathrm{lh}(t)\rangle \in S_*$ and the map $[t \mapsto \langle y \restriction \mathrm{lh}(t), t, z \restriction \mathrm{lh}(t)\rangle]$ shows that $\langle T(y), \subsetneq\rangle$ is order-preserving embeddable into $\mathbb{T} = \langle S_*, \lhd_*\rangle$, where $\lhd_*$ is the natural order on S_*. By the above remarks, this shows that $\mathbb{T}_y \leq \mathbb{T}$ holds for every $y \in A$. □

Proof of Theorem 1.10. Let $\mathbb{P} = \mathbb{P}(\mathcal{TO}_\kappa)$ be the forcing given by Theorem 1.5 that codes the subset $\mathcal{TO}_\kappa$ of ${}^\kappa\kappa$ and G be $\mathbb{P}$-generic over V. By the above remarks and Proposition 7.3, we have $\mathcal{TO}^{\mathrm{V}}_\kappa \subseteq \mathcal{TO}^{\mathrm{V}[G]}_\kappa$ and $\mathcal{TO}^{\mathrm{V}}_\kappa$ is a $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa$ in $\mathrm{V}[G]$. Lemma 8.1 shows that there is a $\mathbb{T}_G \in \mathcal{T}^{\mathrm{V}[G]}_\kappa$ with $\mathbb{T}_x \leq \mathbb{T}_G$ for all $x \in \mathcal{TO}^{\mathrm{V}}_\kappa$. For every $\mathbb{T} \in \mathcal{T}^{\mathrm{V}}_\kappa$ there is an $x \in \mathcal{TO}^{\mathrm{V}}_\kappa$ with $\mathbb{T}$ isomorphic to $\mathbb{T}_x$ and this completes the proof of the theorem. □

9. Two models with a nice structure theory for $\mathbf{\Sigma}^1_1$-subsets

We show that certain fragments of $\mathbf{\Sigma}^1_2$-absoluteness hold in two well-known classes of ZFC-models and derive some consequences about the possible length of $\mathbf{\Sigma}^1_1$-well-orders of subsets of ${}^\kappa\kappa$ in these models. We start with a standard result about Cohen-generic extensions of a ground model.

Lemma 9.1. *Let $\nu > \kappa$ be a cardinal and X be a subset of ν of cardinality κ^+. If G is $\mathrm{Add}(\kappa, \nu)$-generic over* V *and $\bar{G} = G \cap \mathrm{Add}(\kappa, X)$, then there is an elementary embedding*

$$j : \mathrm{L}(\mathcal{P}(\kappa)^{\mathrm{V}[\bar{G}]}) \longrightarrow \mathrm{L}(\mathcal{P}(\kappa)^{\mathrm{V}[G]})$$

with $j \restriction \mathrm{On} = \mathrm{id}_{\mathrm{On}}$ and $j \restriction \mathcal{P}(\kappa)^{\mathrm{V}[\bar{G}]} = \mathrm{id}_{\mathcal{P}(\kappa)^{\mathrm{V}[\bar{G}]}}$.

Proof. We define $P = \mathcal{P}(\kappa)^{V[G]}$ and $\bar{P} = \mathcal{P}(\kappa)^{\mathrm{V}[\bar{G}]}$. By the construction of $\mathrm{L}(\bar{P})$, there is a surjection

$$s : [\mathrm{On}]^{<\omega} \times \bar{P} \longrightarrow \mathrm{L}(\bar{P})$$

definable in $\mathrm{L}(\bar{P})$ by a formula $\varphi \equiv \varphi(u, v_0, v_1, w)$ and the parameter $\bar{P}$. Define

$$j(a) = b \iff (\exists x \in \bar{P})(\exists A \in [\mathrm{On}]^{<\omega}) \\ [\langle \mathrm{L}(\bar{P}), \in\rangle \models \varphi(a, x, A, \bar{P}) \wedge \langle \mathrm{L}(P), \in\rangle \models \varphi(b, x, A, P)].$$

In order to show that j is a well-defined function and an elementary embedding with the above properties, it suffices to show that for all $x_0, \ldots, x_{n-1} \in \bar{P}$, $A \in [\mathrm{On}]^{<\omega}$ and every $\mathcal{L}_\in$-formula $\psi \equiv \psi(u_0, \ldots, u_{n-1}, v_0, \ldots, v_{m-1}, w)$

$$\langle \mathrm{L}(\bar{P}), \in\rangle \models \psi(\vec{x}, A, \bar{P}) \iff \langle \mathrm{L}(P), \in\rangle \models \psi(\vec{x}, A, P).$$

holds. There exist $\bar{G}_0, \bar{G}_1 \in \mathrm{V}[\bar{G}]$ such that $\bar{G}_0$ is $\mathrm{Add}(\kappa, 1)$-generic over V and $\bar{G}_1$ is $\mathrm{Add}(\kappa, \kappa^+)$-generic over $\mathrm{V}[\bar{G}_0]$ with $\vec{x} \in \mathrm{V}[\bar{G}_0]$ and $\mathrm{V}[\bar{G}] = \mathrm{V}[\bar{G}_0][\bar{G}_1]$. Moreover, there is $G_1 \in \mathrm{V}[G]$ that is $\mathrm{Add}(\kappa, \nu)$-generic over $\mathrm{V}[\bar{G}_0]$ with $\mathrm{V}[G] = \mathrm{V}[\bar{G}_0][G_1]$.

Let F be $\mathrm{Col}(\omega, 2^\kappa)^{\mathrm{V}[G]}$-generic over $\mathrm{V}[G]$. We show that there is a $H \in \mathrm{V}[G][F]$ that is $\mathrm{Add}(\kappa, \kappa^+)^{\mathrm{V}}$-generic over $\mathrm{V}[\bar{G}_0]$ and satisfies $\mathcal{P}(\kappa)^{\mathrm{V}[G]} = \mathcal{P}(\kappa)^{\mathrm{V}[\bar{G}_0][H]}$.

We work in $\mathrm{V}[G][F]$. Let $\langle x_n \mid n<\omega\rangle$ enumerate $\mathcal{P}(\kappa)^{\mathrm{V}[G]}$ and let $\langle \alpha_n \mid n<\omega\rangle$ be strictly increasing and cofinal in $\kappa^{+\mathrm{V}}$. We define $\mathbb{P}=\mathrm{Add}(\kappa,1)^{\mathrm{V}[\bar{G}_0]}$, $\mathbb{P}_n=\prod_{i<n}\mathbb{P}$, $\mathbb{Q}=\mathrm{Add}(\kappa,\kappa^+)^{\mathrm{V}[\bar{G}_0]}$ and $\mathbb{Q}_n=\mathrm{Add}(\kappa,\alpha_n)^{\mathrm{V}[\bar{G}_0]}$. Using the factor-property of Cohen-Forcing, it is easy to define a sequence $\langle H_n \mid n<\omega\rangle$ of filters in $\mathbb{P}$ that satisfy the following properties for all $n<\omega$.

(1) $H_n\in\mathrm{V}[G]$.
(2) H_n is $\mathbb{P}$-generic over $\mathrm{V}[\bar{G}_0][H_0,\dots,H_{n-1}]$ and $x_n\in\mathrm{V}[\bar{G}_0][H_0,\dots,H_n]$.
(3) There is a $G'\in\mathrm{V}[G]$ that $\mathrm{Add}(\kappa,\nu)$-generic over $\mathrm{V}[\bar{G}_0][H_0,\dots,H_n]$ and satisfies $\mathrm{V}[G]=\mathrm{V}[\bar{G}_0][H_0,\dots,H_n][G']$.

For all $n<\omega$, we let $e_n:\mathbb{P}_n\longrightarrow\mathbb{P}_{n+1}$ denote the natural inclusion. In $\mathrm{V}[\bar{G}_0]$, there are isomorphisms $i_n:\mathbb{P}_n\longrightarrow\mathbb{Q}_n$ with $i_n=i_{n+1}\circ e_n$ for all $n<\omega$. For all $n<\omega$, $H_0\times\dots\times H_{n-1}$ is $\mathbb{P}_n$-generic over $\mathrm{V}[\bar{G}_0]$ and we can define

$$H=\bigcup_{n<\omega} i_n``(H_0\times\dots\times H_{n-1})\in\mathcal{P}(\mathbb{Q})^{\mathrm{V}[G][F]}.$$

Since $\mathbb{Q}_n\subseteq\mathbb{Q}_{n+1}\subseteq\bigcup_{n<\omega}\mathbb{Q}_n=\mathbb{Q}$, $e_n^{-1}``(H_0\times\dots\times H_n)=H_0\times\dots\times H_{n-1}$, $i_n``H_0\times\dots\times H_{n-1}=H\cap\mathbb{Q}_n$ is a filter in $\mathbb{Q}_n$ for all $n<\omega$, it is easy to see that H is a filter in $\mathbb{Q}$. We show that H is also $\mathbb{Q}$-generic over $\mathrm{V}[\bar{G}_0]$. If $\mathcal{A}\in\mathrm{V}[\bar{G}_0]$ is a maximal antichain in $\mathbb{Q}$, then $\mathcal{A}\subseteq\mathbb{Q}_n$ for some $n<\omega$, because the $\mathbb{Q}$ satisfies the κ^+-chain condition in $\mathrm{V}[\bar{G}_0]$. By the above remarks, $H\cap\mathbb{Q}_n=i_n``(H_0\times\dots\times H_{n-1})$ is $\mathbb{Q}_n$-generic over $\mathrm{V}[\bar{G}_0]$. Therefore, we get $\mathcal{A}\cap H\neq\emptyset$. Since $\mathbb{Q}$ satisfies the κ^+-chain condition, it is easy to see that $\mathcal{P}(\kappa)^{\mathrm{V}[G]}=\mathcal{P}(\kappa)^{\mathrm{V}[\bar{G}_0][H]}$ holds.

The weak homogeneity of $\mathrm{Add}(\kappa,\kappa^+)$ in $\mathrm{V}[\bar{G}_0]$ yields the following equivalences.

$$\begin{aligned}
&\langle\mathrm{L}(\bar{P}),\in\rangle\models\psi(\vec{x},A,\bar{P})\\
\Longleftrightarrow\;&\langle\mathrm{V}[\bar{G}_0][\bar{G}_1],\in\rangle\models(\exists p)\ \left[p=\mathcal{P}(\kappa)\wedge\psi(\vec{x},A,p)^{\mathrm{L}(p)}\right]\\
\Longleftrightarrow\;&\langle\mathrm{V}[\bar{G}_0],\in\rangle\models\left[\mathbb{1}_{\mathrm{Add}(\kappa,\kappa^+)}\Vdash(\exists p)\ \left[p=\mathcal{P}(\check{\kappa})\wedge\psi(\check{\vec{x}},\check{A},p)^{\mathrm{L}(p)}\right]\right]\\
\Longleftrightarrow\;&\langle\mathrm{V}[\bar{G}_0][H],\in\rangle\models(\exists p)\ \left[p=\mathcal{P}(\kappa)\wedge\psi(\vec{x},A,p)^{\mathrm{L}(p)}\right]\\
\Longleftrightarrow\;&\langle\mathrm{L}(P),\in\rangle\models\psi(\vec{x},A,P).
\end{aligned}$$

□

This result has two useful corollaries in our context.

Corollary 9.2. *Let $\nu>\kappa$ be a cardinal and G be $\mathrm{Add}(\kappa,\nu)$-generic over V. Then the* axiom of choice *fails in $\langle\mathrm{L}(\mathcal{P}(\kappa)^{\mathrm{V}[G]}),\in\rangle$. In particular, the graph of a well-order of ${}^\kappa\kappa$ is not a $\mathbf{\Sigma}^1_n$-subset of ${}^\kappa\kappa\times{}^\kappa\kappa$ in $\mathrm{V}[G]$.*

Proof. This follows directly from Lemma 9.1 and [15, Proposition 5.1(b)]. □

Corollary 9.3. *Let λ and ν be cardinals with $\nu>\kappa$. If G is $\mathrm{Add}(\kappa,\nu)$-generic over V, then generic absoluteness for $\mathbf{\Sigma}^1_2$-subsets of ${}^\kappa\kappa$ under $\mathrm{Add}(\kappa,\lambda)$ holds in $\mathrm{V}[G]$.*

Proof. Let $T\in\mathrm{V}[G]$ be a tree on κ^{m+n+1} and assume

$$\mathbb{1}_{\mathrm{Add}(\kappa,\lambda)}\Vdash\text{“}(\exists x_0,\dots,x_n\in{}^{\check{\kappa}}\check{\kappa})(\forall y_1,\dots,y_m\in{}^{\check{\kappa}}\check{\kappa})\langle x_0,\dots,x_n,y_0,\dots,y_m\rangle\notin[\check{T}]\text{”}$$

holds in $\mathrm{V}[G]$. We may assume that $T\in\mathrm{V}[\bar{G}]$ with $\bar{G}=G\cap\mathrm{Add}(\kappa,\kappa^+)$.

Let $\gamma = \max\{\lambda^+, \nu^+\}$ and F be $\mathrm{Add}(\kappa,\gamma)$-generic over V with $G = F \cap \mathrm{Add}(\kappa,\nu)$. There are $H_0, H_1 \in \mathrm{V}[F]$ such that H_0 is $\mathrm{Add}(\kappa,\lambda)$-generic over $\mathrm{V}[\bar{G}]$, H_1 is $\mathrm{Add}(\kappa,\gamma)$-generic over $\mathrm{V}[\bar{G}][H_0]$ and $\mathrm{V}[F] = \mathrm{V}[\bar{G}][H_0][H_1]$. By the above assumption, the statement

$$(\exists x_0, \dots, x_n \in {}^\kappa\kappa)(\forall y_1, \dots, y_m \in {}^\kappa\kappa)\langle x_0, \dots, x_n, y_0, \dots, y_m\rangle \notin [T]$$

holds in $\mathrm{V}[\bar{G}][H_0]$. An application of Proposition 7.3 shows that this statement also holds in $\mathrm{V}[F] = \mathrm{V}[\bar{G}][H_0][H_1]$ and hence in $\mathrm{L}(\mathcal{P}(\kappa)^{\mathrm{V}[F]})$. By Lemma 9.1, it holds in $\mathrm{L}(\mathcal{P}(\kappa)^{\mathrm{V}[\bar{G}]})$ and in $\mathrm{V}[\bar{G}]$. Since $\mathrm{V}[G]$ is either equal to $\mathrm{V}[\bar{G}]$ or an $\mathrm{Add}(\kappa,\nu)$-generic extension of $\mathrm{V}[\bar{G}]$, we can use Proposition 7.3 again to conclude that the statement holds in $\mathrm{V}[G]$. □

Proposition 9.4. *Let $\nu > \kappa$ be a cardinal. If G is $\mathrm{Add}(\kappa,\nu)$-generic over* V *and A is a $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa$ of cardinality bigger than $(2^\kappa)^{\mathrm{V}}$ in $\mathrm{V}[G]$, then A has a perfect subset in $\mathrm{V}[G]$.*

Proof. Fix a tree T on $\kappa\times\kappa$ with $A = p[T]^{\mathrm{V}[G]}$. There are $G_0, G_1 \in \mathrm{V}[G]$ such that G_0 is $\mathrm{Add}(\kappa,\kappa^+)$-generic over V, $T \in \mathrm{V}[G_0]$, G_1 is $\mathrm{Add}(\kappa,\nu)$-generic over $\mathrm{V}[G_0]$ and $\mathrm{V}[G] = \mathrm{V}[G_0][G_1]$. Since $(2^\kappa)^{\mathrm{V}[G_0]} = (2^\kappa)^{\mathrm{V}}$, we get $p[T]^{\mathrm{V}[G_0]} \subsetneq p[T]^{\mathrm{V}[G]}$ and there is a $\exists^x$-perfect embedding into T in $\mathrm{V}[G_0]$ by Lemma 7.6. Proposition 7.5 implies that $p[T]$ has a perfect subset in $\mathrm{V}[G]$. □

By combining the above results with Lemma 7.15, we derive the following corollary. Note that the absoluteness property Theorem 1.5 shows that this bound on the cardinality of order-types is optimal.

Corollary 9.5. *If $\nu > \kappa$ is a cardinal, G is $\mathrm{Add}(\kappa,\nu)$-generic over* V *and R is $\mathbf{\Sigma}^1_1$-well-ordering of a subset of ${}^\kappa\kappa$ in $\mathrm{V}[G]$, then $\mathrm{dom}(R) \neq ({}^\kappa\kappa)^{\mathrm{V}[G]}$ and the order-type of $\langle \mathrm{dom}(R), R\rangle$ has cardinality at most $(2^\kappa)^{\mathrm{V}}$ in $\mathrm{V}[G]$.* □

Using large cardinals and the *Levy-Collapse*, models with a nice structure theory for $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$ can be obtained by mimicking classical constructions. We will repeatedly use the following folkloristic fact.

Lemma 9.6. *Let ν be a cardinal with $\nu = \nu^{<\kappa}$ and $\mathbb{P}$ be a $<\kappa$-closed partial order.*

(1) *If $\mathbb{P}$ has cardinality at most ν, then $\mathrm{Col}(\kappa,\nu)$ and $\mathbb{P}\times\mathrm{Col}(\kappa,\nu)$ are forcing equivalent.*
(2) *If $\mathbb{P}$ has cardinality less than ν and $\lambda^{<\kappa} < \nu$ holds for all $\lambda < \nu$, then $\mathrm{Col}(\kappa,{<}\nu)$ and $\mathbb{P}\times\mathrm{Col}(\kappa,{<}\nu)$ are forcing equivalent.*

Proof. See [9, Corollary 2.3] and [9, Corollary 2.4]. □

Generic absoluteness for $\mathbf{\Sigma}^1_3$-subsets of ${}^\omega\omega$ is equiconsistent with the existence of a Σ_2-reflecting cardinal, i.e. an inaccessible cardinal ν such that $\langle \mathrm{V}_\nu, \in\rangle$ is a Σ_2-elementary submodel of $\langle \mathrm{V}, \in\rangle$ (see [3] and [5]). The consistency of generic absoluteness for $\mathbf{\Sigma}^1_2$-subsets of ${}^\kappa\kappa$ under forcing with $<\kappa$-closed partial orders follows from a direct generalization of the proof of this result.

Lemma 9.7. *Let $\nu > \kappa$ be a Σ_2-reflecting cardinal and γ be a cardinal. If $G\times H$ is $(\mathrm{Col}(\kappa,{<}\nu)\times\mathrm{Add}(\kappa,\gamma))$-generic over* V*, then generic absoluteness for $\mathbf{\Sigma}^1_2$-subsets of ${}^\kappa\kappa$ under $<\kappa$-closed forcings holds in $\mathrm{V}[G][H]$.*

Proof. In $\mathrm{V}[G][H]$, let T be a tree on κ^{m+n+1} and $\mathbb{Q}$ be a $<\kappa$-closed partial order such that

$$p \Vdash \text{“}(\exists x_0, \dots, x_n \in {}^{\check\kappa}\check\kappa)(\forall y_1, \dots, y_m \in {}^{\check\kappa}\check\kappa)\langle x_0, \dots, x_n, y_0, \dots, y_m\rangle \notin [\check T]\text{”}$$

holds for some $p \in \mathbb{Q}$.

We can find $F \in \mathrm{V}[G][H]$, $\bar\nu < \nu$ and $i < 2$ such that F is $(\mathrm{Col}(\kappa, \bar\nu) \times \mathrm{Add}(\kappa, i))$-generic over V, $T \in \mathrm{V}[F]$ and $\mathrm{V}[G][H]$ is a $(\mathrm{Col}(\kappa,\nu) \times \mathrm{Add}(\kappa, \bar\gamma))$-generic extension of $\mathrm{V}[F]$ for some $\bar\gamma \in \{0, \gamma\}$. Then ν is a Σ_2-reflecting cardinal in $\mathrm{V}[F]$. Let $\dot{\mathbb{Q}} \in \mathrm{V}[F]$ be a $(\mathrm{Col}(\kappa,\nu) \times \mathrm{Add}(\kappa,\bar\gamma))$-name for $\mathbb{Q}$ such that

$$\mathbb{1}_{\mathrm{Col}(\kappa,\nu)\times\mathrm{Add}(\kappa,\bar\gamma)} \Vdash \text{“}\dot{\mathbb{Q}} \textit{ is a } {<}\kappa\textit{-closed partial order}\text{”}.$$

By Lemma 9.6, there is a cardinal $\lambda > \bar\nu$ such that the partial order

$$((\mathrm{Col}(\kappa,\nu) \times \mathrm{Add}(\kappa,\bar\gamma)) * \dot{\mathbb{Q}}) \times \mathrm{Col}(\kappa, {<}\lambda)$$

is forcing equivalent to $\mathrm{Col}(\kappa, {<}\lambda)$ in $\mathrm{V}[F]$. Proposition 7.3 and the weak homogeneity of $\mathrm{Col}(\kappa,{<}\lambda)$ imply that

$$(16)\quad \begin{aligned}&(\exists \lambda > \bar\nu)(\exists \mathbb{P})\ \big[\lambda \textit{ is a cardinal, } \mathbb{P} = \mathrm{Col}(\kappa,{<}\lambda) \textit{ and}\\ &\mathbb{1}_{\mathbb{P}} \Vdash \text{“}(\exists x_0, \dots, x_n \in {}^{\check\kappa}\check\kappa)(\forall y_1, \dots, y_m \in {}^{\check\kappa}\check\kappa)\langle x_0, \dots, x_n, y_0, \dots, y_m\rangle \notin [\check T]\text{”}\big]\end{aligned}$$

holds in $\mathrm{V}[F]$. We can apply Σ_2-elementarity to see that (16) holds in $\mathrm{V}_\nu[F]$ and hence there is a cardinal $\lambda_* < \nu$ that witnesses that (16) holds in $\mathrm{V}[F]$. There is $F_* \in \mathrm{V}[G][H]$ such that F_* is $\mathrm{Col}(\kappa,{<}\lambda_*)$-generic over $\mathrm{V}[F]$ and $\mathrm{V}[G][H]$ is a generic extension of $\mathrm{V}[F][F_*]$ by a $<\kappa$-closed partial order. A final application of Proposition 7.3 shows that

$$(\exists x_0, \dots, x_n \in {}^\kappa\kappa)(\forall y_0, \dots, y_m \in {}^\kappa\kappa)\ \langle x_0, \dots, x_n y_0, \dots, y_m\rangle \notin [T]$$

holds in $\mathrm{V}[G][H]$. □

Note that the consistency strength of the above assumption is bounded by the existence of a Mahlo cardinal. Another generalization of the proof of the absoluteness result mentioned above shows that the consistency strength of $\mathbf{\Sigma}^1_2$-absoluteness under $<\kappa$-closed partial orders is exactly a Σ_2-reflecting cardinal.

Lemma 9.8. *Assume that generic absoluteness holds for $\mathbf{\Sigma}^1_2$-subsets of ${}^\kappa\kappa$ under $<\kappa$-closed forcings. Then κ^+ is a Σ_2-reflecting cardinal in* L.

Proof. Let $\nu = \kappa^+$. First, assume, toward a contradiction, that there is an $\alpha < \kappa^+$ such that $\nu = (\alpha^+)^{\mathrm{L}}$. By the results of Section 2, there is a tree T on $\kappa \times \kappa$ such that $p[T]$ is equal to the set of all $x \in {}^\kappa 2$ with

$$\langle \mathrm{H}(\kappa^+), \in\rangle \models \text{“}\langle \kappa, \in_x\rangle \textit{ is a well-order of order-type } \beta \textit{ and there is a surjection } f : \alpha \longrightarrow \beta \textit{ that is an element of } \mathrm{L}\text{”}$$

in V and every generic extension of V by a $<\kappa$-closed forcing.

Let G be $\mathrm{Col}(\kappa,\nu)$-generic over V and $x \in ({}^\kappa 2)^{\mathrm{V}[G]}$ such that $\langle \kappa, \in_x\rangle$ is a well-order of order-type ν. Then $x \notin p[T]^{\mathrm{V}[G]}$ and there is a $x_0 \in ({}^\kappa\kappa)^{\mathrm{V}}$ with $x_0 \notin p[T]^{\mathrm{V}}$, a contradiction. Hence ν is inaccessible in L.

Let $\varphi(u, v, w_0, \dots, w_{n-1})$ be a Δ_0-formula, $z_0, \dots, z_{n-1} \in \mathrm{L}_\nu$ and $\mu > \nu$ such that

$$\langle \mathrm{L}, \in\rangle \models (\forall y)\ \varphi(x, y, z_0, \dots, z_{n-1})$$

holds for some $x \in \mathrm{L}_\mu$. By the results of Section 2, there is a tree T on $\kappa \times \kappa$ such that $p[T]$ is the set of all $x_0 \in {}^\kappa 2$ with

$$\langle \mathrm{H}(\kappa^+), \in \rangle \models (\exists x, y)[x, y \in \mathrm{L} \wedge \neg\varphi(x, y, z_0, \ldots, z_{n-1}) \wedge \psi(x_0, x, \kappa)]$$

in V and every generic extension of V by a $<\kappa$-closed forcing, where $\psi \equiv \psi(u, v, w)$ is the Σ_1-formula defined by (12) in Section 6. Let G be $\mathrm{Col}(\kappa, \mu^+)$-generic over V. Then $x \in \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ witnesses that $p[T]^{\mathrm{V}[G]} \neq ({}^\kappa\kappa)^{\mathrm{V}[G]}$. By $\mathbf{\Sigma}^1_2$-absoluteness, we have $p[T]^{\mathrm{V}} \neq ({}^\kappa\kappa)^{\mathrm{V}}$ and there is an $x_* \in \mathrm{L}_\nu$ with

$$\langle \mathrm{L}_\nu, \in \rangle \models (\forall y)\ \varphi(x_*, y, z_0, \ldots, z_{n-1}).$$

Since $\langle \mathrm{L}_\nu, \in \rangle$ is a Σ_1-elementary submodel of $\langle \mathrm{L}, \in \rangle$, we can conclude

$$\langle \mathrm{L}_\nu, \in \rangle \models (\exists x)(\forall y)\ \varphi(x, y, z_0, \ldots, z_{n-1}).$$

□

Next, we present an argument due to Philipp Schlicht showing that it is consistent that the class of all $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$ has the perfect subset property.

Proposition 9.9. *Let $\nu > \kappa$ be an inaccessible cardinal and γ be a cardinal. If $G \times H$ is $(\mathrm{Col}(\kappa, <\nu) \times \mathrm{Add}(\kappa, \gamma))$-generic over* V *and A is a $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa$ of cardinality greater than κ in $\mathrm{V}[G][H]$, then A contains a perfect subset in $\mathrm{V}[G][H]$.*

Proof. Let $A = p[T]$. As above, there is a $\bar{\nu} < \nu$, $i < 2$ and $F \in \mathrm{V}[G][H]$ such that F is $\mathrm{Col}(\kappa, <\bar{\nu}) \times \mathrm{Add}(\kappa, i)$-generic over V, $T \in \mathrm{V}[F]$ and $\mathrm{V}[G][H]$ is a generic extension of $\mathrm{V}[F]$ by a $<\kappa$-closed forcing. The set $p[T]^{\mathrm{V}[F]}$ has cardinality κ in $\mathrm{V}[G][H]$ and this means $p[T]^{\mathrm{V}[F]} \subsetneq p[T]^{\mathrm{V}[G][H]}$. By Lemma 7.6, there is a $\exists^x$-perfect embedding into T in $\mathrm{V}[F]$ and $p[T]^{\mathrm{V}[G][H]}$ contains a perfect subset in $\mathrm{V}[G][H]$ by Proposition 7.5. □

As above, we can combine these results with Lemma 7.15.

Corollary 9.10. *Let $\nu > \kappa$ be an inaccessible cardinal, γ be a cardinal and assume that either $\gamma > \nu$ holds or ν is a Σ_2-reflecting cardinal. If $G \times H$ is $(\mathrm{Col}(\kappa, <\nu) \times \mathrm{Add}(\kappa, \gamma))$-generic over* V *and R is $\mathbf{\Sigma}^1_1$-well-ordering of a subset of ${}^\kappa\kappa$ in $\mathrm{V}[G][H]$, then the order-type of $\langle \mathrm{dom}(R), R \rangle$ has cardinality at most κ in $\mathrm{V}[G][H]$.* □

10. Open problems

We close this paper with some questions motivated by the above results.

It is natural to ask whether Theorem 1.6 is optimal with respect to the complexity of the coded subset in the generic extension of the ground model.

Question 10.1. *Is there a partial order $\mathbb{P}$ with the following properties?*

1. *$\mathbb{P}$ preserves cofinalities and cardinalities.*
2. *If G is $\mathbb{P}$-generic over* V, *then $({}^\kappa\kappa)^{\mathrm{V}}$ is a κ-Borel subset of ${}^\kappa\kappa$ without a perfect subset in $\mathrm{V}[G]$.*

A positive answer to this question would imply that every subset of ${}^\kappa\kappa$ is κ-Borel in a cofinality-preserving generic extension of the ground model, because such a forcing could be combined with *almost disjoint coding.* In the other direction, an answer to the following question might provide a negative answer to Question 10.1.

Question 10.2. *Does* ZFC *(plus large cardinal axioms) prove nontrivial statements about the possible lengths of well-orders of subsets of* ${}^{\kappa}\kappa$ *whose graph is a* κ*-Borel subset of* ${}^{\kappa}\kappa \times {}^{\kappa}\kappa$ *?*

A positive answer to Question 10.1 would also show that the absoluteness statement of Theorem 5.2 holds for other classes of partial orders.

Question 10.3. *Does the statement of Theorem 5.2 hold if we replace* $\Gamma_{\mathrm{V}}(\mathbb{P}, G, \kappa)$ *by the class of all* $<\kappa$*-closed partial orders?*

If we restrict the canonical well-order of L to ${}^{\kappa}\kappa$, then we get a well-order whose graph is a $\mathbf{\Delta}^1_1$-subset of ${}^{\kappa}\kappa \times {}^{\kappa}\kappa$. Results of Sy-David Friedman and Peter Holy in [7] show that there is a partial order that forces "$2^\kappa = \kappa^+$" and the existence of a $\mathbf{\Delta}^1_1$-well-order of ${}^{\kappa}\kappa$. We may therefore ask whether the existence of a $\mathbf{\Delta}^1_1$-well-order of ${}^{\kappa}\kappa$ is compatible with a failure of the (GCH) at κ.

Question 10.4. *Does the existence of a well-order of* ${}^{\kappa}\kappa$ *whose graph is a* $\mathbf{\Delta}^1_1$*-subset of* ${}^{\kappa}\kappa \times {}^{\kappa}\kappa$ *imply that* $2^\kappa = \kappa^+$ *holds?*

There are many open questions concerning the perfect subset property and weakenings of it. We present two interesting examples.

Question 10.5. *Is it consistent that all* $\mathbf{\Pi}^1_1$*-subsets of* ${}^{\kappa}\kappa$ *have the perfect subset property?*

Question 10.6. *Is it consistent that every subset of* ${}^{\kappa}\kappa$ *in* $\mathrm{L}(\mathcal{P}(\kappa))$ *either has cardinality less than* 2^κ *or contains a perfect subset?*

References

[1] David Asperó and Sy-David Friedman. Definable wellorderings of H_{ω_2} and CH. Preprint available online at http://www.logic.univie.ac.at/∼sdf/papers/joint.aspero.omega-2.pdf.

[2] David Asperó and Sy-David Friedman. Large cardinals and locally defined well-orders of the universe. *Ann. Pure Appl. Logic*, 157(1):1–15, 2009.

[3] Joan Bagaria and Sy D. Friedman. Generic absoluteness. In *Proceedings of the XIth Latin American Symposium on Mathematical Logic (Mérida, 1998)*, volume 108, pages 3–13, 2001.

[4] James Cummings. Iterated forcing and elementary embeddings. In *The Handbook of Set Theory (M. Foreman, A. Kanamori, and M. Magidor, eds.)*, volume 2, pages 775–884. Springer, Berlin, 2010.

[5] Qi Feng, Menachem Magidor, and Hugh Woodin. Universally Baire sets of reals. In *Set theory of the continuum (Berkeley, CA, 1989)*, volume 26 of *Math. Sci. Res. Inst. Publ.*, pages 203–242. Springer, New York, 1992.

[6] Sy-David Friedman. Forcing, combinatorics and definability. In *Proceedings of the 2009 RIMS Workshop on Combinatorical Set Theory and Forcing Theory in Kyoto, Japan, RIMS Kokyuroku No. 1686*, pages 24–40. 2010.

[7] Sy-David Friedman and Peter Holy. Condensation and large cardinals. *Fund. Math.*, 215(2):133–166, 2011.

[8] Sy-David Friedman, Tapani Hyttinen, and Vadim Kulikov. Generalised descriptive set theory and classification theory. Preprint available online at http://www.logic.univie.ac.at/∼sdf/papers/joint.tapani.vadim.pdf.

[9] Gunter Fuchs. Closed maximality principles: implications, separations and combinations. *J. Symbolic Logic*, 73(1):276–308, 2008.

[10] Leo Harrington. Long projective wellorderings. *Ann. Math. Logic*, 12(1):1–24, 1977.

[11] Tapani Hyttinen and Mika Rautila. The canary tree revisited. *J. Symbolic Logic*, 66(4):1677–1694, 2001.

[12] Tapani Hyttinen and Jouko Väänänen. On Scott and Karp trees of uncountable models. *J. Symbolic Logic*, 55(3):897–908, 1990.

[13] Thomas J. Jech. Trees. *J. Symbolic Logic*, 36:1–14, 1971.

[14] R. B. Jensen and R. M. Solovay. Some applications of almost disjoint sets. In *Mathematical Logic and Foundations of Set Theory (Proc. Internat. Colloq., Jerusalem, 1968)*, pages 84–104. North-Holland, Amsterdam, 1970.

[15] Akihiro Kanamori. *The higher infinite*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003. Large cardinals in set theory from their beginnings.

[16] Richard Laver. Certain very large cardinals are not created in small forcing extensions. *Ann. Pure Appl. Logic*, 149(1-3):1–6, 2007.

[17] Alan Mekler and Jouko Väänänen. Trees and Π^1_1-subsets of ${}^{\omega_1}\omega_1$. *J. Symbolic Logic*, 58(3):1052–1070, 1993.

[18] Mark Nadel and Jonathan Stavi. $L_{\infty\lambda}$-equivalence, isomorphism and potential isomorphism. *Trans. Amer. Math. Soc.*, 236:51–74, 1978.

[19] Itay Neeman and Jindřich Zapletal. Proper forcings and absoluteness in $L(\mathbf{R})$. *Comment. Math. Univ. Carolin.*, 39(2):281–301, 1998.

[20] Saharon Shelah and Pauli Väisänen. The number of $L_{\infty\kappa}$-equivalent nonisomorphic models for κ weakly compact. *Fund. Math.*, 174(2):97–126, 2002.

[21] Stevo Todorčević and Jouko Väänänen. Trees and Ehrenfeucht-Fraïssé games. *Ann. Pure Appl. Logic*, 100(1-3):69–97, 1999.

[22] Jouko Väänänen. A Cantor-Bendixson theorem for the space $\omega_1^{\omega_1}$. *Fund. Math.*, 137(3):187–199, 1991.

[23] Jouko Väänänen. Games and trees in infinitary logic: A survey. In *Quantifiers (M. Krynicki, M. Mostowski and L. Szczerba, eds.)*, pages 105–138. Kluwer Academic Publishers, 1995.

Institut für Mathematische Logik und Grundlagenforschung, Fachbereich Mathematik und Informatik, Universität Münster, Einsteinstr. 62, 48149 Münster, Germany

E-mail address: philipp.luecke@uni-muenster.de