Regularity Properties and Inaccessibles

Dissertation

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Eigenanteilserklärung. The candidate acknowledges an amount of support in structuring and writing the thesis given by his supervisor that considerably exceeded what is expected and standard for doctoral dissertations. The overall structure of the thesis and the introductory paragraphs to the chapters and many sections were provided by the supervisor. The supervisor provided a substantial number of corrections of fact, mathematical detail, and argumentative structure. Moreover, the exposition of most of Chapter 2 was written by the supervisor for the joint paper [2] and is included verbatim in this thesis.

The original mathematical research content of this thesis can be found in Chapters 3, 4, and 5.

Chapter 3 is the result of a collaboration with Dr. Michel Gaspar to which both authors contributed equally. The results form part of a pre-publication entitled *Borel chromatic numbers of locally countable* F_σ *graphs and forcing with superperfect trees* jointly authored with Gaspar which is currently in submission and being reviewed [1]. The results and substantial parts of the text of this chapter are included in Gaspar's doctoral dissertation [13] submitted at the Universität Hamburg in August 2022.

Chapter 4 is the result of a collaboration with Dr. Lucas Wansner to which both authors contributed equally. The results will be forming part of a joint paper currently in preparation that has the working title *Amoebas and their regularities*, co-authored by Benedikt Löwe, Lucas Wansner, and the candidate [2]. The results and substantial parts of the text of this chapter are included in Wansner's doctoral dissertation [35] submitted at the Universität Hamburg in March 2023.

Chapter 5 is the sole work of the candidate; it benefitted from discussions with Professor Jörg Brendle during a visit in Hamburg during the month of September 2023.

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Contents

Chapter 1

Introduction

This thesis deals with independence phenomena in set theory of the reals. It is a well-known phenomenon that statements about the well-behaviour of sets of reals, especially at the second and higher levels of the projective hierarchy, are independent of the standard Zermelo-Fraenkel axioms of set theory ZFC. Usually, very simple sets of reals can be proved to be well-behaved, but the statement of well-behaviour of more complex sets are usually false in the smallest transitive model of set theory, Gödel's constructible universe **L**, and can be made true in larger models by adding generics.

One particular example of well-behaviour is the property of being Lebesgue measurable: non-Lebesgue measurable sets tend to be ill-behaved (e.g., they are used in the famous Banach-Tarski decomposition of the sphere that leads to the Banach-Tarski paradox [33]), but we know that these cannot be very simple: Borel sets are by definition Lebesgue measurable and it can be shown that sets at the first level of the projective hierarchy are as well [23, Theorem 12.2], but in **L**, not all sets at the second level of the projective hierarchy are [23, Corollary 13.10].

In general, regularity statements at the second level of the projective hierarchy are independent of ZFC, and statements of this type form a complex implication diagram of different logical strengths. This implication diagram has been studied in the past decades and is reasonably well understood at the second level of the projective hierarchy. (Details will be given in Chapter 2.) The theory of these statements is intricately interwoven with that of the existence of particular combinatorial objects (e.g., generic real numbers, quasi-generic real numbers, or real numbers with other combinatorial properties) and therefore often involves a detailed analysis of particular models of set theory obtained by forcing with a given forcing partial order over **L**. Among statements of this type, the strongest is

"for all $x \in \mathbb{R}$, $\aleph_1^{\mathbf{L}[x]} < \aleph_1$"

also known as "$\aleph_1$ is inaccessible by reals". This statement implies the existence of an inaccessible cardinal in **L** and therefore is strictly stronger in the sense of consistency strength than ZFC. It implies all regularity statements at the second level of the projective hierarchy and can therefore be seen as the strongest such principle (cf. § 2.7.)

The results in this thesis will contribute to the mentioned implication diagram and identify a number of additional regularity properties that have the maximal logical strength, i.e., are equivalent to "$\aleph_1$ is inaccessible by reals".

In particular, in Chapter 2, we shall provide the general framework with definitions and a list of established results and techniques that will be used in the thesis. Due to a general theorem known as *Ikegami's Theorem* (Theorem 2.6.1), we can prove a non-implication between regularity statements by showing that forcing with one forcing notion does not add quasigenerics for the other.

In Chapter 3, we use this technique to separate Laver and Silver regularity, answering two open questions from the published literature (cf. p. 24 for a discussion of the open questions).

In Chapter 4, we prove that a number of regularity properties are all of maximal strength at the $\mathbf{\Sigma}^1_2$ level: they imply that $\aleph_1$ is inaccessible by reals. The forcings discussed are amoeba forcing, amoeba forcing for category, and localisation forcing. Definitions of these forcing notions will be given in § 4.2.

Finally, in Chapter 5, we shall consider the less well-known forcing notions Matet and Willowtree forcing and separate their regularities from the others.

The following theorems are considered to be the main contributions of this thesis (all mentioned forcings, regularity properties, and the corresponding notation will be introduced in Chapter 2).

1. In the Laver model, $\mathbf{\Sigma}^1_2(\mathbb{L})$ holds and $\mathbf{\Delta}^1_2(\mathbb{E}_0)$ and $\mathbf{\Delta}^1_2(\mathbb{V})$ fail (Corollary 3.4.2).

2. In the Laver model, $\mathbf{\Delta}^1_2(\mathbb{V})$ fails, but for every real r, there is a splitting real over $\mathbf{L}[r]$ (Corollary 3.4.4).

3. The statement $\mathbf{\Sigma}^1_2(\mathbb{A})$ is equivalent to the statement "$\aleph_1$ is inaccessible by reals" (Corollary 4.3.5).

4. The statement $\mathbf{\Sigma}^1_2(\mathbb{UM})$ is equivalent to the statement "$\aleph_1$ is inaccessible by reals" (Theorem 4.4.2).

5. The statement $\mathbf{\Sigma}^1_2(\mathbb{LOC})$ is equivalent to the statement "$\aleph_1$ is inaccessible by reals" (Theorem 4.5.2).

6. In the Matet model, $\mathbf{\Delta}^1_2(\mathbb{V})$ fails (Corollary 5.3.8).

7. In the Sacks model, $\mathbf{\Delta}^1_2(\mathbb{W})$ fails (Corollary 5.4.3).

The thesis assumes that the reader is familiar with the theory of forcing as well as the basic theory of the constructible universe. The background can be found in the standard textbook literature such as the monograph [21].

Chapter 2

General Framework

The main object of study in *set theory of the reals* is the set of real numbers, but it is often customary to work with slightly different topological spaces. Traditionally, the set of real numbers $\mathbb{R}$ is defined as the Dedekind (or Cauchy) completion of the rational numbers $\mathbb{Q}$.

A topological space is called *Polish* if it is separable and completely metrisable. Examples of Polish spaces are the classical real numbers $\mathbb{R}$ as well as Cantor space 2^ω and Baire space ω^ω. The two latter examples are topologised with the product topology of the discrete topology on 2 and ω, respectively. Equivalently, the basic open sets are of the form $[s] = \{x \mid x \supseteq s\}$, for $s \in Z^{<\omega}$ and $x \in Z^\omega$ where $Z \in \{2, \omega\}$.

The three mentioned examples are not homeomorphic to each other: Cantor space is compact whereas the other two are not; $\mathbb{R}$ is connected whereas Baire space has a basis of clopen sets. However, Baire space is homeomorphic to the irrational numbers $\mathbb{R}\backslash\mathbb{Q}$ [21, p. 42] and many of the properties we shall be investigating in this thesis hold for one of the three spaces if and only if they hold for the others.

As a consequence, it has become customary in the field of *set theory of the reals* to work mostly over Baire space and to refer to the elements of all three topological spaces as *real numbers* or *reals*.

2.1 Complexity of sets of reals

If X is a Polish space, then the products of the form $(\omega^\omega)^k \times X$ with the product topology are also Polish spaces.

A σ*-algebra* on a Polish space X is a collection of subsets of X closed under countable intersections and unions and complements. A σ*-ideal* on X is a collection of subsets of X closed under subsets and countable unions.

The elements of the smallest σ-algebra containing the open subsets of such a space are its *Borel sets*. Borel sets can be described by specifying how they were obtained from basic open sets by the operations of countable union and complementation: such a description is a well-founded tree $T \subseteq \omega^{<\omega}$ (cf. § 2.2) that can be encoded as a real number; this is known as a *Borel code* and we denote the function that obtains the Borel set from its code by B, i.e., if c is a Borel code, then B_c is the Borel set coded by c; the details of how to do this precisely do not matter for this thesis; they are given, e.g., in [21, pp. 504–507].

If $A \subseteq \omega^\omega \times X$, we write

$$\mathrm{p}(A) := \{x \in X\,;\, \exists y (y,x) \in A\}$$

for the *projection of* A. We define the *projective hierarchy* of sets by recursion on n: if $B \in (\omega^\omega)^k \times X$, then

$$\begin{aligned} B \in \mathbf{\Sigma}^1_1 &\iff \text{there is a Borel set } A \text{ such that } B = \mathrm{p}(A),\\ B \in \mathbf{\Pi}^1_n &\iff ((\omega^\omega)^k \times X)\backslash B \in \mathbf{\Sigma}^1_n,\\ B \in \mathbf{\Sigma}^1_{n+1} &\iff \text{there is a set } A \in \mathbf{\Pi}^1_n \text{ such that } B = \mathrm{p}(A), \text{ and}\\ B \in \mathbf{\Delta}^1_n &\iff B \in \mathbf{\Sigma}^1_n \cap \mathbf{\Pi}^1_n. \end{aligned}$$

We refer to the sets $\mathbf{\Delta}^1_n$, $\mathbf{\Sigma}^1_n$, and $\mathbf{\Pi}^1_n$ as the *nth level of the projective hierarchy*. The projective hierarchy is proper in the sense that $\mathbf{\Delta}^1_n \subsetneq \mathbf{\Sigma}^1_n$, $\mathbf{\Delta}^1_n \subsetneq \mathbf{\Pi}^1_n$, $\mathbf{\Sigma}^1_n \subsetneq \mathbf{\Delta}^1_{n+1}$, and $\mathbf{\Pi}^1_n \subsetneq \mathbf{\Delta}^1_{n+1}$ and it measures the descriptive complexity of sets of reals in second-order arithmetic: roughly, sets on the nth level of the projective hierarchy need an alternating quantifier sequence of length n to be defined. We call a pointclass *projective* if it is one of these pointclasses. For details, cf. [23, Section 12].

2.2 Trees and arboreal forcing notions

We use the usual notation for sequences, i.e., if $s, t \in \omega^{<\omega}$ and $k \in \omega$, we write $s^\frown t$ for the concatenation of s and t and sk for the unique sequence that has s as initial segment and continues with the value k.

As usual, a *tree on* X is a subset of $X^{<\omega}$ closed under initial segments. In our case, X is either 2 or ω. If T is a tree on X, we write $[T] := \{x \in X^\omega \mid \forall n (x{\upharpoonright}n \in T)\}$ for the set of *branches through* T.

If T is a tree and $t \in T$, we say that t *splits in* T if there are at least $k \neq \ell$ such that $t^\frown k \in T$ and $t^\frown \ell \in T$; we say that t *splits infinitely in* T if there are infinitely many k such that $t^\frown k \in T$. Each tree T has a unique element of minimal length that splits in T; we call this the *stem of* T, in symbols, $\mathrm{st}(T)$. A tree T is called a *Sacks tree* or *perfect tree* if for each $t \in T$ there is an n $s \supsetneq t$ such that $s \in T$ splits in T.

Other notions of tree that will play a prominent role in this thesis are Miller and Laver trees: a tree T is called a *Miller tree* or *superperfect tree* if for each $t \in T$ there is a $s \supsetneq t$ such that $s \in T$ splits infinitely in T; a tree T is called a *Laver tree* if it has a splitting node and for each $t \in T$ such that $\mathrm{st}(T) \subseteq t$, the node t is infinitely splitting.

A forcing notion $\mathbb{P}$ is called *arboreal* if each condition is a perfect tree on either 2 or ω, for each $T \in \mathbb{P}$ and $t \in T$, we have that $T_t := \{s \in T\,;\, s \subseteq t \text{ or } t \subseteq s\} \in \mathbb{P}$, and we have that $T \leqslant T'$ implies that $[T] \subseteq [T']$. We have that $[T]$ is a closed subset of either Cantor or Baire space. We write $T \leqslant_0 T'$ if $[T] \subseteq [T']$ and $\mathrm{st}(T) = \mathrm{st}(T')$.[1]

[1]This notation will coincide with the relations $\leqslant_n$ and $\leqslant^A_n$ for fusion sequences, defined in § 2.9.

As mentioned in Chapter 1, we shall refer to [21, Section 14] for the basics of forcing. In this section, we provide the basic definitions needed for the results in this thesis. If G is a generic filter for $\mathbb{P}$, then $\bigcap_{T \in G}[T] = \{g\}$ is a singleton where g is an element of Cantor or Baire space. We identify the generic filter with that real.

Sacks forcing, denoted by $\mathbb{S}$, consists of all perfect trees ordered by inclusion. Note that if $\mathbb{P}$ is arboreal, then $\mathbb{P} \subseteq \mathbb{S}$.

Miller forcing $\mathbb{M}$ and *Laver forcing* $\mathbb{L}$, consist of all Miller or Laver trees, respectively, ordered by inclusion.

Hechler forcing, denoted by $\mathbb{D}$, consists of pairs (s, f), where $s \in \omega^{<\omega}$ and $f \in \omega^{\omega}$, such that $s \subseteq f$. We say that $(t, g) \leqslant (s, f)$ iff $s \subseteq t$ and for all $k \geqslant \mathrm{lh}(s)$, $g(k) \geqslant f(k)$.

Silver forcing, denoted by $\mathbb{V}$, consists of partial functions f from ω to 2, such that $|\omega \backslash \mathrm{dom}(f)| = \omega$. For $g, f \in \mathbb{V}$, we say that $g \leqslant f$ if and only if $f \subseteq g$.

Matet forcing, denoted by $\mathbb{T}$, consists of pairs (s, A), where $s \in \omega^{<\omega}$ is strictly increasing and $A \subseteq [\omega]^{<\omega}$ is infinite, and for all $a \in A$, $\max(\mathrm{ran}(s)) < \min(\mathrm{ran}(a))$. We order it by $(t, B) \leqslant (s, A)$ if and only if

$$s \subseteq t \forall b \in B \exists A' \subseteq A(|A'| < \omega \wedge b = \cup A') \wedge \exists A'' \subseteq A(\mathrm{ran}(t) \backslash \mathrm{ran}(s) \subseteq A'').$$

One can focus only on the Matet conditions which are of the form (s, A) such that there is an enumeration $(a_n)_{n \in \omega}$ of A, such that for all $n \in \omega$, $\max(a_n) < \max(a_{n+1})$.

Note that every Matet condition defines a Miller tree, since for a Matet condition (s, A), we can define a tree T on ω^{ω} as follows:

$$\mathrm{st}(T) = s \text{ and } \forall t \in T, \mathrm{succ}(t) = \{a \in A : \max(t) < \min(a)\}$$

where $\mathrm{succ}(t)$ denotes the set of successors of t.

However, the ordering on $\mathbb{T}$ is not the same as the inclusion on Miller trees. Matet forcing was introduced by Matet in [29].

Willowtree forcing, denoted by $\mathbb{W}$, consists of pairs (f, A), where f is a partial function from ω to 2, such that $\omega \backslash \mathrm{dom}(f) = \bigcup A$, where $A \subseteq [\omega]^{<\omega}$ is infinite, and $\max(a_n) < \min(a_{n+1})$ where $(a_n)_{n \in \omega}$ is an enumeration of A. We order it by $(g, B) \leqslant (f, A)$ if and only if

$$f \subseteq g \text{ and } \forall b \in B \exists A' \subseteq A(b = \bigcup A') \text{ and } \forall a \in A(a \subseteq \mathrm{dom}(g) \implies g \restriction a \text{ is constant }).$$

This forcing notion was introduced and studied in [4].

The forcing notions $\mathbb{W}$ and $\mathbb{V}$, $\mathbb{T}$, and $\mathbb{R}$ are uniform versions of $\mathbb{S}$, $\mathbb{M}$, and $\mathbb{L}$, respectively and were studied in Brendle's [4] where the following implication diagram is given:

$$\begin{array}{ccccc}
\mathbb{S} & \supseteq & \mathbb{M} & \supseteq & \mathbb{L} \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \\
\mathbb{W} & \supseteq & \mathbb{T} & & \rotatebox{90}{$\subseteq$} \\
\rotatebox{90}{$\subseteq$} & & & \rotatebox{45}{$\subseteq$} & \\
\mathbb{V} & & \supseteq & & \mathbb{R}
\end{array}$$

We shall call this diagram the *Uniform Forcings Diagram*; it and its consequence for regularity properties will be the main topic of Chapter 5.

A tree $p \subseteq 2^{<\omega}$ is an E_0*-tree* if and only if it is perfect and for every splitting node $s \in p$, there are $s_0 \supseteq s^\frown 0$ and $s_1 \supseteq s^\frown 1$, of the same length, such that

$$\left\{x \in 2^\omega \mid s_0^\frown x \in [p]\right\} = \left\{x \in 2^\omega \mid s_1^\frown x \in [p]\right\}.$$

The partial order consisting of E_0-trees is called E_0*-forcing*, denoted by $\mathbb{E}_0$. The notions of E_0-trees and E_0-forcing were introduced by Zapletal [36, § 2.3.10] and are closely connected to the equivalence relation E_0, the minimal non-smooth Borel equivalence relation on Baire space. We call an E_0-tree a *Silver tree* if $s_0 = s^\frown 0$ and $s_1 = s^\frown 1$. The partial order of all Silver trees is naturally isomorphic to Silver forcing $\mathbb{V}$.[2]

Definition 2.2.1. *An arboreal forcing notion* $\mathbb{P}$ *has the* pure decision property *if and only if for every* $p \in \mathbb{P}$ *and every sentence* φ *there exists* $q \leqslant_0 p$ *such that* q *decides* φ*.*

Fact 2.2.2. *Sacks forcing* $\mathbb{S}$*, Silver forcing* $\mathbb{V}$*, Miller forcing* $\mathbb{M}$*, Laver forcing* $\mathbb{L}$*, and Matet forcing* $\mathbb{T}$ *have the pure decision property.*

Proof. The cases for Sacks, Silver, Miller, and Laver are classical; the case for Matet forcing will be proved in Theorem 5.2.3. □

If $\mathbb{P}$ is any of our forcing notions, we call the generic extension obtained from $\mathbf{L}$ by a length ω_1 iteration of $\mathbb{P}$ with countable support the $\mathbb{P}$*-model*, i.e., the *Sacks model*, *Miller model*, *Laver model*, etc. Similarly, the generic extension obtained from $\mathbf{L}$ by a length ω_2 iteration of $\mathbb{P}$ with countable support is called the ω_2*-*$\mathbb{P}$*-model*. If $\mathbb{P}$ is a forcing notion that preserves $\aleph_1$, even in an ω_2-iteration, then the ω_2-$\mathbb{P}$-model is a model of $\neg\mathsf{CH}$ and contains $\aleph_2$ many $\mathbb{P}$-generic reals.

Fact 2.2.3. In the $\mathbb{P}$-model, the following statement is true: "for every $x \in \mathbb{R}$, there is a $\mathbb{P}$-generic over $\mathbf{L}[x]$".

Cf. § 2.5 for forcing notions living on other Polish spaces.

2.3 Regularity properties

At the highest level of abstraction, a *regularity property* is just any property of sets of reals. However, we shall consider only regularity properties that are derived from forcing notions. In this, we follow Zapletal's framework of *idealised forcing* [36, 37] as discussed in [25, Chapter 2] and [34, §§ 3 & 4].

[2] In his "historical remark", Zapletal traces this identification to a conversation with Löwe at the *Very Informal Gathering* in Los Angeles in 2003 [36, p. 30].

Getting an ideal from a forcing notion. If $\mathbb{P}$ is an arboreal forcing notion, then we say that $A \subseteq \omega^\omega$ is $\mathbb{P}$-*null* if for each $T \in \mathbb{P}$ there is some $S \leqslant T$ such that $[S] \cap A = \varnothing$. We denote the set of all $\mathbb{P}$-null sets by $\mathcal{N}_\mathbb{P}$ and the σ-ideal generated by $\mathcal{N}_\mathbb{P}$ by $\mathcal{I}_\mathbb{P}$. We say that $A \in \mathcal{I}^*_\mathbb{P}$ if for each $T \in \mathbb{P}$ there is some $S \leqslant T$ such that $[S] \cap A \in \mathcal{I}_\mathbb{P}$. Furthermore, we say that A is $\mathbb{P}$-*measurable* if for every $T \in \mathbb{P}$ there is some $S \leqslant T$ such that either $[S] \backslash A \in \mathcal{I}_\mathbb{P}$ or $[S] \cap A \in \mathcal{I}_\mathbb{P}$.

Getting a forcing notion from an ideal. If I is a σ-ideal over a Polish space X, then we define $\mathbb{P}_I$ as the partial order of Borel sets not in I, ordered by inclusion. A set $A \subseteq X$ is said to be I-*regular* if for every set $B \in \mathbb{P}_I$ there exists $C \leqslant B$, such that either $C \cap A = \varnothing$ or $C \subseteq A$.

If $\mathbb{P}$ is an arboreal forcing notion and I is a σ-ideal, both $\mathbb{P}$-measurability and I-regularity are examples of *regularity properties*. It turns out that for well-behaved arboreal forcing notions, these notions coincide.

Fact 2.3.1. Let $\mathbb{P}$ be one of the arboreal forcings listed in § 2.2. Then a set A is $\mathbb{P}$-measurable if and only if it is $\mathcal{I}^*_\mathbb{P}$-regular.

Proof. Cf. [34, Theorem 4.3.8] where this is proved for all proper and strongly tree-like forcing notions $\mathbb{P}$ [34, Definition 4.2.17]. □

In general, sets on the first level of the projective hierarchy are regular whereas sets on the second level of the projective hierarchy are not in **L**.

Fact 2.3.2. Let $\mathbb{P}$ be one of the arboreal forcings listed in § 2.2. Then all $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ sets are $\mathbb{P}$-measurable. Furthermore, in **L**, there is a $\mathbf{\Delta}^1_2$ set that is not $\mathbb{P}$-measurable.

Proof. Cf. [25, Propositions 2.2.3 & 2.2.4] where this is proved for all σ-ideals I such that $\mathbb{P}_I$ is proper. □

If $\mathbf{\Gamma}$ is one of the projective classes, i.e., $\mathbf{\Delta}^1_n$, $\mathbf{\Sigma}^1_n$, or $\mathbf{\Pi}^1_n$ and $\mathbb{P}$ one of our arboreal forcing notions, we write $\mathbf{\Gamma}(\mathbb{P})$ for "any set in $\mathbf{\Gamma}$ is $\mathbb{P}$-measurable". Fact 2.3.2 implies that those statements for $n \geqslant 2$ imply $\mathbf{V} \neq \mathbf{L}$.

It is one of the aims of the area of set theory of the reals to determine the implications between statements of the form $\mathbf{\Delta}^1_2(\mathbb{P})$ and $\mathbf{\Sigma}^1_2(\mathbb{P})$ for our arboreal forcing notions $\mathbb{P}$. These statements are usually characterised in terms of *transcendence over* **L**, i.e., an equivalence between $\mathbf{\Gamma}(\mathbb{P})$ and the existence of certain objects that cannot exist in **L**.

Topologies and category bases. Some regularity properties are not defined in terms of I-regularity or $\mathbb{P}$-measurability; e.g., the Baire property was originally defined in topological terms. In [35, § 2.2] provides a general framework for these and links them to the combinatorial regularity properties.

Let X be a set and $C \subseteq \mathcal{P}(X)$. We call (X, C) is a *weak category base* if and only if

(i) $X = \bigcup C$ and

(ii) for every $A, A' \in C$, $A \cap A'$ contains an element of C or for every $c \in C$, there is some $c' \in C$, such that $c' \subseteq c$ and $c' \cap (A \cap A') = \varnothing$.

We call (X, C) a *category base* if in addition for every $c \in C$ and every $C' \subseteq C$ consisting of pairwise disjoint sets with $|C'| < |C|$, we have

(i) if $c \cap \bigcup C'$ contains an element of C, then there is some element $c' \in C'$ such that $c \cap c'$ contains an element of C and

(ii) if $c \cap \bigcup C'$ does not contain an element of C, then there is some $c' \subseteq c$ such that $c' \in C$ and $c' \cap \bigcup C' = \varnothing$.

We refer to the elements of C as *regions* and say that $A \subseteq X$ is C*-singular* if and only if for every region c there is a region $c' \subseteq c$ such that $c' \cap A = \varnothing$; a subset $A \subseteq X$ is C*-small* if it is a countable union of C-singular sets; finally, $A \subseteq X$ is said to be C*-measurable* if and only if for every region c, there is a region c' such that $c' \subseteq c$ and either $c' \backslash A$ or $c' \cap A$ is C-small.[3] The collection of all C-small sets is denoted by $\mathcal{I}_C$; this set forms a σ-ideal.

Similarly, for a region c, we say that $A \subseteq X$ is C*-not small in* c if $c \cap A$ is not C-small and that it is C*-not small everywhere in* c if $c' \cap A$ is not C-small for any region $c' \subseteq c$. The ideal I^*_C is defined to be the collection of all subsets of X such that there is no region c for which A is C-not small everywhere in c.[4]

The set C is a partial order with the ordering $\subseteq$; we say that the weak category base (X, C) has the *countable chain condition* if the partial order $(C, \subseteq)$ does. A weak category base (X, C) is called *proper* if $(C, \subseteq)$ is a proper forcing notion, $\mathcal{I}_C$ is a proper σ-ideal, and every region is not C-small.

Proposition 2.3.3 (Wansner)**.**

(a) For each of the forcing notions $\mathbb{P}$ listed in § 2.2, the set $C_{\mathbb{P}} := \{[T]\,;\, T \in \mathbb{P}\}$ forms a weak category base. Furthermore, the notions of $\mathbb{P}$-measurability and $C_{\mathbb{P}}$-measurability are equivalent.

(b) If the regions of a weak category base (X, C) with the countable chain condition form a basis for a topology on X, then the notions of the Baire property in that topology and being C-measurable are equivalent.

Proof. Cf. [35, Proposition 2.2.10 (a) & (d)]. □

Proposition 2.3.4. *If (X, C) is a proper weak category base that has the countable chain condition, then $\mathcal{I}_C = \mathcal{I}^*_C$.*

Proof. Cf. [35, Proposition 2.2.17]. □

Definition 2.3.5. *If X is a Polish space and (X, C) is a weak category base, we say that* C is Borel compatible with X *if every region is Borel and every Borel set is C-measurable.*

[3]Wansner uses the terms "C-meagre" and "C-Baire" for "C-small" and "C-measurable", respectively.
[4]Wansner uses the term "C-abundant" for "C-not small".

Definition 2.3.6. *Let (X, C) and (Y, D) be weak category bases. A function $f : C \to D$ is called a* projection *if*

(i) whenever $c \subseteq c'$, then $f(c) \subseteq f(c')$ and

(ii) for every $c \in C$ and $d \leqslant f(c)$ there is a $c' \subseteq c$ such that $f(c') \leqslant d$.

The following rather technical lemma is the main tool in Chapter 4.

Lemma 2.3.7 (Wansner's Implication Lemma)**.** *Let X and Y be uncountable Polish spaces and (X, C) and (Y, D) be proper weak category bases such that (X, C) and (Y, D) are Borel compatible with X and Y, respectively. Assume that I_D^* is Borel generated, let $\alpha > 0$ be an ordinal, that $\langle h_\beta : \beta < \alpha \rangle$ is a sequence of Borel functions from X to Y, and that $\langle \bar{h}_\beta : \beta < \alpha \rangle$ be a sequence of projections from C to D such that*

(a) for every $\beta < \alpha$ and every $c \in C$, there is some region $c' \subseteq c$ such that $h_\beta[c'] \subseteq \bar{h}_\beta(c)$ and

(b) for every $d \in D$, there are $\beta < \alpha$ and $c \in C$ such that $\bar{h}_\beta(c) \subseteq D$.

Then for every projective pointclass $\boldsymbol{\Gamma}$, we have that if every $\boldsymbol{\Gamma}(X)$ set is C-measurable, then every $\boldsymbol{\Gamma}(Y)$ set is D-measurable.

Proof. [35, Theorem 2.2.46] □

2.4 Quasigenerics

As mentioned in § 2.3, one of the central aim of set theory of the reals is the characterisation of statements of the form $\boldsymbol{\Delta}^1_2(\mathbb{P})$ and $\boldsymbol{\Sigma}^1_2(\mathbb{P})$ by means of *transcendence over* $\mathbf{L}$. The first such transcendence result was in terms of generics:

Theorem 2.4.1 (Solovay; [21, Theorem 26.20])**.** *Every $\boldsymbol{\Sigma}^1_2$ set is Lebesgue measurable if and only if for every $x \in \mathbb{R}$, the set of random reals over $\mathbf{L}[x]$ has measure one.*

Results of this form are known as *Solovay-type characterisations*. Similarly, the characterisations of $\boldsymbol{\Delta}^1_2(\mathbb{P})$ in terms of the existence of generics are called *Judah-Shelah-type characterisations*. In general, the existence of generics is enough to prove regularity at the $\boldsymbol{\Delta}^1_2$-level [25, Proposition 2.2.5] and existence of many generics is enough to prove regularity at the $\boldsymbol{\Sigma}^1_2$-level [25, Proposition 2.2.6].

However, in general, the existence of generics is too strong for proving the equivalence. In [6], Brendle, Halbeisen, and Löwe introduced the crucial notion of *quasigenerics* for this purpose. If I is a σ-ideal and M any model of set theory, we call a real x *I-quasigeneric over M* if for any Borel set $A \in I$ with Borel code in M, we have that $x \notin A$. For an arboreal forcing notion $\mathbb{P}$, we call x *$\mathbb{P}$-quasigeneric over M* if it is $\mathcal{I}^*_{\mathbb{P}}$-quasigeneric over M.

Lemma 2.4.2. *For our arboreal forcing notions, a $\mathbb{P}$-generic over M is a $\mathbb{P}$-quasigeneric over M.*

Proof. Cf. [37, Proposition 2.1.2]. □

Lemma 2.4.3. *If $\mathbb{P}$ has the countable chain condition, then the notions of being $\mathbb{P}$-generic over M and being $\mathbb{P}$-quasigeneric over M coincide.*

Proof. Cf. [25, Lemma 2.3.2]. □

We say that $\mathbb{P}$ has the *Ikegami property* if $\mathbf{\Delta}^1_2(\mathbb{P})$ is equivalent to "for all $x \in \mathbb{R}$, there is a $\mathbb{P}$-quasigeneric over $\mathbf{L}[x]$" and $\mathbf{\Sigma}^1_2(\mathbb{P})$ is equivalent to "for all $x \in \mathbb{R}$, there is a $\mathcal{I}^*_{\mathbb{P}}$-positive set of $\mathbb{P}$-quasigenerics over $\mathbf{L}[x]$".

Proposition 2.4.4. *If $\mathbb{P}$ has the Ikegami property, then the $\mathbb{P}$-model satisfies $\mathbf{\Delta}^1_2(\mathbb{P})$.*

Proof. Fact 2.2.3 gives $\mathbb{P}$-generics over each $\mathbf{L}[x]$. Lemma 2.4.2 shows that they are $\mathbb{P}$-quasigeneric; then the Ikegami property gives the desired conclusion. □

2.5 Some topological spaces

If X is a Polish space, we say that a forcing notion $\mathbb{P}$ *lives on* X if there is a map

$$[\cdot]\colon \mathbb{P} \to \mathcal{P}(X)$$

such that $p \leqslant q$ implies $[p] \subseteq [q]$ and a generic filter G yields a singleton $\bigcap_{p\in G}[p] = \{x\}$ such that $x \in X$. Our arboreal forcing notions from § 2.2 all live on ω^ω, assigning to the tree p the set of branches $[p]$. In Chapter 4, we shall consider forcing notions that live on the spaces defined in this section. Note that the definitions of the notions of quasigenerics and the Ikegami property transfer without any problems to the setting of forcings living on a Polish space X, as long as the definition of the Polish space and the ideal are sufficiently absolute.

The set of pruned trees of half measure. We denote Lebesgue measure on 2^ω by μ. We define $\mathbf{R}$ to be the set of all pruned trees $T \subseteq 2^{<\omega}$ such that $\mu([T]) = \frac{1}{2}$. Via a bijection between ω and $2^{<\omega}$, we can think of the elements of $\mathbf{R}$ as elements of Cantor space, so $\mathbf{R} \subseteq 2^\omega$, topologised with the subspace topology.

Proposition 2.5.1. *The space $\mathbf{R}$ is a Polish space.*

Proof. Cf. [35, Proposition 2.3.3] □

The set of open sets. We consider the standard real line $\mathbb{R}$ with Lebesgue measure also denoted by μ (since no confusion is possible); as usual, we write $\mathbb{R}^+ := \{x \in \mathbb{R}\,;\, x > 0\}$. We write $\mathbf{O}$ for the set of all open subsets of $\mathbb{R}$ and fix a computable coding of the basic open sets of $\mathbb{R}$ (the open intervals with rational endpoints), i.e., a function $C\colon \omega \to \mathbf{O}$ such that $C(n)$ is the nth basic open set. We define $c\colon \mathbf{O} \to 2^\omega$ by $c(O)(n) = 1$ if and only if $C(n) \subseteq O$. We topologise the set $\mathbf{O}$ with the initial topology of the map c (with respect to the standard topology on Cantor space).

Proposition 2.5.2. *The space $\mathbf{O}$ is a Polish space.*

Proof. Cf. [35, Lemma 2.3.22] □

The universally meagre topology. For our second space, we shall consider the partial order $2^{<\omega}$. As usual, a subset $E \subseteq 2^{<\omega}$ is called *dense* if for any $s \in 2^{<\omega}$, there is a $t \in E$ such that $s \subseteq t$; it is called *open* if extensions of elements of E are elements of E.

We also define the set $\mathbf{U}$ as follows: a sequence $x \in (2^{<\omega})^\omega$ is in $\mathbf{U}$ if and only if for every $s \in 2^{<\omega}$ there are infinitely many $n \in \omega$ such that $s \subseteq x(m)$. Via a bijection between $2^{<\omega}$ and ω, we can consider $\mathbf{U}$ as a subset of Baire space. As a $\mathbf{\Pi}^0_2$ subset of Baire space, $\mathbf{U}$ with its subspace topology is a Polish space.

Let $\sigma = (\sigma(0), ..., \sigma(n-1))$ be a finite sequence of elements of $2^{<\omega}$ and E be an open dense subset of $2^{<\omega}$. We define

$$[\sigma, E] := \{x \in \mathbb{U} : \sigma \subseteq x \text{ and } \forall n \geqslant \mathrm{lh}(\sigma)(x(n) \in E)\}$$

and let U be the collection of those sets.

Proposition 2.5.3. *Every element of U is clopen in the Polish space $\mathbf{U}$. Furthermore, the set U forms a topology basis on $\mathbf{U}$.*

Proof. It is easy to see that the sets $[\sigma, E]$ are closed in the Polish space $\mathbf{U}$. To see that they are clopen, one just checks that

$$\mathbf{U}\backslash[\sigma, E] = \bigcup\{[\sigma', 2^{<\omega}] : (\sigma \nsubseteq \sigma' \text{ and } \sigma' \nsubseteq \sigma) \text{ or } \exists n \in \mathrm{dom}(\sigma'\backslash\sigma)(\sigma'(n) \notin E)\}.$$

In order to see that C is a topology base, let $[\sigma, E]$ and $[\sigma, E']$ be two elements of $C_{\mathbb{U}}$. We assume without loss of generality that $\sigma \subseteq \sigma'$, then if $x \in [\sigma, E] \cap [\sigma, E']$, we have $\sigma' \subseteq x$ and for every $n \geqslant \mathrm{lh}(\sigma)$, $x(n) \in E$. Hence for every $n \in \mathrm{dom}(\sigma'\backslash\sigma)$, $\sigma'(n) \in E$ and so $(\sigma', E \cap E') \leqslant (\sigma, E), (\sigma', E')$. Therefore, $[\sigma', E \cap E'] = [\sigma, E] \cap [\sigma, E']$. □

The topology generated by U is called the *universally meagre topology.* It is not a Polish topology. We shall see in Proposition 4.2.5 that L forms a proper weak category base that is Borel compatible with $\mathbf{U}$.

The localisation topology. Finally, we say that a function $f : \omega \to [\omega]^{<\omega}$ is a *slalom* if every $n \in \omega$, $|f(n)| \leqslant n + 1$. The set of slaloms is denoted by $\mathbf{Loc}$. Using canonical bijections from ω to $[\omega]^{\leqslant n+1}$, $\mathbf{Loc}$ is in bijection with Baire space, so we can consider it as a homeomorphic copy of Baire space.

Let F be a finite subset of Baire space and $\sigma = (\sigma(0), ..., \sigma(n-1))$ a finite sequence of elements of $\omega^{<\omega}$ such that $|\sigma(k)| = k + 1$ for all $k < n$ and $|F| \leqslant n + 1$. Define

$$[\sigma, F] := \{f \in \mathbf{Loc}\,;\ f{\restriction}\mathrm{lh}(\sigma) = \sigma \text{ and } \forall x \in F \forall n \geqslant \mathrm{lh}(\sigma)(x(n) \in f(n))\}$$

and let L be the collection of those sets.

Proposition 2.5.4. *Every element of L is clopen in the space* $\mathbf{Loc}$ *(i.e., Baire space). Furthermore, the set L forms a topology basis on* $\mathbf{Loc}$.

Proof. It is easy to see that sets of the form $[\sigma, F]$ are closed. Let us show that they are clopen:

$$\mathbf{Loc}\backslash[\sigma, F] = \bigcup\{[\sigma', \varnothing] : (\sigma \nsubseteq \sigma' \wedge \sigma' \nsubseteq \sigma) \wedge \exists x \in F \exists n \in \mathrm{dom}(\sigma' \backslash \sigma)(x(n) \notin \sigma'(n))\}.$$

In order to see that L is a topology basis, let $[\sigma, E] \cap [\sigma', E'] \neq \varnothing$. Then, we let $m = |E|+|E'|$. We show that $[\sigma, E] \cap [\sigma', E'] = A = \bigcup\{[f\restriction m, E \cup E'] : f \in [\sigma, E] \cap [\sigma, E']\}$. Clearly, $[\sigma, E] \cap [\sigma', E'] \subseteq A$. On the other hand if $f \in A$ and $f' \in [\sigma, E] \cap [\sigma', E']$ such that $f \in [f'\restriction m, E \cup E']$, then for every $x \in E \cup E'$, for every $n \geqslant \mathrm{lh}(\sigma)$, $x(n) \in f'(n)$ and for every $n \geqslant \mathrm{lh}(\sigma')$, $x(n) \in f'(n)$. Also $f\restriction m = f'\restriction m$ and for every $x \in E \cup E'$ and every $n \geqslant m$, $x(n) \in f(n)$. Hence, $f \in [\sigma, E] \cap [\sigma', E']$. □

The topology generated by L is called the *localisation topology*. It is not a Polish topology. We shall see in Proposition 4.2.7 that it is a proper weak category base that is Borel compatible with **Loc**.

2.6 Ikegami's Theorem

Ikegami's theorem is the main structural theorem of the field, connecting regularity properties to quasigenerics. It was originally proved in Ikegami's doctoral dissertation [20] and then streamlined by Khomskii [25] and generalised by Wansner [34].

Theorem 2.6.1 (Ikegami, 2010). *Let $\mathbb{P}$ be one of the arboreal forcings listed in § 2.2. Then $\mathbb{P}$ has the Ikegami property.*

Proof. Cf. [25, Theorem 2.3.7] where this was proved for a class of forcings that includes all of the mentioned ones. □

Our main use of Ikegami's theorem is to separate regularity properties from each other.

Corollary 2.6.2. *If $\mathbb{P}$ and $\mathbb{Q}$ are two forcing notions with the Ikegami property, then if an ω_1-iteration of $\mathbb{Q}$ does not add $\mathbb{P}$-quasigenerics, then the $\mathbb{Q}$-model satisfies $\mathbf{\Delta}^1_2(\mathbb{Q}) \wedge \neg\mathbf{\Delta}^1_2(\mathbb{P})$.*

Proof. Since $\mathbb{Q}$ has the Ikegami property, the $\mathbb{Q}$-model satisfies $\mathbf{\Delta}^1_2(\mathbb{Q})$ by Proposition 2.4.4. Since the ω_1-iteration that produced the $\mathbb{Q}$-model added no $\mathbb{P}$-quasigenerics, there is no $\mathbb{P}$-quasigeneric over $\mathbf{L}$ in the $\mathbb{Q}$-model. Since $\mathbb{P}$ has the Ikegami property, this means that $\mathbf{\Delta}^1_2(\mathbb{P})$ must fail and we have separated the two regularity properties. □

Ikegami's Theorem 2.6.1 will be used extensively in Chapters 3 & 5; in Chapter 4, we shall need generalisations of Ikegami's Theorem due to Wansner in the context of weak category bases.

Theorem 2.6.3 (Wansner). *Let X be an uncountable Polish subspace of ω^ω and (X, C) be a proper weak category base that is Borel compatible with X and such that the set of Borel codes of elements of $\mathcal{I}^*_C$ is $\mathbf{\Sigma}^1_2$. Then:*

(a) *Every* $\mathbf{\Delta}^1_2(X)$ *set is* C*-measurable if and only if for every* $r \in \omega^\omega$ *such that* X *is coded in* $\mathbf{L}[r]$ *there is an* $\mathcal{I}^*_C$*-quasigeneric over* $\mathbf{L}[r]$*.*

(b) *Every* $\mathbf{\Sigma}^1_2(X)$ *set is* C*-measurable if and only if for every* $r \in \omega$ *such that* X *is coded in* $\mathbf{L}[r]$*, the set* $\{x \in X\,;\, x \text{ is not } \mathcal{I}^*_C\text{-quasigeneric over } \mathbf{L}[r]\}$ *is* $\mathcal{I}^*_C$*-small.*

Proof. Cf. [35, Corollary 2.2.29]. □

2.7 Inaccessibility by reals

The statement "for all $x \in \mathbb{R}$, we have $\aleph_1^{\mathbf{L}[x]} < \aleph_1$" is known as $\aleph_1$ *is inaccessible by reals*. It means that each of the models $\mathbf{L}[x]$ is wrong about the value of $\aleph_1$ and in particular implies that the true $\aleph_1$ is inaccessible in all of these models.

Theorem 2.7.1. *If* $\aleph_1$ *is inaccessible by reals, then for each* $x \in \mathbb{R}$*, there is an inaccessible cardinal in* $\mathbf{L}[x]$*.*

Proof. As mentioned, we shall show that $\aleph_1$ is inaccessible in $\mathbf{L}[x]$. By downwards absoluteness of regularity and GCH in $\mathbf{L}[x]$, if $\aleph_1$ is not inaccessible in $\mathbf{L}[x]$, it must be a successor cardinal, i.e., there is some $\xi < \aleph_1$ such that $\mathbf{L}[x]$ has surjections from ξ to any countable ordinal. But since ξ is countable, there is some y that codes a wellorder of length ξ. Then $\mathbf{L}[x, y]$ is a model of "ξ is countable" and every countable ordinal has size at most ξ, so $\aleph_1^{\mathbf{L}[x,y]} = \aleph_1$. □

This is a transcendence property over $\mathbf{L}$ which implies all others at the second level of the projective hierarchy in the presence of Ikegami's theorem.

Proposition 2.7.2. *If* $\mathbb{P}$ *has the Ikegami property, then* $\aleph_1$ *being inaccessible by reals implies* $\mathbf{\Sigma}^1_2(\mathbb{P})$*.*

Proof. There are $\aleph_1^{\mathbf{L}[x]}$ many Borel codes in $\mathbf{L}[x]$, so if $\aleph_1$ is inaccessible by reals, countably many; thus

$$N_x := \{A\,;\, A \in \mathcal{I}^*_{\mathbb{P}} \text{ is a Borel set with code in } \mathbf{L}[x]\}$$

is countable. Every real that is not $\mathbb{P}$-quasigeneric over $\mathbf{L}[x]$ must be in some set in N_x, so contained in $\bigcup N_x$ which is a countable union of elements of $\mathcal{I}^*_{\mathbb{P}}$, and therefore itself in $\mathcal{I}^*_{\mathbb{P}}$. By the Ikegami property, this is equivalent to $\mathbf{\Sigma}^1_2(\mathbb{P})$. □

Most of the regularity statements are strictly weaker than inaccessibility by reals; in fact, they have the consistency strength of ZFC. Some regularity statements have large cardinal strength at the third level of the projective hierarchy; the most famous example is Lebesgue measurability [31]. Very few properties already have large cardinal strength at the second level of the projective hierarchy. This was proved for Hechler forcing and eventually different forcing by Brendle and Löwe [9, 10]. The result about Hechler forcing will be relevant in Chapter 4.

Fact 2.7.3 (Brendle-Löwe 1999; [9, Proposition 5.13])**.** *If every* $\mathbf{\Sigma}^1_2$ *set is Hechler measurable, then* $\aleph_1$ *is inaccessible by reals.*

2.8 Brendle-Łabędzki lemmas

One way to prove inaccessibility by reals from regularity properties is with the help of *Brendle-Łabędzki lemmas.* In this section, we shall give a very abstract definition.

Definition 2.8.1. *An* assignment *is a pair of formulas* (Ψ, Φ) *such that in every transitive model of set theory* M*, we have that if* $A \in M$ *and* $M \models \Psi(A)$*, then, in* M*,* Φ *defines an injection* $a \mapsto c_a^A$ *with domain* A*, i.e.,*

$$M \models \Phi(a, c, A) \iff a \in A \wedge \\ \big(a \in A \wedge b \in A \wedge a \neq b \wedge \Phi(a, c, A) \wedge \Phi(b, d, A)\big) \Rightarrow c \neq d$$

where we let c_a^A *be the unique* c *such that* $\Phi(a, c, A)$*.*

We fix an ideal I on some Polish space X.

Definition 2.8.2. *An assignment* (Ψ, Φ) *is called* I-canonical *if for each transitive model* M *of set theory, we have that*

$$M \models \exists A(\Psi(A) \wedge |A| = 2^{\aleph_0})$$

and if $M \models \Psi(A)$ *and* $a \in A$*, then* c_a^A *is a Borel code in* M *such that the Borel set* B_a^A *coded by* c_a^A *is in* I*.*

In the following, we shall say that an ideal I *satisfies a Brendle-Łabędzki lemma* if there is an I-canonical assignment (Φ, Ψ) such that for each $Z \in I$ and any set A we have that

$$\{B_a^A\,;\, a \in A \wedge B_a^A \subseteq Z\}$$

is countable. A forcing notion $\mathbb{P}$ *satisfies a Brendle-Łabędzki lemma* if the ideal $\mathcal{I}_{\mathbb{P}}^*$ does.

The name derives from the fact that the first lemma of this type was implicitly used in an argument about eventually different forcing by Brendle and subsequently written up and published by Łabędzki in [26, Theorem 4.7] who also proved a similar lemma for Hechler forcing [27, Theorem 6.2].[5] These two lemmas were then used to prove that $\aleph_1$ is inaccessible by reals in [9, Theorem 5.9] and [10, Theorem 2], respectively. The abstract proof below of the following theorem follows precisely the lines of these two proofs.

Theorem 2.8.3. *Let* $\mathbb{P}$ *be a forcing notion with the Ikegami property that satisfies a Brendle-Łabędzki lemma, then* $\mathbf{\Sigma}_2^1(\mathbb{P})$ *implies that* $\aleph_1$ *is inaccessible by reals.*

Proof. Fix any $x \in \mathbb{R}$. By the Ikegami property, the assumption gives us that the set

$$N_x := \{z \in X\,;\, z \text{ is not } \mathcal{I}_{\mathbb{P}}^*\text{-quasigeneric over } \mathbf{L}[x]\}$$

is in $\mathcal{I}_{\mathbb{P}}^*$. By definition of quasigenericity, every Borel set B with Borel code in $\mathbf{L}[x]$ satisfies $B \subseteq N_x$. By the canonicity assumption, we find a particular A in $\mathbf{L}[x]$ such that $\mathbf{L}[x] \models \Psi(A)$ and $|A| = \aleph_1^{\mathbf{L}[x]}$ and for all $a \in A$, we have $B_a^A \subseteq N_x$. But by the Brendle-Łabędzki lemma, the set $\{B_a^A\,;\, a \in A \wedge B_a^A \subseteq N_x\}$ is countable, so $\aleph_1^{\mathbf{L}[x]}$ is countable which is what we aimed to show. □

[5] In the first mentioned result on $\mathbb{E}$, Ψ was "is a family of pairwise eventually different functions f_a" and B_a^A was the set $\{x\,;\, x \text{ and } f_a \text{ agree on infinitely many values}\}$. In the second result on $\mathbb{D}$, Ψ was "is an almost disjoint family" and B_a^A was the set $\{x\,;\, \text{ran}(x) \text{ does not contain any elements of } a\}$.

2.9 Fusion sequences

The conditions of arboreal forcings consisting of perfect or superperfect trees can be identified with $2^{<\omega}$ or $\omega^{<\omega}$, respectively. In this section, we provide the necessary notation for this and introduce fusion sequences.

Trees on 2. Let $p \in \mathbb{P} \subseteq 2^{<\omega}$. We start by identifying the nodes of p with that of the full tree $2^{<\omega}$, defining an injection $\sigma \mapsto \sigma *_0 p$ between $2^{<\omega}$ and p by $\langle\rangle *_0 p := \mathrm{st}(T)$ and if $\sigma *_0 p$ is defined, there is a splitting node in T above $\sigma *_0 p$, i.e., some τ such that both $\tau^\frown 0$ and $\tau^\frown 1$ are in T; let $(\sigma^\frown 0) *_0 p := \tau^\frown 0$ and $(\sigma^\frown 1) *_0 p := \tau^\frown 0$. Then, for $x \in 2^\omega$, we define $x *_0 p$ to be $\bigcup_{n\in\omega}(x\restriction n) *_0 p$. Furthermore, we define

$$p *_0 \sigma := \{\tau *_0 p \mid \tau \subseteq \sigma \text{ or } \sigma \subseteq \tau\}$$

to be the $*_0$-*restriction of* p *to* σ.

If we write $\mathrm{L}_n(p)$ for the set of nth splitting nodes of p (in particular, $\mathrm{L}_0(p) := \mathrm{st}(p)$), then we say that $p \leqslant_n q$ if $p \subseteq q$ and $\mathrm{L}_n(p) = \mathrm{L}_n(q)$. A sequence $\{p_n\,;\, n \in \omega\}$ of perfect trees such that $p_{n+1} \leqslant_n p_n$ for all $n \in \omega$ is called a *fusion sequence*. For fusion sequences, the set $q := \bigcap_{n\in\omega} p_n$ is a perfect tree with $q \leqslant_n p_n$ for every $n \in \omega$.

Trees on ω. The mechanism for trees on ω is very similar but comes with a number of additional technicalities since we need to talk about frontiers. For this, let $\mathbb{P}$ be an arboreal forcing notion such that every tree $p \in \mathbb{P}$ is a superperfect tree on ω. Once more, we identify the nodes of p with the elements of $\omega^{<\omega}$ via an injection $\sigma \mapsto \sigma *_0 p$ defined recursively by $\langle\rangle *_0 p = \mathrm{st}(p)$ and $(\sigma^\frown\langle n\rangle) *_0 p$ is the minimal splitting node of p extending the nth immediate successor of $\sigma *_0 p$, in the lexicographic order (for $n \in \omega$). For $x \in \omega^\omega$, we write $x *_0 p := \bigcup_{n\in\omega}(x\restriction n) *_0 p$. Furthermore, we define

$$p *_0 \sigma := \{\tau *_0 p \mid \tau \subseteq \sigma \text{ or } \sigma \subseteq \tau\}$$

to be the $*_0$-*restriction of* p *to* σ and

$$\mathrm{L}_n(p) := \{\sigma *_0 p \mid \sigma \in n^n\}$$

to be the *nth* $*_0$-*diagonal level of* p.

A set $B \subseteq \omega^{<\omega}$ is a p-*frontier* iff for every $a \in [p]$, there exist $x \in \omega^\omega$ and a unique $n \in \omega$ such that $x *_0 p = a$, and $x\restriction n \in B$. If $\sigma \in \omega^{<\omega}$ and B is a p-frontier, then $\mathrm{proj}_B(\sigma) = \{\tau \in B \mid \sigma \subseteq \tau\}$ is the *projection of* σ *to* B. We write $B[n]$ for the nth element in a frontier in a fixed enumeration of $\omega^{<\omega}$.

A sequence of frontiers $A = \{A_n\,;\, n \in \omega\}$ is a p-*chain* iff for all $\sigma \in A_{n+1}$ there exists a unique $\tau \in A_n$ such that $\tau \subsetneq \sigma$. Given a p-chain $A = \{A_n\,;\, n \in \omega\}$, we define a technical, but important operation denoted by $*_1$ by recursion:

- $\langle\rangle *_1 (p, A) = \mathrm{st}(p)$,
- $\langle n\rangle *_1 (p, A) = A_0[n]$, and

- $(\sigma^\frown\langle n\rangle) *_1 (p,A)$ is the nth immediate successor of $\sigma *_1 (p,A)$, in the lexicographic order, if $|\sigma| > 0$ is odd; and $(\sigma^\frown\langle n\rangle) *_1 (p,A)$ is the set $\mathrm{proj}_{A_{|\sigma|}}(\sigma *_1 (p,A))[n]$, if $|\sigma| > 0$ is even.

As before, this operation is extended to $x \in \omega^\omega$ by

$$x *_1 (p,A) = \bigcup_{n\in\omega}(x\restriction n) *_1 (p,A).$$

As in the case of $*_0$, we write

$$\begin{aligned}(p,A) *_1 \sigma &:= \{\tau *_1 (p,A) \mid \tau \subseteq \sigma \text{ or } \sigma \subseteq \tau\} \text{ and}\\ \mathrm{L}_n^A(p) &:= \{\sigma *_1 (p,A) \mid \sigma \in n^n\}\end{aligned}$$

for the *$*_1$-restriction of p to σ* and the *nth $*_1$-diagonal level of* (p,A), respectively. This allows us to define

$$q \leqslant_n^A p :\Leftrightarrow q \leqslant p \text{ and } \mathrm{L}_n^A(q) = \mathrm{L}_n^A(p).$$

Note that $\mathrm{L}_0^A(p)$ does not depend on A (since it just consists of the stem of p) and so $\leqslant_0^A$ does not either. We can therefore write $\leqslant_0$ for $\leqslant_0^A$.[6]

A sequence $\{p_n\,;\, n \in \omega\} \subseteq \mathbb{P}$ such that $p_{n+1} \leqslant_n^A p_n$ for all $n \in \omega$ is called a *fusion sequence*. For fusion sequences, the set $q := \bigcap_{n\in\omega} p_n$ is a superperfect tree with $q \leqslant_n^A p_n$ for every $n \in \omega$.

Definition 2.9.1. *The following recursively defined function* $\mathrm{j} : \omega^{<\omega} \to \omega^{<\omega}$ *will be called the* auxiliary map. *We let* $\mathrm{j}(\langle\rangle) = \langle\rangle$ *and suppose that* $\mathrm{j}(\sigma)$ *has been defined and* $\mathrm{j}(\sigma^\frown\langle k\rangle)$ *has been defined for all* $k < n^2$. *Let* $\{k_m\,;\, m < 2n+1\}$ *be an increasing enumeration of* $(n+1)^2\backslash n^2$ *and set*

$$\mathrm{j}(\sigma^\frown\langle k_m\rangle) := \begin{cases}\mathrm{j}(\sigma)^\frown\langle m\rangle^\frown\langle k_m\rangle, & \text{if } m < n \text{ and}\\ \mathrm{j}(\sigma)^\frown\langle n\rangle^\frown\langle k_m\rangle, & \text{if } n \leqslant m < 2n+1.\end{cases}$$

Definition 2.9.2. *A map* $i\colon \omega^{<\omega} \to \omega^{<\omega}$ *is called* height-preserving *if for each* $\sigma \in \omega^{<\omega}$, *we have that* $|i(\sigma)| = |\sigma|$. *It is called* j-stable *if*

(a) $i(\langle\rangle) = \langle\rangle$*; and*

(b) for every $n \in \omega$*, there are infinitely many natural numbers* k_n*'s such that*

$$\mathrm{j}(i(\sigma^\frown\langle k_n\rangle))\restriction 2|i(\sigma)|+1 = \mathrm{j}(i(\sigma))^\frown\langle n\rangle.$$

Note that if i is height-preserving and j-stable, then $|\mathrm{j}(i(\sigma))| = 2|\sigma|$.

Lemma 2.9.3. *Suppose that* i *is* j*-stable and height-preserving. Let* $p \in \mathbb{L}$ *and* $A = (A_n)_{n\in\omega}$ *be a* p*-chain. Then* $\mathrm{ran}(\mathrm{j}\circ i) *_1 (p,A)$ *is a stem-preserving Laver subtree of* p.

Proof. Follows from the construction. □

[6] Cf. Footnote 1.

2.10 Iterations of arboreal forcings

In this section, we shall be providing the main technical tools for dealing with iterations of arboreal forcings. As mentioned, we are interested in separation results using Corollary 2.6.2, so we should like to prove that all reals in the $\mathbb{P}$-model or ω_2-$\mathbb{P}$-model have a certain property.

For this, let $\mathbb{P}$ be any arboreal forcing notion with the pure decision property (cf. Definition 2.2.1) and α any ordinal. We denote by $\mathbb{P}_\alpha$ the countable support iteration of $\mathbb{P}$ of length α. Any element of Cantor space in the generic extension will have some name $\dot{x}$ and some condition p that forces "$\dot{x} \in 2^\omega$". We fix this name and condition for the remainder of this section.

Guiding reals and continuous reading of names. The following result shows that for forcings of our type any real added by the forcing $\mathbb{P}$ can be approximated by ground model reals that we call *guiding reals*.

Theorem 2.10.1 (Existence of guiding reals)**.** *Let $\mathbb{P}$ be an arboreal forcing notion with the pure decision property whose conditions are trees on ω and $\sigma \in \omega^{<\omega}$ be a finite sequence. For any name $\dot{x}$ for an element of Cantor space and any $p \in \mathbb{P}$ there exist $q \leqslant p$ and a ground model real g such that*

$$q *_0 \sigma^\frown k \Vdash \dot{x}\upharpoonright(|\sigma| + k) = g\upharpoonright(|\sigma| + k)$$

for all $k \in \omega$. Such a real g is called a guiding real *with respect to $\dot{x}$, σ, and q.*

Proof. The proof is a direct application of the pure decision property and the compactness of Cantor space; cf., e.g., [13, Claim 3.2.2]. □

Consequently, if we are working in a generic extension by the generic filter G and $p \in G$ was the original condition that forces "$\dot{x}$ is a real", we can extend p to a $q \in G$ such that there is a guiding real for each σ. Without loss of generality, we can assume that $p = q$. In this context, we now write x_σ for the guiding real with respect to $\dot{x}$, σ, p, and q, suppressing the rest of the notation. Similarly, if T is a tree with stem $\mathrm{st}(T)$, we write $x_T := x_{\mathrm{st}(T)}$. In this setting, we define for any $r \leqslant p = q$ the *tree of r-interpretations for* $\dot{x}$ by

$$T_r(\dot{x}) = \{s \in \omega^{<\omega} \mid \exists r' \leqslant r(r' \Vdash s \subseteq \dot{x})\}.$$

We can use the guiding reals to define the function $f : [q] \to \omega^\omega$ by $f(a *_0 q) := \lim_{n\in\omega} x_{a\upharpoonright n}$ (if the conditions are trees on ω; with the obvious modification if they are trees on 2). This function is continuous and if $\dot{x}_{\mathrm{gen}}$ is a name for the generic real, then $q \Vdash f(\dot{x}_{\mathrm{gen}}) = \dot{x}$. This is also known as *continuous reading of names* [37, Definition 3.1.1].[7]

Fact 2.10.2. *If $\mathbb{P} = \mathbb{L}$, then the function witnessing continuous coding of names is injective.*

[7] Cf. also [13, p. 31].

Proof. This follows from a result by Gray proved in [8, Theorem 16].[8] Cf. also [13, Fact 3.2.3 & p. 32]. □

The existence of guiding reals can be generalised to the iteration case.

Theorem 2.10.3 (Existence of guiding reals for iterations)**.** *Let $\mathbb{P}$ be an arboreal forcing notion with the pure decision property, α any ordinal, and $p \in \mathbb{P}_\alpha$. For each $\beta \in \alpha$ and $\sigma \in \omega^{<\omega}$, there exists $p^\beta_\sigma \leqslant p$ and a $\mathbb{P}_\beta$-name for a real $\dot{x}^\beta_\sigma$ such that for all $\gamma \in \beta$, $p^\beta_\sigma \restriction \gamma \Vdash p^\beta_\sigma(\gamma) = p(\gamma)$ and*

$$p^\beta_\sigma \restriction \beta \Vdash p^\beta_\sigma(\beta) \leqslant_0 p(\beta)$$

and if we define $q(\sigma,\beta) := (p^\beta_\sigma(\beta) *_0 \sigma^\frown k)^\frown p^\beta_\sigma \restriction (\beta,\alpha)$, then

$$q(\sigma,\beta) \Vdash \dot{x}\restriction(|\sigma|+k) = \dot{x}^\beta_\sigma \restriction (|\sigma|+k).$$

Proof. Note that the pure decision property means that for any statement there exists $r \leqslant p$ such that for all $\gamma \in \beta$, $r\restriction\gamma \Vdash r(\gamma) = p(\gamma)$ and $r\restriction\beta \Vdash r(\beta) \leqslant_0 p(\beta)$ and $r\restriction[\gamma,\alpha)$ decides the statement. This way we get $p^\beta_\sigma \leqslant p$ such that, for every $k \in \omega$:

$$p^\beta_\sigma \restriction \beta \Vdash p^\beta_\sigma(\beta) *_0 (\sigma^\frown k)^\frown p^\beta_\sigma \restriction (\beta,\alpha) \text{ decides } \dot{x}\restriction(|\sigma|+k).$$

The claim can now be proved with a compactness argument similar to the one used in Theorem 2.10.1 (cf. [13, Claim 3.2.2] for a proof). □

Faithfulness. As mentioned, we try to preserve a property of reals in iterations. For this, we shall define a notion of *faithfulness*. We fix some finite subset $F \subseteq \alpha$ and a function $\eta \colon F \to \omega$. For $p, q \in \mathbb{P}_\alpha$, we write

$$q \leqslant_{(F,\eta)} p \text{ if and only if } q \leqslant p \text{ and for all } \gamma \in F \text{ and } \sigma \in \prod_{\gamma\in F} \eta(\gamma)^{\eta(\gamma)}, \text{ we have that } q\restriction\gamma \Vdash q *_0 \sigma(\gamma) = p *_0 \sigma(\gamma).$$

For forcings whose conditions are trees on 2 rather than ω, we need to restrict σ to $\prod_{\gamma\in F} 2^{\eta(\gamma)}$ for this definition.

We extend the definitions of $*_0$ and $*_1$ to the conditions $p \in \mathbb{P}_\alpha$ in a specific situation as follows. Fix $F \subseteq \alpha$ finite, $\eta \colon F \to \omega$, $\sigma \in \prod_{\gamma\in F} \eta(\gamma)^{\eta(\gamma)}$, and A^γ a chain of $p(\gamma)$-frontiers for each $\gamma \in F$ (writing $A := \{A^\gamma \,;\, \gamma \in F\}$). Then we define $p *_0 \sigma$ such that

$$\forall\gamma \in F\, ((p *_0 \sigma)\restriction\gamma \Vdash (p *_0 \sigma)(\gamma) = p(\gamma) *_0 \sigma(\gamma))$$

and $(p, A) *_1 \sigma$ such that

$$\forall\gamma \in F\, (((p, A) *_1 \sigma)\restriction\gamma \Vdash ((p, A) *_1 \sigma)(\gamma) = (p(\gamma), A^\gamma) *_1 \sigma(\gamma))\,.$$

Definition 2.10.4. *Let A^γ a chain of $p(\gamma)$-frontiers for each $\gamma \in F$ and $\varphi(x, y)$ be a formula in two free variables. We say that $p \in \mathbb{P}_\alpha$ is φ-(F,η)-*faithful *if for any $\sigma, \sigma' \in \prod_{\gamma\in F} \eta(\gamma)^{\eta(\gamma)}$ such that $\sigma \neq \sigma'$, we have that*

$$p\restriction\max(F) \Vdash \varphi(x_{\sigma *_1 p}, x_{\sigma' *_1 p}).$$

[8] Cf. [17].

Again, for forcings whose conditions are trees on 2 rather than ω, we need to restrict σ to $\prod_{\gamma\in F} 2^{\eta(\gamma)}$ for this definition and instead of $*_1$, we need to consider only $*_0$, as the frontiers will just be the splitting levels and the technicalities involved are much simpler.

Also for Matet forcing although it's conditions are trees on ω, we shall be considering $*_0$ instead of $*_1$ since the technicalities involved in this are simpler regarding this aspect but have other intricacies of it's own which we shall discover in Chapter 5

Definition 2.10.5. *A sequence* $\{(p_n, F_n, \eta_n)\,;\, n \in \omega\}$ *is called an* augmented fusion sequence *if*

(i) F_n *is a finite subset of* α*,*

(ii) $\eta_n\colon F_n \to \omega$*,*

(iii) $F_n \subseteq F_{n+1}$*,*

(iv) $\eta_{n+1}(\gamma) \geqslant \eta_n(\gamma)$ *for* $\gamma \in F_n$*, and*

(v) $p_{n+1} \leqslant_{(F_n,\eta_n)} p_n$*.*

(vi) for all $n \in \omega$ *and* $\gamma \in \operatorname{supp}(p_n)$ *there exists* $m \in \omega$ *such that* $\gamma \in F_m$ *and* $\eta_m(\gamma) \geqslant n$*.*

We say that q *is the* fusion *of the augmented fusion sequence if for all* $\gamma \in \alpha$*,* $q{\upharpoonright}\gamma \Vdash q(\gamma) = \bigcap_{n\in\omega} p_n(\gamma)$*.*

Our objective in Chapters 3 & 5 will be to obtain a fusion sequence such that q_n is φ-(F_n, η_n)-faithful. This will guarantee that the fusion of this sequence will be faithful as well.

2.11 Implications between regularity properties

As mentioned, one of the themes of the research area of set theory of the reals has been the study of the implication diagram between the regularity properties derived from arboreal forcings. In Figure 2.1, we give the state of knowledge as it had been established before this thesis. This diagram is complete in the sense that for any two statements in the diagram, there is an implication between them if and only if there is an arrow in the transitive closure of the diagram.

The notable exception was the non-implication between $\mathbf{\Delta}^1_2(\mathbb{L})$ and $\mathbf{\Delta}^1_2(\mathbb{V})$ which had been open and explicitly listed as an open question by Fischer, Friedman, and Khomskii [12, Question 6.3], Brendle and Löwe in [10, Figure 1] and by Ikegami in [20, Figure 2.1]. This problem is solved in this thesis; cf. Corollary 3.4.2.

In Figure 2.2, we add some of the regularities for the forcings $\mathbb{A}$, $\mathbb{E}_0$, $\mathbb{C}$, $\mathbb{T}$, and $\mathbb{W}$ to the diagram and mark the various implication questions that we tackle in this thesis. Chapter 3 will deal with $\mathbb{L}$, $\mathbb{V}$, and $\mathbb{E}_0$; Chapter 4 will deal with $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{D}$; and Chapter 5 will deal with $\mathbb{T}$ and $\mathbb{W}$. In particular, the following implications and non-implications will be proved:

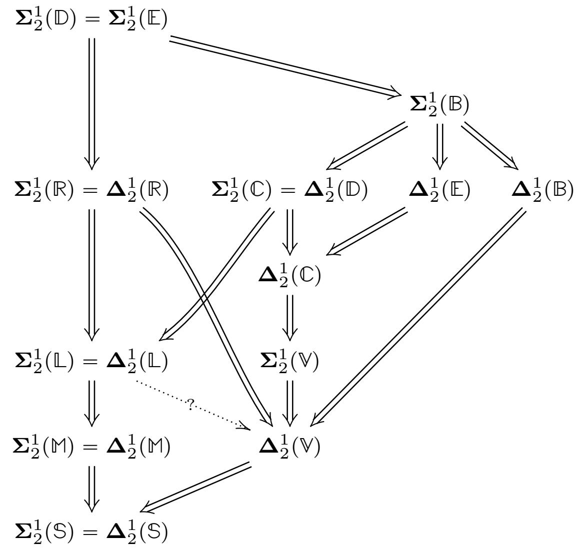


Figure 2.1: Complete implication diagram of regularity properties for the forcings $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}$, $\mathbb{E}$, $\mathbb{L}$, $\mathbb{M}$, $\mathbb{R}$, $\mathbb{S}$, and $\mathbb{V}$ from [35, Figure 1.1]. Note that the non-implication between $\mathbf{\Delta}^1_2(\mathbb{L})$ and $\mathbf{\Delta}^1_2(\mathbb{V})$, marked with a "?" was unknown before the result in this thesis (Corollary 3.4.2).

(1) $\mathbf{\Delta}^1_2(\mathbb{L}) \not\Rightarrow \mathbf{\Delta}^1_2(\mathbb{E}_0)$ (Corollary 3.4.2).

(2) $\mathbf{\Sigma}^1_2(\mathbb{A}) \Rightarrow \mathbf{\Sigma}^1_2(\mathbb{D})$, whence $\mathbf{\Sigma}^1_2(\mathbb{B}) \not\Rightarrow \mathbf{\Sigma}^1_2(\mathbb{A})$ (Corollary 4.3.5).

(3) $\mathbf{\Delta}^1_2(\mathbb{T}) \not\Rightarrow \mathbf{\Delta}^1_2(\mathbb{V})$ (Corollary 5.3.8).

(4) $\mathbf{\Delta}^1_2(\mathbb{S}) \not\Rightarrow \mathbf{\Delta}^1_2(\mathbb{W})$ (Corollary 5.4.3).

For reference, we list some of the results represented in the diagrams that will be used in this thesis.

Theorem 2.11.1. *The statements* $\mathbf{\Delta}^1_2(\mathbb{L})$ *and* $\mathbf{\Sigma}^1_2(\mathbb{L})$ *are equivalent.*

Proof. Cf. [9, Theorem 4.1]. □

Theorem 2.11.2. *If* $\mathbf{\Delta}^1_2(\mathbb{V})$*, then* $\mathbf{\Delta}^1_2(\mathbb{E}_0)$.

Proof. Cf. [7, p. 1350]. □

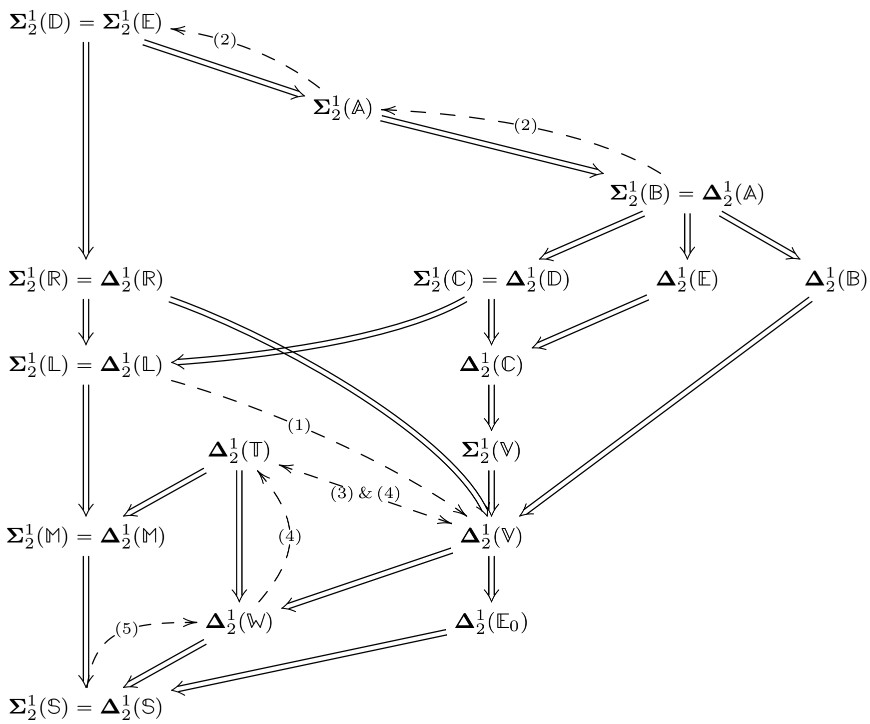


Figure 2.2: Implication diagram of regularity properties with open questions that are solved in this thesis marked by the number in the list of results.

Chapter 3

Laver forcing

3.1 Introduction

The main result of this chapter is the separation of Laver-measurability and Silver-measurability by analysing the Laver model. Whether this separation is possible has been asked several times in the published literature.[1]

The analysis of the Laver model is closely related to the study of Borel chromatic numbers of graphs. The systematic study of definable graphs started in [24] as a descriptive set-theoretic approach to concepts and results from graph theory, and this field is nowadays called *descriptive graph combinatorics*.

If X be a Polish space, we call G a *graph on* X if $G \subseteq X \times X$ is irreflexive and symmetric. Since a graph is a subset of the Polish space $X \times X$, it can be closed, F_σ, Borel, or analytic. A graph is called *locally countable* if the set $\{y \in X \mid (x, y) \in G\}$ is countable, for every $x \in X$.

If G is a graph on a Polish space X, and $\alpha \geqslant 1$ is an ordinal, then an *α-colouring* of G is a function $c\colon X \to \alpha$ such that $c(x) \neq c(y)$, for all $(x, y) \in E$. The sets $c^{-1}(\{\beta\})$ for $\beta < \alpha$ are called the *maximally monochromatic sets* for c. We say that an α-colouring c is a *Borel colouring* if all maximally monochromatic sets are Borel.

The Borel chromatic number of G, denoted by $\chi_{\mathrm{B}}(G)$, is the least cardinality of an ordinal α for which there exists a Borel α-colouring of G. Since we assumed X to be a Polish space, all Borel chromatic numbers are bounded by $2^{\aleph_0}$. We shall see later that uncountable Borel chromatic numbers may assume different values in different models of set theory.[2]

If E is an equivalence relation over X, we can think of E as a graph by making it irreflexive, i.e., considering $E \backslash \mathrm{Id}_X$ where $\mathrm{Id}_X := \{(x, x)\,;\, x \in X\}$ is the identity on X. We use the above notation for equivalence relations, i.e., we write $\chi_{\mathrm{B}}(E)$ for the Borel

[1] Cf. p. 24; [12, Question 6.3], [10, Figure 1], and [20, Figure 2.1].

[2] For ZFC-results about Borel chromatic numbers, we refer the reader to [24]; for consistency results, to [15, 14]. At the heart of the field of descriptive graph combinatorics is the G_0-dichotomy: it says that there exists a closed graph G_0 which is minimal for analytic graphs of uncountable Borel chromatic numbers, i.e., if G is analytic and $\chi_{\mathrm{B}}(G)$ is uncountable, then $\chi_{\mathrm{B}}(G_0) \leqslant \chi_{\mathrm{B}}(G)$ [24, Theorem 6.6].

chromatic number of the equivalence relation E.[3]

If $x, y \in 2^\omega$, we can consider them as sets of natural numbers and define their symmetric difference $x \triangle y := \{k\,;\, x(k) \neq y(k)\}$. This operation gives rise to one of the most interesting relations for us. We define

$$\begin{aligned} x E_0 y :&\iff \forall^\infty n\ (x(n) = y(n)) \\ &\iff x \triangle y \text{ is finite.} \end{aligned}$$

This is an F_σ equivalence relation.

Zapletal connected the equivalence relation E_0 to E_0-trees and the forcing notion $\mathbb{E}_0$. This will become relevant in our applications of the main result later (cf. § 3.4, in particular Theorem 3.4.1).

Theorem 3.1.1 (Zapletal). *A real is $\mathbb{E}_0$-quasigeneric over M if and only if it avoids all Borel E_0-independent sets coded in M.*

Proof. Zapletal uses different terminology, but the key lemma is [36, Lemma 2.3.29]; cf. also [13, Fact 1.3.2]. □

Gaspar and Geschke asked [14, Question 5.2] whether $\chi_B(E_0)$ is consistently smaller than the bounding number $\mathfrak{b}$; we give a positive answer to that question in Corollary 3.2.8. The key technical ingredient in our proof is a preservation theorem for Laver forcing, Theorem 3.2.7. Theorem 3.2.7 (a) was independently proved by Zapletal, but for closed graphs instead. His methods rely on the heavy machinery of his *idealized forcing* (cf. [37]), as well as iterable properties for "sufficiently definable and homogeneous ideals". The approach we take here is completely different and we resort only to classical combinatorial arguments of the forcings involved.

As an additional consequence, our result proves the separation of Laver and Silver measurability (the mentioned open question posed by Fischer, Friedman, and Khomskii): in the Laver model, all $\mathbf{\Sigma}^1_2$ sets are Laver measurable, but not all $\mathbf{\Delta}^1_2$ sets are Silver measurable (cf. Corollary 3.4.2).

Furthermore, we apply our preservation theorem to answer a question of Brendle, Halbeisen, and Löwe: whether the existence of splitting reals (cf. p. 36) implies Silver measurability [6, Question 2]. The answer is 'No' as we show in Corollary 3.4.4.

3.2 Definitions and the main result

Let G be a graph on a Polish space X.

Definition 3.2.1. *A set $A \subseteq X$ is called* G-independent *if $A^2 \cap G = \varnothing$.*

Note that if c be an α-colouring and A is maximally monochromatic for c (i.e., of the form $c^{-1}(\{\beta\})$ for some $\beta < \alpha$), then A is G-independent. Therefore, we observe that we can reformulate the definition of Borel chromatic numbers.

[3] The descriptive graph combinatorics of equivalence relations has been extensively studied in [19, 7] and other papers.

Fact 3.2.2. *For every graph G, $\chi_{\mathrm{B}}(G)$ is the least cardinality of a family $\mathcal{F}$ of Borel G-independent sets such that $\bigcup \mathcal{F} = X$.*

If G is a graph, we call $\mathcal{C} = (C_n)_{n\in\omega}$ a *cover of* G if $G = \bigcup_{n\in\omega} C_n$. Note that each $C_n \subseteq X^2 \setminus \mathrm{Id}_X$ where $\mathrm{Id}_X := \{(x,x) \,;\, x \in X\}$ is the identity on X; this space is a Polish space. We call a cover *closed* if all its elements are closed subsets of $X^2 \setminus \mathrm{Id}_X$.

Fact 3.2.3. *A graph G on a Polish space X is F_σ if and only if there is a closed cover of G.*

Definition 3.2.4. *If $\mathcal{C} = (C_n)_{n\in\omega}$ is a cover of G, the function defined by*

$$\ell_{\mathcal{C}}(x,y) = \begin{cases} \min\{n+1 \mid (x,y) \in C_n\}, & \textit{if } (x,y) \in G \\ 0, & \textit{if } x = y. \\ \omega, & \textit{if } (x,y) \notin G \cup \mathrm{Id}_X. \end{cases}$$

*is called the G-*locator of $\mathcal{C}$*, corresponding to the fixed enumeration $(c_n)_{n\in\omega}$. We shall not mention the enumeration because in each and every of our proofs the enumeration will be fixed.*

Clearly, the G-locator of $\mathcal{C}$ is identically ω on a set A if and only if A is G-independent. We write

$$\ell_{\mathcal{C}}(A,B) := \min\{\ell_{\mathcal{C}}(a,b) \mid (a,b) \in A \times B\}$$

for $A, B \subseteq X$.

Definition 3.2.5. *Let $\mathcal{C}$ be a cover for G. We say that G is $\ell_{\mathcal{C}}$-*unbounded *iff for every $(x,y) \in X^2$ and n natural number such that $\ell_{\mathcal{C}}(x,y) > n$, there exists an open neighbourhood O of y such that $\ell_{\mathcal{C}}(x,z) > n+1$, for every $z \in O \setminus \{y\}$.*

As mentioned, the main graph considered here is $E_0 \setminus \mathrm{Id}_{2^\omega}$. This is an F_σ graph with the closed cover defined by

$$C_n = \big\{(x,y) \in (2^\omega)^2 \mid 0 < |x \triangle y| \leqslant n+1\big\}$$

and it is $\ell_{\mathcal{C}}$-unbounded.

Note that the locator for this cover is an infinite version of the usual distance on the set of vertices—i.e., the distance between two vertices is the shortest length of a path between them—, and this will be further discussed in § 3.5.

Proposition 3.2.6. If $\mathcal{C}$ is a cover for G and G is $\ell_{\mathcal{C}}$-unbounded graph, then it is locally countable.

Proof. Let us consider C_0, and let $x \in X$. Then since X is compact $\{y \in X : (x,y) \in C_0\}$ would have a limit point if the above set is uncountable. Let this limit point be z. Then, one can never find an open set O, with $z \in O$, such that for all $r \in O \setminus \{z\}$, $\ell_{\mathcal{C}}(x,y) > 1$. This means that C_0 is countable. But one can easily notice that there is nothing special about C_0 and that this argument applies to all the $C_n's$. Therefore x has at most countably many G-edges. □

Theorem 3.2.7. *Let G be an F_σ graph on a totally disconnected compact Polish space X and $\mathcal{C}$ be a closed cover of G.*

(a) If G is locally countable then, in the ω_2-Miller model, every point in the completion of X is contained in a Borel G-independent set coded in the ground model; and

(b) if G is $\ell_{\mathcal{C}}$-unbounded then, in the ω_2-Laver model, every point in the completion of X is contained in a Borel G-independent set coded in the ground model.

As mentioned before, (a) was proved by Zapletal independently but for closed graphs.

The bounding number $\mathfrak{b}$ is the smallest cardinality of an unbounded set, i.e., $\mathfrak{b} := \min\{|F|\,;\, F \subseteq \omega^\omega$ and for all $f \in \omega^\omega$ there is some $g \in F$ such that $\{n\,;\, f(n) < g(n)\}$ is infinite$\}$.

Corollary 3.2.8. *It is consistent with the axioms of* ZFC *that $\chi_{\mathrm{B}}(E_0) < \mathfrak{b}$.*

Proof. This happens in the ω_2-Laver model: it is well known that in that model $\mathfrak{b} = \aleph_2$ [3, Model 7.6.13]. But by Theorem 3.2.7, we have that $\chi_{\mathrm{B}}(G) \leqslant |\omega^\omega \cap \mathbf{L}| = \aleph_1$. □

For a diagram involving common small cardinal characteristics of the continuum, and a few Borel chromatic numbers, cf. [14, Figure 1].

3.3 Technical lemmas

The reason why Theorem 3.2.7 can be proved for totally disconnected compact Polish spaces is that they are the continuous injective image of 2^ω when they lack isolated points:

Claim 3.3.1. Let X be homeomorphic to 2^ω, and $\varphi : 2^\omega \to X$ be one such homeomorphism, and G be a graph on X with cover $\mathcal{C}$. Then G is F_σ iff

$$\varphi^*[G] = \left\{\left(\varphi^{-1}(x), \varphi^{-1}(y)\right) \in (2^\omega)^2 \mid (x, y) \in G\right\}$$

is an F_σ graph on 2^ω. Moreover,

(a) G is locally countable iff $\varphi^*[G]$ is locally countable; and

(b) G is $\ell_{\mathcal{C}}$-unbounded iff $\varphi^*[G]$ is $\ell_{\mathcal{C}}$-unbounded.

In any case, we have that $\chi_{\mathrm{B}}(\varphi^*[G]) = \chi_{\mathrm{B}}(G)$.

Proof. Follows directly from the fact that φ is a homeomorphism. □

Single step. Now, in order to prove Theorem 3.2.7, we first investigate what happens when we add only one generic real to the universe. This corresponds to the successor stage of the forcing iteration. So, let $\dot{x}$ be a name for a real and assume that p is a condition (either a Miller or a Laver tree) that forces "$\dot{x}$ is a real" and that is strong enough to guarantee that all guiding reals are defined (cf. § 2.10).

This gives us the function f witnessing continuous reading of names (cf. 22). In the case of Laver forcing we know by Fact 2.10.2 that f is injective. This means for any $p \in \mathbb{L}$ that $f``[p]$ is a Borel set coded in the ground model since f is injective and $[p]$ is a closed set (cf., e.g., [30, Exercise 2E9]).

Lemma 3.3.2. *Let G be an F_σ graph on 2^ω, with a closed countable cover $\mathcal{C}$.*

(a) If G is locally countable and $\mathbb{P} = \mathbb{M}$, then there is a stem-preserving extension $q \leqslant p$ such that $f``[q]$ is a G-independent set.

(b) If G is $\ell_{\mathcal{C}}$-unbounded and $\mathbb{P} = \mathbb{L}$, then there is a stem-preserving extension $q \leqslant p$ such that $f``[q]$ is a G-independent set.

Proof. Let us prove (a) first. In the Miller case, we define an order-preserving injection $i : \omega^{<\omega} \to \omega^{<\omega}$, and a strictly increasing sequence $(k_n)_{n\in\omega}$ of natural numbers such that, for all $\sigma, \tau \in n^{\leqslant n}$,

(1) $\ell_{\mathcal{C}}\left(\left[x_{i(\sigma)}\restriction|i(\sigma)| + k_n\right], \left[x_{i(\tau)}\restriction|i(\tau)| + k_n\right]\right) \geqslant |\sigma| - |\tau|$, if $\tau \subseteq \sigma$,

(2) $\ell_{\mathcal{C}}\left(\left[x_{i(\sigma)}\restriction|i(\sigma)| + k_n\right], \left[x_{i(\tau)}\restriction|i(\tau)| + k_n\right]\right) \geqslant |\sigma|+|\tau|-2|\sigma\cap\tau|$, if σ and τ are distinct (here $\sigma \cap \tau$ denotes the longest common initial segment of σ and τ), and

(3) for all $\sigma' \in ((n+1)^2)^{\leqslant(n+1)}\setminus(n^2)^{\leqslant n}$, such that $\sigma \subseteq \sigma'$ the closure of $f``[p *_0 i(\sigma')]$ is a subset of $\left[x_{i(\sigma)}\restriction|i(\sigma)| + k_n\right]$.

Once this is done with care, we can ensure that $q = \operatorname{ran}(i) *_0 p$ is our desired Miller tree. In fact, if $a, b \in [q]$ are distinct, then $f(a)$ and $f(b)$ do not form an edge: in fact, for every $n \in \omega$, there exists $\sigma_{a,n}, \sigma_{b,n}$ such that $|\sigma_{a,n}| = |\sigma_{b,n}| = n + 1$, $i(\sigma_{a,n}) *_0 p \subseteq a$ and $i(\sigma_{b,n}) *_0 p \subseteq b$. Then

$$\ell_{\mathcal{C}}(f(a), f(b)) \geqslant \ell_{\mathcal{C}}\left(x_{i(\sigma_{a,n})}, x_{i(\sigma_{b,n})}\right) \geqslant 2(n + 1 - |\sigma_{a,n} \cap \sigma_{b,n}|);$$

and the sequence $|\sigma_{a,n} \cap \sigma_{b,n}|$ is constant. Hence, $\ell_{\mathcal{C}}(f(a), f(b)) = \omega$.

This construction can be carried out for Miller forcing if G is locally countable: assume $i\restriction n^{\leqslant n}$ has been defined and let $\prec$ denote the lexicographic order on $\omega^{<\omega}$. By induction on σ, also assume $i(\tau)$ has been defined, for all $\tau \prec \sigma$. Since $f``[p *_0 i(\sigma\restriction|\sigma|-1)]$ is uncountable (because $\dot{x}$ is not in the ground model), there exists $a \in \omega^\omega$ such that $i(\sigma\restriction|\sigma| - 1) \subseteq a$; and $\left(f(a *_0 p), x_{i(\tau)}\right) \notin G$. In particular, it follows from the closedness of the C_n's, and from the continuity of f, that there exists an initial segment of a, which we choose to be $i(\sigma)$, such that

$$\ell_{\mathcal{C}}\left(x_{i(\sigma)}, x_{i(\tau)}\right) > \begin{cases} |\sigma| - |\tau|, & \text{if } \tau \subseteq \sigma; \text{ and} \\ |\sigma| + |\tau| - 2|\sigma \cap \tau|, & \text{if } \tau \text{ and } \sigma \text{ are incompatible.} \end{cases}$$

This finishes the inductive construction.

We prove (b); in the case of Laver forcing, we assume that G is $\ell_{\mathcal{C}}$-unbounded: first, let $A = (A_n)_{n\in\omega}$ a p-chain as in Claim 2.10.2, witnessing the injectivity of f. Similarly to the case of Miller, we need to construct some order-preserving injection $i : \omega^{<\omega} \to \omega^{<\omega}$, and a strictly increasing sequence $(k_n)_{n\in\omega}$ of natural numbers, but we need some changes:

(1) i is height-preserving and j-stable;

for $\sigma, \tau \in \omega^{<\omega}$,

(2) $\ell_{\mathcal{C}}\left(\left[x_{\sigma(i,p,A)}{\restriction}|\sigma(i,p,A)| + k_n\right], \left[x_{\tau(i,p,A)}{\restriction}|\tau(i,p,A)| + k_n\right]\right) \geqslant \lceil(|\sigma| - |\tau|)/2\rceil$, if $\tau \subseteq \sigma$;

(3) $\ell_{\mathcal{C}}\left(\left[x_{\sigma(i,p,A)}{\restriction}|\sigma(i,p,A)| + k_n\right], \left[x_{\tau(i,p,A)}{\restriction}|\tau(i,p,A)| + k_n\right]\right) \geqslant \lceil(|\sigma| + |\tau| - 2|\sigma \cap \tau|)/2\rceil$

where $\sigma(i,p,A)$ is the unique element of $\omega^{<\omega}$ such that

$$\sigma(i,p,A) *_0 p = \mathrm{j}(i(\sigma)) *_1 (p, A).$$

We shall proceed by induction on the set of even natural numbers, that is $\{2k : k \in \omega\}$. So, assume $i{\restriction}(n^2)^{\leqslant n}$ has been defined for some $n > 0$. Moreover, let assume $i(\tau)$ has been defined for some σ and all $\tau \prec \sigma$, where $\sigma, \tau \in \left((n+1)^2\right)^{\leqslant n+1} \setminus (n^2)^{\leqslant n}$.

Let $\sigma^- = \sigma{\restriction}|\sigma| - 1$ (thus $|\sigma^-| = |\sigma| - 1$), a node for which i is defined according to our induction hypothesis — that is, for each $\tau \prec \sigma$ and $z \in \left[x_{\sigma^-(i,p,A)}{\restriction}|\sigma^-(i,p,A)| + k_n\right]$, we have that

$$\ell_{\mathcal{C}}\left(z, x_{\tau(i,p,A)}\right) \geqslant \begin{cases} \lceil(|\sigma| - |\tau|)/2\rceil - 1, & \text{if } \tau \subseteq \sigma\text{; and} \\ \lceil(|\sigma| + |\tau| - 2|\Delta(\sigma,\tau)|)/2\rceil - 1, & \text{if } \tau \text{ and } \sigma \text{ are incompatible.} \end{cases}$$

Now using $\ell_{\mathcal{C}}$-unboundedness, for each such τ, we let O_τ be an open set around $x_{\sigma^-(i,p,A)}$ such that for all $z \in O_\tau \setminus \left\{x_{\sigma^-(i,p,A)}\right\}$:

$$\ell_{\mathcal{C}}\left(z, x_{i(\tau)}\right) \geqslant \begin{cases} \lceil(|\sigma| - |\tau|)/2\rceil, & \text{if } \tau \subseteq \sigma\text{; and} \\ \lceil(|\sigma| + |\tau| - 2|\Delta(\sigma,\tau)|)/2\rceil, & \text{if } \tau \text{ and } \sigma \text{ are incompatible.} \end{cases}$$

Since $\bigcap_{\tau\prec\sigma} O_\tau$ is an open neighborhood of $x_{\sigma^-(i,p,A)}$, by choosing $i(\sigma)$ such that $\left[x_{\sigma(i,p,A)}{\restriction}|\sigma(i,p,A)|\right] \subseteq \bigcap_{\tau\prec\sigma} O_\tau$ we get

$$\ell_{\mathcal{C}}\left(x_{\sigma(i,p,A)}, x_{\tau(i,p,A)}\right) \geqslant \begin{cases} \lceil(|\sigma| - |\tau|)/2\rceil, & \text{if } \tau \subseteq \sigma\text{; and} \\ \lceil(|\sigma| + |\tau| - 2|\Delta(\sigma,\tau)|)/2\rceil, & \text{if } \tau \text{ and } \sigma \text{ are incompatible.} \end{cases}$$

In any case, we use the closedness of the C_n's one more time if necessary to get a natural number k_{n+1} such that

$$\begin{aligned} &\ell_{\mathcal{C}}\left(\left[x_{\sigma(i,p,A)}{\restriction}|\sigma(i,p,A)| + k_{n+1}\right], \left[x_{\tau(i,p,A)}{\restriction}|\sigma(i,p,A)| + k_{n+1}\right]\right) \\ &\qquad \geqslant \ell_{\mathcal{C}}\left(x_{\tau(i,p,A)}, x_{\tau(i,p,A)}\right). \end{aligned}$$

□

Iteration. Our goal now is to prove some version of Lemma 3.3.2 for countable support iterations of Laver forcing. For an ordinal $\alpha \geqslant 1$, let $\mathbb{P}_\alpha$ denote the countable support iteration of $\mathbb{P}$ (where $\mathbb{P}$ is either $\mathbb{M}$ or $\mathbb{L}$). Let F be a finite subset of α and $\eta : F \to \omega$. For $p, q \in \mathbb{P}_\alpha$, we say that $q \leqslant_{F,\eta} p$ iff

$$\forall \gamma \in F \left(q \restriction \gamma \Vdash q(\gamma) \leqslant_{\eta(\gamma)} p(\gamma) \right).$$

For the rest of this section, $\dot{x}$ is a name for an element of 2^ω not added by any proper initial segment of the iteration and p is a condition forcing "$\dot{x}$ is a real".

Theorem 3.3.3. *Suppose that α is a limit ordinal and $p \in \mathbb{P}_\alpha$. Then there is $q \leqslant p$ such that for every coordinate $\beta \in \alpha$, and $\sigma, \tau \in \omega^{<\omega}$, there are chains of frontiers $A^\beta := (A^\beta_n ; n \in \omega)$ such that if σ and τ are such that $q \restriction \beta$ forces that $\mathrm{st}(q(\beta) *_0 \sigma)$ and $\mathrm{st}(q(\beta) *_0 \tau)$ are immediate successors of nodes of frontiers*[4] *and $q(\beta) *_0 \sigma \neq q(\beta) *_0 \tau$ then*

$$q \restriction \beta \Vdash x_{q(\beta)*_0\sigma} \neq x_{q(\beta)*_0\tau}.$$

Proof. This is a direct consequence of Theorem 2.10.3, Lemma 3.3.2 and the fact that $\dot{x}$ is not added by any proper initial segment of the iteration. □

Theorem 3.3.4. *If α is a successor ordinal say $\delta + 1$ and $r \in \mathbb{L}_\alpha$, then there exists $p \leqslant r$ such that for $\sigma, \tau \in \omega^{<\omega}$, there are chains of frontiers $A^\alpha := (A^\alpha_n ; n \in \omega)$ such that if σ and τ are such that $p \restriction \delta + 1$ forces that $\mathrm{st}(p(\alpha) *_0 \sigma)$ and $\mathrm{st}(p(\alpha) *_0 \tau)$ are immediate successors of nodes of frontiers and $p(\alpha) *_0 \sigma \neq p(\alpha) *_0 \tau$ then*

$$p \restriction \alpha \Vdash x_{p(\alpha)*_0\sigma} \neq x_{p(\alpha)*_0\tau}.$$

For $\beta \in \delta + 1$ the splitting levels form a chain of frontiers but they do not necessarily satisfy the above inequality.

Proof. It follows from Theorem 2.10.3 and Lemma 3.3.2 that $r \restriction \delta + 1$ forces that there is $p(\delta + 1) \leqslant r(\delta + 1)$ such that $p(\delta + 1)$ has frontiers satisfying the inequality mentioned in the theorem's statement. □

For any condition $r \in \mathbb{P}_\alpha$, r decides some (proper) initial segment of the values of the real with name $\dot{x}$. We write $\dot{x}_r$ for the maximal initial segment decided by the condition r. If F is any finite set, $\eta \colon F \to \omega^{<\omega}$, and $\sigma, \tau \in \prod_{\gamma \in F} \eta(\gamma)^{\eta(\gamma)}$, we define

$$\ell^{\sigma,\tau}_{\max} := \max_{\gamma \in F} \{ |\sigma(\gamma)| + |\tau(\gamma)| - 2|\sigma(\gamma) \cap \tau(\gamma)| \} \text{ and}$$
$$\ell^{\eta}_{\max} := \max\{ \ell^{\sigma,\tau}_{\max} ; \sigma, \tau \in \prod_{\gamma \in F} \eta(\gamma)^{\eta(\gamma)} \}.$$

Let G be a graph with cover $\mathcal{C}$, $q \leqslant p$, F a finite subset of α, with a chain of frontiers A^γ for each co-ordinate $\gamma \in F$ and $\eta \colon F \to \omega$. We say that q is G-(F, η)-*faithful* iff

$$\ell_{\mathcal{C}} \left([\dot{x}_{q *_1 (\sigma, A_\gamma)}], [\dot{x}_{q *_1 (\tau, A_\gamma)}] \right) \geqslant \lceil \ell^{\sigma,\tau}_{\max} / 2 \rceil$$

for all distinct $\sigma, \tau \in \prod_{\gamma \in F} \eta(\gamma)^{\eta(\gamma)}$ and $\gamma \in F$.

[4] We remind the reader that for the sake of ease of reading, we defined guiding reals of trees as follows: $x_T := x_{\mathrm{st}(T)}$.

Lemma 3.3.5. *If α is a successor ordinal say $\delta+1$, and σ, $\tau \in \omega^{<\omega}$, then $p{\restriction}\delta+1$ forces $\ell_{\mathcal{C}}([\dot{x}_{p(\delta)*_1\sigma}],[\dot{x}_{p(\delta)*_1\tau}]) \geqslant |\sigma|+|\tau|-2|\sigma\cap\tau|$.*

Proof. Follows from the proof of Lemma 3.3.2. □

Lemma 3.3.6. *Let G be an F_σ graph, $\mathcal{C}$ be a closed cover for G, and G be $\ell_{\mathcal{C}}$-unbounded. Let F be a finite subset of α (containing $\alpha-1$ if α is a successor ordinal), $\beta \in F$, $\bar{\gamma}=\max(F)$, $\eta_{\max} := \max\{\eta(\gamma)\,;\,\gamma\in F\}$, and $\eta' : F\to\omega$ be defined by*

$$\eta'(\gamma) := \begin{cases} \eta(\gamma) & \text{if } \gamma \notin \{\beta,\bar{\gamma}\},\\ \min\{2k : 2k > \eta(\beta) \text{ and } k\in\omega\} & \text{if } \gamma=\beta\neq\bar{\gamma},\\ \min\{2k+1\,;\,2k+1 > \eta_{\max}+\ell^{\eta}_{\max}+1\}+1 & \text{if } \gamma=\bar{\gamma}.\end{cases}$$

Let $q \leqslant_{F,\eta} p$ be a G-(F,η)-faithful condition. Then there exists a G-(F,η')-faithful condition $r \leqslant_{F,\eta} q$.

We remark that one can check that the proof for Miller forcing only requires that G is locally countable (rather than $\ell_{\mathcal{C}}$-unbounded; cf. Proposition 3.2.6).

Proof. Since $\dot{x}$ is not added at a proper initial stage, every stage of the iteration has a chain of frontiers associated to it that satisfies Theorem 3.3.3 or Theorem 3.3.4 as the case may be. Let $\{\sigma_0,\dots,\sigma_{m-1}\}$ be an enumeration of $\prod_{\gamma\in F\setminus\{\bar{\gamma}\}}\eta(\gamma)^{\eta(\gamma)}$. We define $\eta'' : F\to\omega$ such that $\eta''{\restriction}F\setminus\{\bar{\gamma}\}=\eta'$ and $\eta''(\bar{\gamma})=\eta'(\bar{\gamma})-1$.

We define a $\leqslant_{F,\eta}$-decreasing sequence $(p_j)_{j<m}$ by recursion. Assume we have constructed p_{j-1}; using ideas from the proof of Lemma 3.3.2 and Lemma 3.3.5, we define an order-preserving injection i on $\omega^{\leqslant\eta''(\bar{\gamma})}$, a strictly increasing sequence k_n of natural numbers, and a $p_j \leqslant_{F,\eta} q_j$ with the following condition:

We denote by $\tau(i,p,A)$ the unique element of $\omega^{<\omega}$ such that $\tau(i,p,A)*_0 p = \mathrm{j}(i(\tau))*_1(p,A)$ and let $\tau,\tau'\in\omega^{\leqslant\eta''(\bar{\gamma})}$. Then $(p_j *_1 \sigma_j){\restriction}\bar{\gamma}$ forces

(i) i is height-preserving and j-stable,

(ii) $\tau\subseteq\tau'$, it forces $\ell_{\mathcal{C}}([x_{\tau(i,p,A)}{\restriction}|\tau(i,p,A)|+k_n],[x_{\tau'(i,p,A)}{\restriction}|\tau'(i,p,A)|+k_n]) \geqslant \lceil(|\tau|-|\tau'|)/2\rceil$, and

(iii) if τ and τ' are incompatible, it forces

$$\ell_{\mathcal{C}}([x_{\tau(i,p,A)}{\restriction}|\tau(i,p,A)|+k_n],[x_{\tau'(i,p,A)}{\restriction}|\tau'(i,p,A)|+k_n]) \geqslant \lceil(|\tau|+|\tau'|-2|\tau\cap\tau'|)/2\rceil.$$

In particular,

$$\ell_{\mathcal{C}}([x^{\bar{\gamma}}_{i(\tau)}{\restriction}|i(\tau)|+k_{\bar{n}}],[x^{\bar{\gamma}}_{i(\tau')}{\restriction}|i(\tau')|+k_{\bar{n}}]) \geqslant \lceil(\ell^{\eta'}_{\max}+1)/2\rceil$$

when $|\tau|=|\tau'|=\lceil\eta_{\max}+(\ell^{\eta}_{\max}+1)/2\rceil$, and $|\tau\cap\tau'|\leqslant\eta_{\max}$.

If $\beta=\bar{\gamma}$, simply let $r=p_{m-1}$; if $\beta\neq\bar{\gamma}$, let $\{I_\tau \mid \tau\in\eta''(\beta)^{\eta''(\beta)}\}$ denote a partition of ω into finitely many infinite pieces. Then $r\leqslant_{F,\eta} p_{m-1}$ is defined such that

(1) $r{\restriction}\bar{\gamma}=p_{m-1}{\restriction}\bar{\gamma}$,

(2) for all coordinatewise extensions $\sigma' \in \prod_{\gamma \in F \setminus \{\bar\gamma\}} \eta''(\gamma)^{\eta''(\gamma)}$, of the restricted product of nodes $\sigma \in \prod_{\gamma \in F\setminus\{\bar\gamma\}} \eta(\gamma)^{\eta(\gamma)}$, for all $\bar\sigma \in \eta(\bar\gamma)^{<\eta(\bar\gamma)}$,

$$(r *_1 \sigma')\restriction\bar\gamma \Vdash \mathrm{succ}(\mathrm{st}(r(\bar\gamma) *_1 \bar\sigma)\setminus\{0,...,\eta(\bar\gamma)-1\}^* = I^*_{\sigma'(\beta)},$$

where $\{0,...,k-1\}^*$ denotes the first k immediate successors of the stem of the restriction of $r(\bar\gamma)$ to $\bar\sigma$, $r(\bar\gamma) *_0 \bar\sigma$; for all $\bar\sigma \in \eta(\bar\gamma)^{\eta(\bar\gamma)}$, $I^*_{\sigma'(\beta)} = \{r(\gamma) *_0 \sigma(\gamma)^\frown k' : k' \in I_{\sigma'(\beta)}\}$ and

(3) $r\restriction(\bar\gamma+1) \Vdash r\restriction(\bar\gamma,\alpha) = p_{m-1}\restriction(\bar\gamma,\alpha)$

□

3.4 Proof of the main result and applications

We can now prove Theorem 3.2.7 (b).[5] With Lemma 3.3.6 and some bookkeeping, we can construct a fusion sequence $(p_n, F_n, \eta_n)_{n\in\omega}$ such that

(i) for all $\gamma \in \mathrm{supp}(p_n)$, there is $m \in \omega$ such that $\gamma \in F_m$ and $\eta_m(\gamma) \geqslant n$; and

(ii) p_n is (F_n, η_n)-faithful.

Let $q \in \mathbb{L}_\alpha$ be defined recursively such that for all $\gamma < \alpha$, we have $(q\restriction\gamma \Vdash q(\gamma) = \bigcap_{n\in\omega} p_n(\gamma))$, let $(x(\gamma)\,;\,\gamma \in \mathrm{supp}(q))$ be a sequence in $(\omega^\omega)^{\mathrm{supp}(q)}$, and define a function f by

$$f\left(\left(x(\gamma)_{\gamma\in\mathrm{supp}(q)}\right)\right) := \bigcup_{n\in\omega} \dot x_{q*_0(x(\gamma)\restriction\eta_n(\gamma))_{\gamma\in F_n}}.$$

The function $f\colon (\omega^\omega)^{\mathrm{supp}(q)} \to 2^\omega$ is a ground model continuous injection mapping the generic sequence to $\dot x$ — i.e., $q \Vdash f(x_{\mathrm{gen}}(\gamma))_{\gamma\in\mathrm{supp}(q)} = \dot x$. Due to the above property of q being a fusion of the faithful sequence (p_n, F_n), we have $\ell_{\mathcal{C}}(f(x), f(y)) = \omega$, for all distinct $x, y \in (\omega^\omega)^{\mathrm{supp}(q)}$. Hence, $f\left[(\omega^\omega)^{\mathrm{supp}(q)}\right]$ is a ground model Borel G-independent set. This finishes the proof of Theorem 3.2.7.

We can now harvest the fruits of our labour and provide the promised solutions of the two open questions.

Theorem 3.4.1. *In the Laver model, if r is a real, then there are no $\mathbb{E}_0$-quasigenerics over $\mathbf{L}[r]$.*

Proof. Let x be any real in the Laver model. Since E_0 is $\ell_{\mathcal{C}}$-unbounded, Theorem 3.2.7 (b) says that x is contained in a Borel E_0-independent set coded in the ground model. But then it cannot be $\mathbb{E}_0$-quasigeneric over the ground model (and hence not over any $\mathbf{L}[r]$) by Theorem 3.1.1. □

Corollary 3.4.2. *In the Laver model, $\mathbf{\Sigma}^1_2(\mathbb{L})$ holds and $\mathbf{\Delta}^1_2(\mathbb{E}_0)$ and $\mathbf{\Delta}^1_2(\mathbb{V})$ fail.*

[5]The proof of Theorem 3.2.7 (a) is the same, using the remark after Lemma 3.3.6.

Proof. Follows from Theorem 2.11.1, Proposition 2.4.4, and Theorems 2.6.1 & 3.4.1. □

A set $s \in [\omega]^\omega$ (interpreted as an increasing element of Baire space) is called a *splitting real* over M if for every $x \in [\omega]^\omega \cap M$, both $x \backslash s$ and $s \cap x$ are infinite.

Theorem 3.4.3. *If $\mathbf{\Delta}^1_2(\mathbb{V})$, then for every real r, there is a splitting real over $\mathbf{L}[r]$.*

Proof. Cf. [6, Proposition 2.4]. □

Brendle, Halbeisen and Löwe asked whether the converse of Theorem 3.4.3 holds [6, Question 2]. Our result implies that the answer is negative.

Corollary 3.4.4. *In the Laver model, $\mathbf{\Delta}^1_2(\mathbb{V})$ fails, but for every real r, there is a splitting real over $\mathbf{L}[r]$.*

Proof. The first part follows from Corollary 3.4.2. Laver forcing adds dominating reals over $\mathbf{L}[r]$ (cf. [3, Lemma 7.3.28]) and the existence of a dominating real implies the existence of a splitting real (cf. [18, Fact 21.1]). □

3.5 Questions

As said earlier, the notion of $\mathcal{C}$-locator is a generalisation of the graph distance (i.e., the shortest length of a path between them). If G is a closed locally countable graph on a Polish space X, let E_G be the equivalence relation whose classes are the connected components of G. Then $E_G \backslash \mathrm{Id}_X$ is a locally countable F_σ-graph with closed cover $\mathcal{C} = (C_n)_{n\in\omega}$ defined by

$$C_n = \left\{(x, y) \in (2^\omega)^2 \mid 0 < d(x, y) \leqslant n + 1\right\},$$

for all $n \in \omega$, where d here denotes the usual distance in G (so, $G = C_0$). Say that G *has unbounded distance* if $E_G \backslash \mathrm{Id}_X$ is $\ell_{\mathcal{C}}$-unbounded.

Question 3.5.1. *Is there a closed locally countable graph defined on a Polish space that does not have unbounded distance? More generally, is there an F_σ locally countable graph that is not $\ell_{\mathcal{C}}$-unbounded* for all *its closed covers?*

Even if the answer to Question 3.5.1 is positive, it could be that Theorem 3.2.7 (b) still holds for all locally countable graphs. However, this could not be proved with the method presented here.

Question 3.5.2. *Does Theorem 3.2.7 (b) still hold if G is an arbitrary locally countable graph?*

We were not able to find a counterexample for Theorem 3.2.7 when the set of vertices is not compact, or not extremely disconnected.

Question 3.5.3. *Does Theorem 3.2.7 still hold if X is not compact (e.g., $X = \omega^\omega$)? What if X is not extremely disconnected (e.g., $X = [0, 1]$, or $X = \mathbb{R}$)?*

Finally, we do know what happens for graphs of different complexities, such as G_δ, $\mathrm{G}_{\delta\sigma}$, $\mathrm{F}_{\sigma\delta}$, etc.

Question 3.5.4. *Does Theorem 3.2.7 still hold if G is an analytic graph?*

Chapter 4

Regularity properties and inaccessible cardinals

As discussed in § 2.7, it is rare that the measurability of all $\mathbf{\Sigma}^1_2$ sets gives the strongest of the transcendence properties, "$\aleph_1$ is inaccessible by reals". One of the few examples of this is Hechler regularity (cf. Fact 2.7.3) which we are going to use in our proofs here.

It had been conjectured since the late 1990s that the same holds for amoeba regularity. However, the fact that amoeba forcing does not live on the reals and that amoeba regularity was not defined in the usual way, made it difficult to analyse it: the analysis required the general framework due to Wansner described in §§ 2.3 & 2.6 from [35].

In this chapter, we introduce various notions of amoeba forcing and prove that $\mathbf{\Sigma}^1_2$ measurability for each of them implies that $\aleph_1$ is inaccessible by reals.

4.1 Being an amoeba

If $\mathbb{P}$ is a forcing notion with the Ikegami property, one way to obtain $\mathbf{\Sigma}^1_2(\mathbb{P})$ is to iteratively add co-null sets of quasigeneric reals in an iteration of length ω_1. In order to do this, we would like to have natural forcing notions adding these large sets of quasigenerics, usually called *amoebas* of the original forcing.

Definition 4.1.1. *Let $\mathbb{P}$ be an arboreal forcing notion and $\mathbb{Q}$ any other forcing notion.*

1. *We say that $\mathbb{Q}$ is a* weak Amoeba *of $\mathbb{P}$ if $\mathbf{\Delta}^1_2(\mathbb{Q})$ implies $\mathbf{\Sigma}^1_2(\mathbb{P})$;*

2. *We say that $\mathbb{Q}$ is a* quasigeneric Amoeba *of $\mathbb{P}$ if for any $T \in \mathbb{P}$, any $\mathbb{Q}$-generic G, and any model $M \supseteq V[G]$, we have that*

$$M \models \exists T' \leqslant T \forall x (x \in [T'] \rightarrow x \textit{ is } \mathbb{P}\textit{-quasigeneric over } V);$$

3. *we say that $\mathbb{Q}$ is a* quasi-Amoeba *of $\mathbb{P}$ if for any $T \in \mathbb{P}$ and any $\mathbb{Q}$-generic G we have that*

$$V[G] \models \exists T' \leqslant T \forall x (x \in [T'] \rightarrow x \textit{ is } \mathbb{P}\textit{-generic over } V); \textit{ and}$$

4. *we say that* $\mathbb{Q}$ *is a* (generic) Amoeba *of* $\mathbb{P}$ *if for any* $T \in \mathbb{P}$*, any* $\mathbb{Q}$*-generic* G*, and any model* $M \supseteq V[G]$*, we have that*

$$M \models \exists T' \leqslant T \forall x (x \in [T'] \rightarrow x \text{ is } \mathbb{P}\text{-generic over } V).$$

Proposition 4.1.2. *Let* $\mathbb{P}$ *be an arboreal forcing notion. Then every Amoeba for* $\mathbb{P}$ *is a quasi-Amoeba for* $\mathbb{P}$ *and every quasi-Amoeba for* $\mathbb{P}$ *is a quasigeneric Amoeba for* $\mathbb{P}$*.*

Proof. Follows directly from the definitions. □

Proposition 4.1.3. *If* $\mathbb{P}$ *is an arboreal forcing with the Ikegami property, every quasi-generic Amoeba (and therefore by Proposition 4.1.2 every Amoeba and every quasi-Amoeba) is a weak Amoeba of* $\mathbb{P}$*.*

Proof. Follows directly from the definitions. □

In general, the various notions of Amoebas do not coincide: for Sacks, Miller, and Laver forcing, the regularity of all $\mathbf{\Delta}^1_2$ sets is equivalent to the regularity of all $\mathbf{\Sigma}^1_2$ sets [9, Theorems 4.1, 6.1, & 7.1]. As a consequence all of these forcings are their own weak Amoebas. Sacks forcing and Miller forcing are quasi-Amoebas, but not Amoebas for themselves [5, Theorem 4, Corollary 5, & Proposition 7], and Laver forcing is not even a quasi-Amoeba for itself [5, Theorem 5]. This situation changes for c.c.c. forcing notions as the following theorem shows.

Theorem 4.1.4. *For c.c.c. forcing notions* $\mathbb{P}$*, every quasi-Amoeba for* $\mathbb{P}$ *is an Amoeba for* $\mathbb{P}$*. (In other words, Amoeba and quasi-Amoeba are equivalent.)*

Proof. Cf. [12, p. 712]. □

In § 4.2, we shall introduce various Amoeba forcings for c.c.c. forcing notions. These forcing notions do not live on Baire space, but on slightly different Polish spaces that we shall define in the following section.

4.2 Definitions of amoebas

Using the spaces from the previous section, we now give the definitions of the various amoebas that we consider in this chapter. We use the spaces **O**, **U**, and **Loc** defined in § 2.5. As in § 2.5, the symbol μ denotes Lebesgue measure, either on 2^ω or $\mathbb{R}$; it will be clear from the context which measure is intended.

Definition 4.2.1. Amoeba forcing*, denoted by* $\mathbb{A}$*, consists of the set of all pruned trees* $T \subseteq 2^{<\omega}$ *such that* $\mu([T]) > \frac{1}{2}$*, ordered by inclusion.*

Amoeba forcing was introduced by Martin and Solovay in [28]. It is an Amoeba for random forcing $\mathbb{B}$. It lives on the Polish space **R** in the sense of § 2.5 by means of the following function:

$$\langle T \rangle := \{S \in \mathbf{R}\,;\, [S] \subseteq [T]\}$$

(for details, cf. [35, pp. 55–56]). The collection $C_{\mathbb{A}}$ of these sets forms a proper weak category base on $\mathbf{R}$ that has the countable chain condition and is Borel compatible with the the Polish space $\mathbf{R}$ (cf. [35, Proposition 2.3.6]).

We shall also be using a variant of amoeba forcing. For this, we write $\mathbb{R}^+_\infty$ for $\mathbb{R}^+ \cup \{\infty\}$.

Definition 4.2.2. Amoeba infinity forcing*, denoted by $\mathbb{A}_\infty$ consists of the set of all pairs $(O, \varepsilon) \in \mathbf{O} \times \mathbb{R}^+_\infty$ such that $\mu(O) < \varepsilon\}$ ordered by $(O', \varepsilon') \leqslant (O, \varepsilon)$ iff $O \subseteq O'$ and $\varepsilon' \leqslant \varepsilon$.*

This forcing notion lives on the Polish space $\mathbf{O}$ by means of the following function:

$$[O, \varepsilon] := \{U \in \mathbf{O} \,;\, O \subseteq U \text{ and } \mu(U) \leqslant \varepsilon\}$$

for $(O, \varepsilon) \in \mathbb{A}_\infty$. We write $C_{\mathbb{A}_\infty}$ for the collection of these sets.

Proposition 4.2.3. *The pair $(\mathbf{O}, C_{\mathbb{A}_\infty})$ is a c.c.c category base which is Borel compatible with $\mathbf{O}$ and the ideal of $C_{\mathbb{A}_\infty}$-small sets is Borel generated.*

Proof. If X is a Polish space and (X, C) is a proper weak category base which satisfies c.c.c, then for a C-singular set A, there is a maximal antichain $\mathcal{A}$ that is countable and $A \subseteq X \backslash \bigcup \mathcal{A}$. But $X \backslash \bigcup \mathcal{A}$ is Borel. Therefore it follows that $\mathcal{I}^*_C$ is Borel generated.

Therefore we need to first prove that it is a category base satisfying c.c.c and is Borel compatible with $\mathbf{O}$. Clearly, $\mathbf{O} = \bigcup C_{\mathbb{A}_\infty}$. Let $c \in C_{\mathbb{A}_\infty}$ and $C \subseteq C_{\mathbb{A}_\infty}$ be a disjoint family, with $|C| < |C_{\mathbb{A}_\infty}|$. Since $\mathbb{A}_\infty$ satisfies c.c.c, C is countable.

Case 1. The set $c \cap \bigcup C$ contains some element of $C_{\mathbb{A}_\infty}$

If $[O, \varepsilon] \in C_{\mathbb{A}_\infty}$. If there isn't any $[O', \varepsilon'] \in C$, such that $[O, \varepsilon] \cap [O', \varepsilon']$ contains an element of $C_{\mathbb{A}_\infty}$, then for every $[O', \varepsilon'] \in C_{\mathbb{A}_\infty}$, $\mu(O \cup O') \geqslant \min\{\varepsilon, \varepsilon'\}$. Hence, for every $[O', \varepsilon'] \in C_{\mathbb{A}_\infty}$, either $[O, \varepsilon] \cap [O', \varepsilon'] = \varnothing$ or for every $U \in [O, \varepsilon] \cap [O', \varepsilon']$, $\mu(U) = \min\{\varepsilon, \varepsilon'\}$. Let $U \in [O, \varepsilon]$ such that $\mu(U) < \varepsilon$ and $\mu(U) \neq \varepsilon'$ for every $[O', \varepsilon'] \in C$. Such U exists as C is countable. Then $U \notin [O, \varepsilon] \backslash \bigcup C$, which is a contradiction.

Case 2. The set $A \cap \bigcup C$ does not contain some element of $C_{\mathbb{A}_\infty}$.

Hence, $[O, \varepsilon] \cap [O', \varepsilon']$ also does not contain any element of $C_{\mathbb{A}_\infty}$. Then for every $[O', \varepsilon']$, $\mu(O \cup O') \geqslant \min\{\varepsilon, \varepsilon'\}$. Since C is countable, we can find some $U \in [O, \varepsilon]$ such that $\mu(U) < \varepsilon$ and for every $[O', \varepsilon'] \in C$, $\mu(U \cup O') > \min\{\varepsilon, \varepsilon'\}$. Then $[U, \varepsilon] \subseteq [O, \varepsilon]$ and for every $[O', \varepsilon'] \in C$, $[U, \varepsilon] \cap [O', \varepsilon'] = \varnothing$.

Now we turn to prove the properness of $(\mathbf{O}, C_{\mathbb{A}_\infty})$. The partial order $(C_{\mathbb{A}_\infty}, \subseteq)$ is proper due to the fact that $\mathbb{A}_\infty$ is c.c.c. We show that every singleton is $C_{\mathbb{A}_\infty}$-small. Let $U \in \mathbf{O}$ and $[O, \varepsilon] \in C_{\mathbb{A}_\infty}$. Without loss of generality $U \in [O, \varepsilon]$. Then we can either decrease ε or increase O to obtain $(O', \varepsilon') \leqslant (O, \varepsilon)$ such that $U \notin [O', \varepsilon']$. Therefore $\{U\}$ is singular. We now show that every region is $C_{\mathbb{A}_\infty}$-not small. Suppose that there is $[O, \varepsilon]$ such that it is $C_{\mathbb{A}_\infty}$-small. Then, $[O, \varepsilon] \subseteq \bigcup_{n\in\omega} S_n$ where S_n are all singular. Now, due to singularity, we can find a decreasing sequence $(O_n, \varepsilon_n)_{n\in\omega}$ such that $(O_0, \varepsilon_0) = (O, \varepsilon)$ and $S_n \cap [O_{n+1}, \varepsilon_{n+1}] = \varnothing$. Let $U = \bigcup_{n\in\omega}(O_n)$. Then for every $n \in \omega$, $O_n \subseteq U$ and

$\mu(U) \leqslant \varepsilon_n$. Hence, $U \in [O_n, \varepsilon_n]$ for every $n \in \omega$. But this is a contradiction, since $[O, \varepsilon] \cap \bigcup_{n>0}[O_n, \varepsilon_n] = \varnothing$. Therefore $[O, \varepsilon]$ is not $C_{\mathbb{A}_\infty}$-small.

Finally, we prove the Borel compatibility part. Let $[O, \varepsilon]$ be a region. We wish to show that it is closed. Let $U \notin [O, \varepsilon]$ and s a code for U. Then, either $O \nsubseteq U$ or $\mu(U) > \varepsilon$.

If $O \nsubseteq U$, then there is some $n \in \omega$ such that $(a_n, b_n) \subseteq O$ but $(a_n, b_n) \nsubseteq U$. Hence, $[s \restriction (n+1)] \cap \mathbf{O}$ is open in $\mathbf{O}$, contains U, and is disjoint from $[O, \varepsilon]$.

If $\mu(U) > \varepsilon$, then there is some $n \in \omega$ such that $\mu(\bigcup\{(a_k, b_k) : k < n \text{ and } s(k) = 1\})$. Hence, $[s \restriction n] \cap \mathbf{O}$ is open in $\mathbf{O}$ contains U and is disjoint from $[O, \varepsilon]$.

In both cases there is an open set containing U and disjoint from $[O, \varepsilon]$. Therefore, $[O, \varepsilon]$ is closed in $\mathbb{R}_\infty$.

Finally, we need to show that every Borel set in $\mathbf{O}$ is $C_{\mathbb{A}_\infty}$-measurable. One can see from the definition that $C_{\mathbb{A}_\infty}$ do form a σ-algebra. Therefore it is enough to show that every open set in $\mathbf{O}$ is measurable. Let $t \in 2^{<\omega}$ and $[O, \varepsilon]$ a region and let $s \in 2^\omega$ be a code for O. Then, we are going to make a case distinction:

Case 1. $t \subseteq s$. Let $\varepsilon' \in \mathbb{R}$ be such that for every $n < \mathrm{lh}(t)$ with $t(n) = 0$, $\mu(O) < \varepsilon' < \mu(O \cup (a_n, b_n))$. Then, $(O, \varepsilon') \leqslant (O, \varepsilon)$ and $[O, \varepsilon'] \subseteq [t] \cap \mathbb{R}_\infty$.

Case 2. There exists $n < \mathrm{lh}(t)$ such that $t(n) = 1 \neq s(n)$. Let $\varepsilon' \in \mathbb{R}$ be such that $\mu(O) < \varepsilon' < \mu(O \cup (a_n, b_n))$. Then $(O, \varepsilon') \leqslant (O, \varepsilon)$ and $[O, \varepsilon'] \cap ([t] \cap \mathbb{R}_\infty)$ is empty.

Case 3. There is some $n < \mathrm{lh}(t)$ such that $t(n) = 0 \neq s(n)$. Then $[O, \varepsilon] \cap ([t] \cap \mathbb{R}_\infty)$ is empty.

□

Definition 4.2.4. Amoeba forcing for category*, also known as* universally meagre forcing *and denoted by* $\mathbb{UM}$*, consists of the set of all* (σ, E) *such that* $\sigma = (\sigma(0), ..., \sigma(n-1))$ *is a finite sequence of elements of* $2^{<\omega}$ *and* E *is an open dense subset of* $2^{<\omega}$*. This set is partially ordered as follows:*

$$(\sigma', E') \leqslant (\sigma, E) \textit{ iff } \sigma \subseteq \sigma' \textit{ and } \forall n \in \mathrm{dom}(\sigma' \backslash \sigma)(\sigma'(n)) \in E.$$

Amoeba forcing for category is an Amoeba for $\mathbb{C}$. It lives on the Polish space $\mathbf{U}$ via the the collection U of sets $[\sigma, E]$ for $(\sigma, E) \in \mathbb{UM}$ already considered in § 2.5.

Proposition 4.2.5. *The pair* $(\mathbf{U}, U)$ *is a proper weak category base which is Borel compatible with* $\mathbf{U}$*.*

Proof. In Proposition 2.5.3, we proved that U forms a topology base on $\mathbf{U}$; thus it is also a weak category base. It is easy to verify that Baire property in this topology is the same as U-measurability. We now move on to prove that it is a proper weak category base and Borel compatible with the subspace topology.

We already have that every region is closed in the subspace topology. Therefore it remains to be checked only that sets that are Borel in the subspace topology are U-measurable. Notice that every open set in the subspace topology can easily be written as a countable union of regions, they are open in the universally meagre topology, too. Therefore Borel sets in the subspace topology are Borel in the universally meagre topology, too, and hence measurable.

Only the properness now remains to be checked. Since, $\mathbb{UM}$ is a c.c.c. forcing notion, $(U, \subseteq)$ is proper.as a forcing notion. Every singleton is clearly U-singular. So, we need to only check that every region is not U-small. So, let there be $[\sigma, E]$ which is U-small. Then, there is a sequence $(N_n)_{n\in\omega}$ of U-singular sets such that $[\sigma, E] \subseteq \bigcup_{n\in\omega} N_n$. Now, one can define a decreasing sequence $[\sigma_n, E_n]$ such that $N_n \cap [\sigma_{n+1}, E_{n+1}] = \varnothing$. Let $x = \bigcup_{n\in\omega} \sigma_n$. Then, $x \in [\sigma_n, E_n]$ for every $n \in \omega$. Thus $x \in [\sigma, E]$, but then $x \in N_n$ for some $n \in \omega$. But that is a contradiction. □

Definition 4.2.6. Localisation forcing, *denoted by* $\mathbb{LOC}$, *consists of the set of all pairs* (σ, F) *such that* $\sigma = (\sigma(0), ..., \sigma(n-1))$ *is a finite sequence of elements of* $[\omega]^{<\omega}$ *and* F *is a finite set of elements of* ω^ω *such that for every* $k < n$, $|\sigma(k)| = k + 1$ *and* $|F| \leqslant n + 1$. *The set is partially ordered as follows:*

$$(\sigma', F') \leqslant (\sigma, F) \textit{ iff } \sigma \subseteq \sigma' \textit{ and } F \subseteq F' \textit{ and } \forall x \in F \forall n \in \operatorname{dom}(\sigma' \backslash \sigma)(x(i) \in \sigma'(i)).$$

Again, in § 2.5, we had considered the collection L of sets $[\sigma, F]$ for $(\sigma, F) \in \mathbb{LOC}$.

Proposition 4.2.7. *The pair* $(\mathbf{Loc}, L)$ *forms a proper weak category base such that it is Borel compatible with* $\mathbf{Loc}$.

Proof. In Proposition 2.5.4, we showed that $(\mathbf{Loc}, L)$ is a topological space; thus, it is also a weak category base. It is also easy to verify that a subset has the Baire property in localisation topology if and only if it is L-measurable and that it is meagre in the localisation topology if and only if it is L-small.

We now move on to the Borel compatibility. Every region is clearly closed in the subspace topology. Also every Borel set in the subspace topology is also Borel in the localisation topology. Therefore every Borel set in the subset topology is L-measurable.

We now show that it is proper. Since $\mathbb{LOC}$ is c.c.c., $(L, \subseteq)$ is a proper forcing notion. Clearly, every singleton set is singular. So, the only thing left to be checked is that every region is not L-small: Let us assume that $[\sigma, E]$ is small. Then, there exists N_n for every $n \in \omega$ such that $[\sigma, E] \subseteq \bigcup_{n\in\omega} N_n$. So, there is a decreasing sequence (σ_n, E_n), such that $(\sigma_0, E_0) = (\sigma, E)$ and $N_n \cap [\sigma_{n+1}, E_{n+1}] = \varnothing$. Let, $x = \bigcup_{n\in\omega} \sigma_n$. Then for every $n \in \omega$, $x \in [\sigma_n, E_n]$. Thus $x \in [\sigma, E]$ and therefore there exists $n \in \omega$, such that $x \in N_n$. But this is impossible since $N_n \cap [\sigma_{n+1}, E_{n+1}] = \varnothing$. Therefore every region is not L-small. □

The corresponding regularity properties are usually defined in terms of the mentioned weak category bases (cf. [22]); i.e., *Amoeba regularity* is $C_{\mathbb{A}}$-measurability, *Amoeba infinity regularity* is $C_{\mathbb{A}_\infty}$-measurability, *universally meagre regularity* is U-measurability, and *localisation regularity* is L-measurability, in symbols $\mathbf{\Gamma}(\mathbb{A})$, $\mathbf{\Gamma}(\mathbb{A}_\infty)$, $\mathbf{\Gamma}(\mathbb{UM})$, and $\mathbf{\Gamma}(\mathbb{LOC})$.

4.3 Amoeba and inaccessibles

We shall prove the following implications:

$$\mathbf{\Sigma}^1_2(\mathbb{A}) \implies \mathbf{\Sigma}^1_2(\mathbb{A}_\infty) \implies \mathbf{\Sigma}^1_2(\mathbb{D}),$$

where the last one implies that $\aleph_1$ is inaccessible by reals (Fact 2.7.3), thus proving the conjecture.

Theorem 4.3.1. *For any projective pointclass* $\mathbf{\Gamma}$, $\mathbf{\Gamma}(\mathbb{A}_\infty) \implies \mathbf{\Gamma}(\mathbb{D})$.

Proof. We apply Wansner's Implication Lemma 2.3.7 to the weak category bases $(\mathbf{O}, C_{\mathbb{A}_\infty})$ and $(\omega^\omega, C_{\mathbb{D}})$. Note that the latter is a proper category base satisfying the countable chain condition that is Borel compatible with ω^ω and the meagre ideal in the dominating topology is Borel generated. Furthermore, for any dense $D \subseteq \mathbb{A}_\infty$, we can define $C_D := \{[O,\varepsilon]\,;\,(O,\varepsilon) \in D\}$ and obtain that $(\mathbf{O}, C_D)$ is a proper category base with is Borel compatible with $\mathbf{O}$ and equivalent to $(\mathbf{O}, C_{\mathbb{A}_\infty})$.

By Lemma 2.3.7, it is therefore enough to find a dense subset $D \subseteq \mathbb{A}_\infty$, a Borel function $h : \mathbf{O} \to \omega^\omega$ and a projection $\bar{h} : D \to \mathbb{D}$ such that

(i) for every $(O,\varepsilon) \in D$, $h[O,\varepsilon] \subseteq [\bar{h}(O,\varepsilon)]$ and

(ii) $\bar{h}[D]$ is dense in $\mathbb{D}$.

Let $\{I^n_k \subseteq \mathbb{R} : n,k \in \omega\}$ be a recursive family of pairwise disjoint open intervals with rational endpoints such that for every $n,k \in \omega$ $\mu(I^n_k) = 2^{-2n}$. Then

$$h(U)(n) = \begin{cases} 0 & \text{if } \mu(U) = \infty, \\ \min\{k \in \omega : \forall \ell \geqslant k(I^n_\ell)\} & \text{otherwise.} \end{cases}$$

The first job is to show that h is Borel. That is for a hechler condition (m,f) such that $h^{-1}([m,f])$ is Borel in $\mathbb{R}_\infty$. Then $U \in h^{-1}([m,f])$ iff $\mu(U) = \infty$ and $\langle 0 : n \in \omega\rangle \in [m,f]$ or $\mu(U) < \infty$ and for $n < m$ $f(n) = \min\{k \in \omega : \forall \ell \geqslant k(I^n_\ell)\}$ and for all $n \geqslant m$, $f(n) \leqslant \min\{k \in \omega : \forall \ell \geqslant k(I^n_\ell)\}$. Therefore $h^{-1}([m,f])$ is Borel. Next up we define the domain of $\bar{h}$ by

$$D = \{(O,\varepsilon) \in \mathbb{A}_\infty : \exists n \in \omega(\sum_{m \geqslant n} 2^{-2m} < \varepsilon - \mu(O) < 2^{-2(n-1)})\}.$$

The task now is to show that D is dense in $\mathbb{A}_\infty$. Let $(O,\varepsilon) \in \mathbb{A}_\infty \backslash D$. Without loss of generality, $\varepsilon \leqslant 1$. Let $n \in \omega$ be minimal such that $\varepsilon - \mu(O) \geqslant 2^{-2(n-1)}$. Hence

$$\varepsilon - \mu(O) \geqslant 2^{-2(n-1)} > 2^{-2(n-1)}/3 = \sum_{m \geqslant n} 2^{-2m}.$$

Then $(O,\varepsilon') \leqslant (O,\varepsilon)$ and $(O,\varepsilon') \in D$. Therefore D is dense.

The domain of $\bar{h}$ is supposed to be a subset of D. For every (O,ε), there is $n_{(O,\varepsilon)} \in \omega$ such that

$$\sum_{m \geqslant n_{(O,\varepsilon)}} 2^{-2m} < \varepsilon - \mu(O) < 2^{-2(n_{(O,\varepsilon)}-1)}$$

We define the domain of $\bar{h}$ to be

$$D' = \{(O,\varepsilon) \in D : \forall U \in [O,\varepsilon](h(O)\restriction n_{(O,\varepsilon)} = h(U))\restriction n_{(O,\varepsilon)}\}$$

Our attempt is to show that D' is dense. Then, if $(O,\varepsilon) \in D$. Without loss of generality, $n_{(O,\varepsilon)} > 0$. Let $g \in \omega^\omega$ such that $n \in \omega$, $g(n) = \max\{k \in \omega : \mu(I_k^n \backslash O) < 2^{1-2n}\}$. We define

$$O' = O \cup \bigcup \{I_{g(n)}^n : n \geqslant n_{(O,\varepsilon)}\}$$

Let $\varepsilon', \varepsilon'' > 0$ such that $\varepsilon'' < \varepsilon' < \varepsilon$, there is some $n \in \omega$ such that

$$\sum_{m \geqslant n} < \varepsilon'' - \mu(O') < \varepsilon' - \mu(O') < 2^{1-2n}$$

for every $n < n_{(O,\varepsilon)}$ and every $k \geqslant h(O')(n)$, $\mu(O' \cup I_n^k) \geqslant \varepsilon'$. Then, $(O', \varepsilon'') \leqslant (O, \varepsilon') \leqslant (O,\varepsilon)$ and $(O',\varepsilon''), (O',\varepsilon) \in D$. We show that $(O',\varepsilon'') \in D'$. Let $U \in [O',\varepsilon']$ and let $n < n_{(O',\varepsilon'')}$. If $n < n_{(O,\varepsilon)}$, then for every $k \geqslant h(O')(n)$, $\mu(U \cup I_n^k) \geqslant \varepsilon' > \varepsilon''$. Thus $h(O')(n) = h(U)(n)$. If $n \geqslant n_{(O,\varepsilon)}$, then $h(O')(n) = g(n)$. Since $n < n_{(O',\varepsilon'')}$, $\varepsilon'' - \mu(O') < 2^{1-2n}$. Hence for every $k \geqslant g(n)$ $\mu(O' \cup I_k^n) > \varepsilon''$ and so $h(O')(n) = g(n) = h(U)(n)$. So, $(O',\varepsilon'') \in D'$. So, D' is dense.

We now define $\bar{h} : D' \to \mathbb{D}$ as $\bar{h}(O,\varepsilon) = (n_{(O,\varepsilon),h(O)})$. Firstly let us show that $\bar{h}$ is a projection. Let, (O,ε) and (O',ε') be such that $(O,\varepsilon) \leqslant (O',\varepsilon')$. Then for every $n \in \omega$, $h(O)(n) \leqslant h(O')(n)$. Since $\varepsilon' - \mu(O') \leqslant \varepsilon - \mu(O)$, $n_{(O,\varepsilon)} \leqslant n_{(O',\varepsilon')}$. Since, $(O,\varepsilon) \in D'$, $h(O)\restriction n_{(O,\varepsilon)} = h(O')\restriction n_{(O,\varepsilon)}$. Therefore $\bar{h}(O',\varepsilon') \leqslant \bar{h}(O,\varepsilon)$ and so $\bar{h}$ is order-preserving.

Let $(O,\varepsilon) \in D'$ and $(n,f) \in \mathbb{D}$ such that $(n,f) \leqslant \bar{h}(O,\varepsilon)$. We define $O' = O \cup \bigcup\{I_k^{n'} : n' \geqslant n_{(O,\varepsilon)}$ and $f(n') = k+1\}$. Then $(O',\varepsilon) \leqslant (O,\varepsilon)$ and $h(O') = f$. We can find $\varepsilon' \leqslant \varepsilon$ such that $(O',\varepsilon') \leqslant (O,\varepsilon)$ and for every $n' < n$ and every $k \geqslant h(O')(n)$ $\mu(O' \cup I_k^{n'}) > \varepsilon'$. Since D' is dense in $\mathbb{A}_\infty$, there is some $(O'',\varepsilon'') \leqslant (O',\varepsilon')$ such that $(O'',\varepsilon'') \in D'$. Then for every $n' \in \omega$, $f(n) = h(O')(n') \leqslant h(O'')(n')$. Moreover $n < n_{(O'',\varepsilon'')}$ and $f\restriction n = h(O'')\restriction n$. Hence, $\bar{h}(O'',\varepsilon'') \leqslant (m,f)$ and so $\bar{h}$ is a projection.

Now, let $(O,\varepsilon) \in D'$ and let $x \in h[O,\varepsilon]$. Then there is a $U \in [O,\varepsilon]$ such that $h(U) = x$. Since $O \subseteq U$, $h(O)(n) \leqslant h(U)(n)$ for all $n \in \omega$. Since, $(O,\varepsilon) \in D'$, $h(O)\restriction n_{(O,\varepsilon)} = h(U)\restriction n_{(U,\varepsilon)}$. Hence, $x \in [\bar{h}(O,\varepsilon)]$.

Let $(m,f) \in \mathbb{D}$. Without loss of generality, $m > 0$. We define $O = \bigcup\{I_k^n : f(n) = k+1\}$ and $\varepsilon = \mu(O) + 2^{-2(m-1)}$. Then, $(O,\varepsilon) \in D'$ and $\bar{h}(O,\varepsilon) = (m,f)$ □

Now, we show that $\mathbf{\Gamma}(\mathbb{A}) \implies \mathbf{\Gamma}(\mathbb{A}_\infty)$. We fix a recursive family $\{A_k^n : n,k \in \omega\}$ of open independent subsets of 2^ω with $\mu(A_k^n) = 2^{(-n+1)}$. Moreover let $I^{\ell,n}$ be the set of finite unions of intervals with rational endpoints with measure $\leqslant 4^{\ell-n}$ and let $\{U_k^{\ell,n} : k,\ell,n \in \omega\}$ be a recursive family of open sets such that for $\ell,n \in \omega$, $\{U_k^{\ell,n} : k \in \omega\}$ enumerates $\mathcal{I}^{\ell,n}$ and each element of $I^{\ell,n}$ occurs infinitely often in this enumeration. For every $\ell \in \omega$, we define functions $h_\ell : \mathbf{R} \cup \mathbb{A} \to \mathbf{O}$ and $\bar{h}_\ell : \mathbb{A} \to \mathbb{A}_\infty$ by

$$h_\ell(S) = \bigcup\{U_k^{\ell,n} : \mu(A_k^n \cap [S]) = 0\} \text{ and}$$
$$\bar{h}_\ell(T) = (h_\ell(T), \sup\{\mu(h_\ell(S)) : S \leqslant T\})$$

Lemma 4.3.2 (Truss 1988)**.** *We define $K_m(S) = \{(n,k) \in m \times \omega : \mu(A_k^n \cap [S]) = 0\}$ for every pruned tree S on 2 and $m \in \omega$.*

(a) For every $\ell \in \omega$, $\bar{h}_\ell$ is a projection.

(b) For every $\ell \in \omega$, if $T \in \mathbb{A}$ and $\mu(h_\ell(T)) < \varepsilon$, then there is some $S \leqslant T$ such that $\bar{h}_\ell(S) \leqslant (h_\ell(T), \varepsilon)$.

(c) For every $S \in \mathbf{R} \cup \mathbb{A}$ and every $n \in \omega$, the set $\{k : \mu(A_k^n \cap [S]) = 0\}$ has size $\leqslant 2^{n+1}$.

(d) For every $T \in \mathbb{A}$ and every $n < m \in \omega$, there is some $k \in \omega$ such that for every $j \geqslant k$, $K_m(T') = K_m(T) \cup \{(n,j)\}$, where $S \leqslant T$ is such that $[S] = [T] \backslash A_j^n$.

Proof. Cf. [32, Proof of Theorem 4.3 & Lemmas 4.4, 4.5, and 4.8]. Also consider the remarks on [35, p. 65]. □

Lemma 4.3.3. *Let $T \in \mathbb{A}$, $\ell \in \omega$ and let $\bar{h}_\ell(T) = (O, \varepsilon)$. Then there is some $T' \leqslant T$ such that for every $S \in \langle T' \rangle$, $\mu(h_\ell(S)) \leqslant \varepsilon$.*

Proof. Let $m \in \omega$ such that

$$\mu(h_\ell(T)) + \sum_{n \geqslant m} 2^{n+1}/4^{\ell-n} \leqslant \varepsilon$$

Now we leave it to the reader to verify that for every $T' \in \mathbb{A}$ and every $n < m$, that

$$\lim_{k\to\infty}\mu(A_k^n \cap [T']) = 1/2^{n+1}\mu([T']) > 1/2^{n+1}1/2 \geqslant 1/2^{m+1}$$

Now one can use the last point of the previous lemma repeatedly to obtain $T' \leqslant T$ such that $K_m(T') = K_m(T)$ and $\mu([T]) - 1/2 < 1/2^{m+1}$. Then there are only finitely many pairs $(n,m) \in m \times \omega$ such that $0 < \mu(A_k^n \cap [T']) \leqslant \mu([T']) - 1/2$. One can again use the last point of the previous lemma repeatedly to obtain $T'' \leqslant T$ such that $K_m(T'') = K_m(T)$ and for every $k \in \omega$ and every $n < m$, if $\mu(A_k^n \cap [T']) > 0$, then $\mu([T''] \cap A_k^n) > \mu([T'']) - 1/2$. Then for every $S \in \langle T'' \rangle$, $K_m(S) = K_m(T'') = K_m(T)$. Hence by the third point of the previous lemma we have

$$\mu(h_\ell(S)) \leqslant \mu(h_\ell(T)) + \sum_{n \geqslant m} |\{k : \mu(A_k^n \cap S) = 0\}|/4^{l-n} \leqslant \mu(h_\ell(T)) + \sum_{n \geqslant m} 2^{n+1}/4^{l-n} \leqslant \varepsilon$$

Therefore, $T'' \leqslant T$ and for every $S \in \langle T'' \rangle$, $\mu(h_\ell(S)) \leqslant \varepsilon$. □

Theorem 4.3.4. *For any projective pointclass $\mathbf{\Gamma}$, $\mathbf{\Gamma}(\mathbb{A}) \implies \mathbf{\Gamma}(\mathbb{A}_\infty)$*

Proof. By Wansner's Implication Lemma 2.3.7, we need to prove the following:

(i) $\{h_{\ell:\ell\in\omega}\}$ is a sequence of Borel functions;

(ii) $\{\bar{h}_\ell : \ell \in \omega\}$ is a sequence of projections from $\mathbb{A}$ to $\mathbb{A}_\infty$;

(iii) for every $\ell \in \omega$ and every $T \in \mathbb{A}$, there is a $T' \leqslant T$ such that $h_\ell(\langle T'\rangle) \subseteq [\bar{h}_\ell(T)]$; and

(iv) $\bigcup_{\ell\in\omega} \bar{h}_\ell[\mathbb{A}]$ is dense in $\mathbb{A}_\infty$

In order to prove that h_ℓ is Borel, define formulas ψ and ψ' by

$$\begin{aligned}\psi(S,i) :\iff& \exists k,n(\mu(A^n_k \cap [S]) = 0 \wedge (a_i,b_i) \subseteq U^{\ell,n}_k),\\ \psi'(S,x) :\iff& \forall i(\psi(P,i) \implies x(i) = 1).\end{aligned}$$

Note that ψ' is arithmetical. Let c be a code for an element of $\mathbf{O}$. Then

$$\begin{aligned}(S,c) \in h_\ell{\restriction}\mathbb{R} &\leftrightarrow \psi'(S,c) \wedge \forall x \in \omega^\omega(\psi(S,y) \to \forall i \in \omega(c(i) = 1 \to x(i) = 1))\\ &\leftrightarrow \psi'(S,c) \wedge \forall i\Big(c(i) = 1 \Rightarrow\\ &\qquad \exists x \in \omega^\omega\big(\forall n\psi(S,z(n)) \wedge (a_i,b_i) \subseteq \bigcup_{n\in\omega} a_{z(n),b_{z(n)}}\big)\Big).\end{aligned}$$

Hence, $h_\ell{\restriction}\mathbb{R}$ is a $\mathbf{\Delta}^1_1$ set and therefore h_ℓ is Borel.

Claim (ii) follows from Lemma 4.3.2. We prove (iii):if $\ell \in \omega$ and $T \in \mathbb{A}$, then there exists $T' \leqslant T$ such that for every $S \in \langle T'\rangle$, $\mu(h_\ell(S)) \leqslant \varepsilon$. Then, $h_\ell[\langle T'\rangle] \subseteq [\bar{h}_\ell(T)]$.

Finally, to show (iv), let $(O,\varepsilon) \in \mathbb{A}_\infty$. Then there is some $\ell \in \omega$ and some $x \in \omega^\omega$ such that $O = \bigcup_{n>1} U^{\ell,n}_{x(n)}$. Let T be the tree such that $[T] = 2^\omega \backslash \bigcup_{n>1} A^n_{x(n)}$. Then $T \in \mathbb{A}$ and $h_\ell(T) = O$. Then there is some $T' \leqslant T$ such that $\bar{h}_\ell(T') \leqslant (h_\ell(T'),\varepsilon) = (O,\varepsilon)$. Thus $\bar{h}_\ell(T') \leqslant (O,\varepsilon)$. □

Corollary 4.3.5. *The following are equivalent:*

(i) $\mathbf{\Sigma}^1_2(\mathbb{A})$,

(ii) for every $r \in \omega^\omega$, $\{x\,;\, x$ *is not Amoeba generic over* $\mathbf{L}[r]\}$ *is* $C_{\mathbb{A}}$*-small, and*

(iii) $\aleph_1$ *is inaccessible by reals.*

Proof. The equivalence of (i) and (ii) is Theorem 2.6.3 (using Lemma 2.4.3); the direction (iii)⇒(i) is Proposition 2.7.2. Finally, if (i) holds, we get $\mathbf{\Sigma}^1_2(\mathbb{A}_\infty)$ by Theorem 4.3.4 whence we obtain $\mathbf{\Sigma}^1_2(\mathbb{D})$ by Theorem 4.3.1 and thus that $\aleph_1$ is inaccessible by reals by Fact 2.7.3. □

4.4 Amoeba for category and inaccessibles

Following the proof strategy of § 4.3, we shall now prove the analogous result for Amoeba for category forcing $\mathbb{UM}$.

Theorem 4.4.1. *For every projective pointclass* $\mathbf{\Gamma}$, $\mathbf{\Gamma}(\mathbb{UM}) \implies \mathbf{\Gamma}(\mathbb{D})$

Proof. For every $n \in \omega$, let t_n denote the sequence with n consecutive 0's followed by a 1. We define $h : \mathbf{U} \to \omega^\omega$ and $\bar{h} : \mathbb{UM} \to \mathbb{D}$ by

$$h(x)(n) := \min\{|s| : s \in \operatorname{ran}(s) \text{ and } t_n \subseteq s\} \text{ and}$$
$$\bar{h}(\sigma, E) := (n_{(\sigma,E)}, f_{(\sigma,E)})$$

where $n_{(\sigma,E)}$ is maximal such that for every $x, x' \in [\sigma, E]$, $h(x)\restriction n = h(x')\restriction n$ and $f_{(\sigma,E)}(n) = \min\{h(x)(n) : x \in [\sigma, E]\}$. By Wansner's Implication Lemma 2.3.7, we prove the following in order to prove the theorem.

(i) $\bar{h}$ is a projection;

(ii) for every (σ, E), $h[\sigma, E] \subseteq [\bar{h}(\sigma, E)]$;

(iii) $h[\mathbb{UM}]$ is dense in $\mathbb{D}$; and

(iv) h is Borel.

Evidently, $\bar{h}$ is order-preserving. Let $(\sigma, E) \in \mathbb{UM}$ and let $(n, f) \leqslant \bar{h}(\sigma, E)$. Then for every $n_{(\sigma,E)} \leqslant m < n$, there is an $s_m \in E$ such that $t_m \subseteq s_m$, and $\operatorname{lh}(s) = f(m)$. We define $\sigma' = \sigma^\frown\langle s_m : n_{(\sigma,E)} \leqslant m < n\rangle$ and $E' = \{s \in E : \exists m \in \omega(t_m \subseteq s) \text{ and } \operatorname{lh}(s) \geqslant f(n)\}$. E' is still dense in $2^{<\omega}$. So, $(\sigma', E') \leqslant (\sigma, E)$. By definition, $n_{(\sigma',E')} \geqslant n$, $f_{(\sigma',E')}\restriction n = f\restriction n$ and for every $m \geqslant n$, $f_{(\sigma',E')}(m) \geqslant f(m)$. Hence, $h(\overline{\sigma'}, E') \leqslant (n, f)$.

We prove (ii), by letting $(\sigma, E) \in \mathbb{UM}$ and $x \in [\sigma, E]$. Then $h(x)\restriction n_{(\sigma,E)} = f_{(\sigma,E)}\restriction n_{(\sigma,E)}$ and for every $n \in \omega$, $h(x)(n) \geqslant f_{(\sigma,E)}$. Hence, $x \in [\bar{h}(\sigma, E)]$.

We now prove (iii): let $(n, f) \in \mathbb{D}$ and for every $m < n$, $s_m \in 2^{<\omega}$ such that $t_m \subseteq s_m$ and $\operatorname{lh}(s_m) = f(m)$. We define $\sigma = \langle s_m : m < n\rangle$ and $E = \{s \in 2^{<\omega} : \exists m \in \omega(t_m \subseteq s \text{ and } \operatorname{lh}(s) \geqslant f(m))\}$. Then $n_{(\sigma,E)} = n$ and for every $m \in \omega$, $f_{(\sigma,E)}(m) = f(m)$. Hence $\bar{h}(\sigma', E') = (n, f)$.

Finally, for (iv), let $s \in \omega^{<\omega}$. Then $h^{-1}([s]) = \bigcap\{[\sigma] : s \subseteq f_{(\sigma,E)\restriction n_{\sigma,E}}\}$. So, $h^{-1}([s])$ is closed in $\mathbb{U}$. Hence, h is Borel. □

Theorem 4.4.2. *The following are equivalent:*

(i) $\mathbf{\Sigma}^1_2(\mathbb{UM})$,

(ii) for every $r \in \mathbb{R}$, *the set* $\{P \in \mathbf{U}\,;\, P$ *is not an* $\mathcal{I}^*_{\mathbb{UM}}$*-quasigeneric over* $\mathbf{L}[r]\}$ *is* U*-small, and*

(iii) $\aleph_1$ *is inaccessible by reals.*

Proof. The equivalence of (i) and (ii) is Theorem 2.6.3; the direction (iii)⇒(i) is Proposition 2.7.2. Finally, if (i) holds, we get $\mathbf{\Sigma}^1_2(\mathbb{D})$ by Theorem 4.4.1 and from that we obtain that $\aleph_1$ is inaccessible by reals by Fact 2.7.3. □

4.5 Localisation and inaccessibles

In this section, we shall prove that $\mathbf{\Sigma}^1_2(\mathbb{LOC})$ implies that $\aleph_1$ is inaccessible by reals. However, this proof differs from the proofs in §§ 4.3 & 4.4: instead of using Wansner's Implication Lemma 2.3.7, we shall use the technique of Brendle-Łabędzki lemmas from § 2.8.

In order to prove a Brendle-Łabędzki lemma, we need a formula Ψ such that in each model of set theory there is a set of size $2^{\aleph_0}$ with property Ψ and an assignment of Borel sets coded in the model to that set.

For every real $x \in \omega^\omega$, we define X_x to be the set $\{f \in \mathbb{LOC} : \exists^\infty n \in \omega(x(n) \notin f(n))\}$. It is easy to see that X_x is Borel in $\mathbb{LOC}$ and since $D_x = \{(\sigma, E) : x \in E\}$ is dense, X_x is always nowhere dense.

We let Ψ be the property "A is a pairwise eventually different family" and if $a \in A$, we let c^A_a be a Borel code for X_a. The following lemma is a Brendle-Łabędzki lemma for $\mathbb{LOC}$.

Lemma 4.5.1 (Brendle-Łabędzki Lemma for $\mathbb{LOC}$)**.** *Let $\mathcal{E}$ be a pairwise eventually different family and let $A \subseteq$ **Loc** be meagre in the localisation topology. Then there are only countably many $g \in \mathcal{E}$ such that $X_g \subseteq A$.*

Proof. There are maximal antichains $\mathcal{A}_n$ such that

$$A \cap \bigcap_{n\in\omega} \bigcup \{[\sigma, E] : (\sigma, E) \in \mathcal{A}_n\} = \varnothing$$

Since $\mathbb{LOC}$ satisfies the c.c.c., every $\mathcal{A}_n$ is of the form $\{(\sigma^n_m, E^n_m) : m \in \omega\}$. For every finite subset M of ω^2, we set $E_M = \{E^n_m : (n, m) \in M\}$. M is said to *cover* $g \in \omega^\omega$ if for all but finitely many $k \in \omega$, there is a $x \in E_M$ such that $x(k) = g(k)$. Since $\mathcal{E}$ is an eventually different family, each M can cover at most finitely many $g \in \mathcal{E}$. Hence, at most countably many $g \in \mathcal{E}$ are covered by some or the other M.

Let g be such that it is not covered by any M. Then we seek to construct a sequence $\langle \tau_n : n \in \omega \rangle$ such that

(i) $\tau_n \in \mathrm{dom}(\mathbb{LOC})$,

(ii) $\tau_n \subsetneq \tau_{n+1}$,

(iii) there is some $k \in \mathrm{dom}(\tau_{n+1} \backslash \tau_n)$ such that $g(k) \notin \tau_{n+1}(k)$,

(iv) for every $k < n$, there is an $m_k \in \omega$ such that $\sigma^k_{m_k} \subseteq \tau_n$ and for every $x \in E^k_{m_k}$ and every $\ell \in \mathrm{dom}(\tau_n \backslash \sigma^k_{m_k})$, $x(\ell) \in \tau_k(\ell)$, and

(v) $(\tau_n, E_{\{(k,m_k):k<n\}}) \in \mathbb{LOC}$.

If $\langle \tau_n : n \in \omega \rangle$ satisfies the above properties, then $\bigcup_{n\in\omega} \tau_n \in X_g \cap \bigcap_{n\in\omega} \bigcup \{[\sigma, E] : (\sigma, E) \in \mathcal{A}_n\}$. Therefore $X_g \nsubseteq A$. Therefore we just inductively define such a sequence. Let $\tau_0 = \varnothing$. Assume that τ_n has already been defined. Let $M = \{(k, m_k) : k \leqslant n\}$.

Since M does not cover g, there are infinitely many $k \in \omega$ such that for every $x \in E_M$, $x(k) \neq g(k)$. Let $\mathrm{lh}(\tau_n)$ be the minimal such. Then there is some $\tau'_n \in \mathrm{dom}(\mathbb{LOC})$ such that $(\tau'_n, E_M) \leqslant (\tau_n, E_M)$ and $g(\ell) \notin \tau'_n(\ell)$. Since $\mathcal{A}_n$ is a maximal antichain, there is some $m \in \omega$ such that $(\tau'_n, E_M) \leqslant (\tau_n, E_M)$ and $g(\ell) \notin \tau'_n(\ell)$. Since $\mathcal{A}_n$ is a maximal antichain, there is some $m \in \omega$ such that (τ'_n, E_M) and (σ^n_m, E^n_m). Let $(\sigma, E) \leqslant (\tau'_n, E_M)$ and (σ^n_m, E^n_m) be a witness. We set $m_n = m$ and $\tau_{n+1} = \sigma$. □

We can now apply Lemma 4.5.1 to obtain the desired result.

Theorem 4.5.2. *The following are equivalent:*

(i) $\mathbf{\Sigma}^1_2(\mathbb{LOC})$*,*

1. *for every* $r \in \mathbb{R}$*, the set* $\{\ell \in \mathbf{Loc}\,;\, \ell$ *is not* $\mathcal{I}^*_{\mathbb{LOC}}$*-quasigeneric over* $\mathbf{L}[r]\}$ *is L-small, and*
2. $\aleph_1$ *is inaccessible by reals.*

Proof. The equivalence of (i) and (ii) is Theorem 2.6.3; the direction (iii)⇒(i) is Proposition 2.7.2. Finally, (i)⇒(iii) in the presence of a Brendle-Łabędzki lemma by Theorem 2.8.3; but Lemma 4.5.1 provides exactly that. □

Chapter 5

Matet and Willowtree forcing

5.1 Implication diagrams

We remind the reader of Brendle's *Uniform Forcings Diagram* from p. 10:

$$
\begin{array}{ccccc}
\mathbb{S} & \supseteq & \mathbb{M} & \supseteq & \mathbb{L} \\
\rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} & & \\
\mathbb{W} & \supseteq & \mathbb{T} & & \rotatebox{90}{$\supseteq$} \\
\rotatebox{90}{$\supseteq$} & & & \rotatebox{45}{$\supseteq$} & \\
\mathbb{V} & & \supseteq & & \mathbb{R}
\end{array}
$$

The inclusions in this diagram immediately give rise to implications between the corresponding regularity properties

$$
\begin{array}{ccccc}
\mathbf{\Delta}^1_2(\mathbb{S}) & \Longleftarrow & \mathbf{\Delta}^1_2(\mathbb{M}) & \Longleftarrow & \mathbf{\Delta}^1_2(\mathbb{L}) \\
\Uparrow & & \Uparrow & & \Uparrow \\
\mathbf{\Delta}^1_2(\mathbb{W}) & \Longleftarrow & \mathbf{\Delta}^1_2(\mathbb{T}) & & \\
\Uparrow & & & \nwarrow & \\
\mathbf{\Delta}^1_2(\mathbb{V}) & & \Longleftarrow & & \mathbf{\Delta}^1_2(\mathbb{R})
\end{array}
$$

which we shall call the *Uniform Regularities Diagram*. We believe that the Uniform Regularities Diagram is complete in the sense of § 2.11. The status of the subdiagram with Matet and Willowtree forcing removed was known, i.e., that the diagram

$$
\begin{array}{ccccc}
\mathbf{\Delta}^1_2(\mathbb{S}) & \Leftarrow & \mathbf{\Delta}^1_2(\mathbb{M}) & \Leftarrow & \mathbf{\Delta}^1_2(\mathbb{L}) \\
\Uparrow & & & & \Uparrow \\
\mathbf{\Delta}^1_2(\mathbb{V}) & & \Longleftarrow & & \mathbf{\Delta}^1_2(\mathbb{R})
\end{array}
$$

is complete in the above sense (cf. Figure 2.1). We should like to emphasise that one component of this completeness is the fact that $\boldsymbol{\Delta}^1_2(\mathbb{L})$ does not imply $\boldsymbol{\Delta}^1_2(\mathbb{V})$ which is the main result of Chapter 3 (Corollary 3.4.2).

Observation 5.1.1. The Uniform Regularities Diagram is complete if the following non-implications hold:

(a) $\boldsymbol{\Delta}^1_2(\mathbb{T}) \not\Rightarrow \boldsymbol{\Delta}^1_2(\mathbb{V})$,

(b) $\boldsymbol{\Delta}^1_2(\mathbb{T}) \not\Rightarrow \boldsymbol{\Delta}^1_2(\mathbb{L})$, and

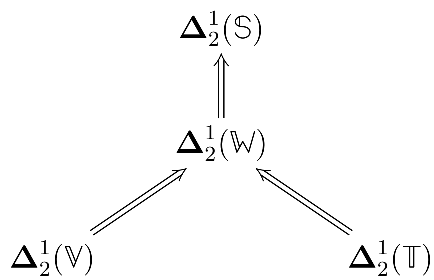
(c) $\boldsymbol{\Delta}^1_2(\mathbb{L}) \not\Rightarrow \boldsymbol{\Delta}^1_2(\mathbb{W})$.

Proof. We go through all non-implications that need to be checked.

By the completeness of the subdiagram with Matet and Willowtree forcing removed, we only need to show the non-implications with the two additional forcings for the forcings $\mathbb{S}$, $\mathbb{M}$, $\mathbb{L}$, and $\mathbb{V}$. (The forcing $\mathbb{R}$ does not have any non-implications in the Uniform Regularities Diagram.) Sacks, Miller and Laver regularity cannot imply either Matet or Willowtree regularity by (c) and transitivity. Since Silver forcing does not does not add unbounded reals (cf. [6, Proposition 4.2]), we have $\boldsymbol{\Delta}^1_2(\mathbb{V}) \not\Rightarrow \boldsymbol{\Delta}^1_2(\mathbb{M})$; thus Silver regularity cannot imply Matet regularity by transitivity.

Again, since $\boldsymbol{\Delta}^1_2(\mathbb{V}) \not\Rightarrow \boldsymbol{\Delta}^1_2(\mathbb{M})$, by transitivity, Willowtree regularity cannot imply Miller regularity (and therefore not Matet, Laver, or Mathias regularity). Finally, the Matet non-implications all follow directly from (a) and (b). □

In this chapter, we shall prove two of the assumptions of Observation 5.1.1: statement (a) in Corollary 5.3.8 and a weaker version of (c), viz. $\boldsymbol{\Delta}^1_2(\mathbb{S}) \not\Rightarrow \boldsymbol{\Delta}^1_2(\mathbb{W})$ (cf. Corollary 5.4.3). The combination of Corollaries 5.3.8 & 5.4.3 implies that the following subdiagram is complete:



Note that statement (a) follows from the fact that Matet forcing preserves p-points which was proved by [11, Theorem 4]. Our proof is more direct and combinatorial.

5.2 Fusion techniques for Matet forcing

As in Chapter 3, the fusion technique will be at the heart of our argument. In this section, we shall introduce the special situation and necessary terminology for fusion arguments for Willowtree and Matet forcing.

We shall denote by FU(A) the set of all finite concatenations of elements of A. For A, B subsets of $[\omega]^{<\omega}$, $A \sqsubseteq B$ (read "A is a condensation of B") if and only if every $a \in A$ is an element of FU(B).

For any finite subset t of ω, A past $t := \{a \in A\,;\, \min(a) > \max(t)\}$. By abuse of notation, we shall write $t < a$ when $\min(a) > \max(t)$.

Lemma 5.2.1. *If (s, A) is a Matet condition and (A_n) is a sequence of subsets of $[\omega]^{<\omega}$ such that $A_{n+1} \sqsubseteq A_n$ past a_n^0, where a_n^0 is the first element of A_n with respect to $<$, and if $B = \{a_n^0 : n \in \omega\}$, then $(s, B) \leqslant (s, A)$.*

Proof. Follows directly from the definition of the ordering. □

Definition 5.2.2. *Let $\mathbb{P}$ be any forcing adding a generic real. We say that $\mathbb{P}$* adds a generic of minimal degree among reals *if for every ground model M, every $\mathbb{P}$-generic c over M, and every real $x \in M[c]$, we have that $c \in M[x]$.*

Pure decision and reals of minimal degree. Our first objective is to show that Matet forcing has *pure decision property* and *adds reals of minimal degree*. The proof of the first can be found in [11, Lemma 2.6] but we include it here as it is an integral part of the argument. The proofs of the above two shall also illustrate the fusion technique in detail.

Theorem 5.2.3. *Let φ be a sentence and (s, A) a Matet condition. Then there is an extension (s, B) such that for any $t \in$ FU(B), $(s, B$ past $t)$ decides φ.*

Proof. We shall by induction define a sequence A_n of subsets of $[\omega]^{<\omega}$ starting with $A_{-1} = A$, such that

1. $A_{n+1} \sqsubseteq A_n$ past a_n^0 and
2. $(s^\frown a_n^0, A_n$ past $a_n^0)$ decides φ.

Given A_n, let's call the increasing enumeration of A_n to be $(a_n^k)_{k\in\omega}$ we can simply find an extension of $(s^\frown a_n^1, A_n$ past $a_n^1)$ say $(s^\frown t, A_n')$ such that it decides φ. We let $A_{n+1} = A_n' \cup \{t\}$.

We now set $A' = \{a_n^0 : n \in \omega\}$. Now, either for infinitely many of $t \in A'$, $(s^\frown t, A'$ past $t) \Vdash \varphi$ or for infinitely many of them $(s^\frown t, A'$ past $t) \Vdash \neg\varphi$. We set $B = \{t \in A' : (s^\frown t, A'$ past $t) \Vdash \varphi\}$ or $B = \{t \in A' : (s^\frown t, A'$ past $t) \Vdash \neg\varphi\}$ depending on which is infinite. Therefore, (s, B) is the required extension. □

Theorem 5.2.4 (Eisworth; [11, Lemma 2.7])**.** *Let $\dot{x}$ be a name for a non-ground model real and (s, A) be a Matet condition forcing that, then there is an extension (s, C) such that for every $t \in$ FU(C), there is a ground model real called the guiding real x_t, such that if $(t_k)_{k\in\omega}$ is an increasing enumeration of C past t then for all $k \in \omega$ we have:*

$$(s^\frown t, C \text{ past } t_k) \Vdash \dot{x}\restriction k = x_t\restriction k$$

Proof. First of all we notice that, it is possible to inductively define a sequence A_n, starting with $A_{-1} = A$, such that $A_{n+1} \sqsubseteq A_n$ past a_n^0, and (s, A_n) decides $\dot{x}$ up to n. Now, considering (s, B), where $B = \{a_n^0 \ : \ n \in \omega\}$, we have that there is a guiding real corresponding to s, say x_s.

Now, we shall use the above argument repetitively in an inductive manner to arrive at the required condition. We need to define a sequence B_n, starting with $B_{-1} = B$ such that:

1. $B_{n+1} \sqsubseteq B_n$ past b_n^0.
2. for all $t \in \mathrm{FU}(\{b_k^0 \ : \ k \leqslant n\})$, we have $(s^\frown t, B_{n+1}$ past $t)$ satisfying the condition that there is a guiding real corresponding to t, say x_t.

Given B_n, we simply enumerate the elements of $\mathrm{FU}(\{b_k^0 \ : \ k \leqslant n\})$ as $t_0, ..., t_m$. Set $C_0 = B_n$ past b_n^0. Given C_i, set $C_{i+1} \sqsubseteq C_i$ past t_{i+1}, such that for $(s^\frown t_{i+1}, C_{i+1})$, there is a guiding real $x_{t_{i+1}}$. We set $B_{n+1} = C_m$. Setting $C = \{b_n^0 \ : \ n \in \omega\}$, we have (s, C) to be the required condition. □

Theorem 5.2.5. *Matet forcing adds a generic of minimal degree among reals.*

Proof. Let $\dot{x}$ be a name for a non ground model real and (s, A) a Matet condition. Now for every $t \in \mathrm{FU}(A)$, we shall denote the guiding real corresponding to $s^\frown t$ and (s, A) as $x_{s^\frown t}$. We choose not to mention (s, A) since for any extension (s, B) of (s, A), the guiding real corresponding to (s, B) and $s^\frown t$ and that of (s, A) and $s^\frown t$ are the same.

We are going to build a sequence A_n, starting with $A_{-1} = A$ and for every $t \in \mathrm{FU}(\{a_j^0 : j \leqslant n\})$, $\dot{x}_{s^\frown t}^n$ denotes the maximal initial segment decided by $(s^\frown t, (A_{n+1} \cup \{a_j^0 : j \leqslant n+1\})$ past $t)$ and $\ell(s^\frown t)$, denotes the largest set according to $<$ in $A_{n+1} \cup \{a_j^0 : j \leqslant n+1\}$ which is a subset of $s^\frown t$.

Now, that we have set up the terminology, we can proceed with the proof. We require that for every $n \in \omega$, $A_{n+1} \sqsubseteq A_n$ past a_n^0 and for every $t \in \mathrm{FU}(\{a_j^0 : j \leqslant n\})$,

$$(s^\frown t, (A_{n+1} \cup \{a_j^0 : j \leqslant n+1\}) \text{ past } t) \Vdash \dot{x}_{s^\frown t}^n \neq x_{s^\frown t \restriction \min(\ell(s^\frown t))} \restriction |\dot{x}_{s^\frown t}^n|$$

and

$$(s^\frown t \restriction \min(\ell(s^\frown t)), (A_{n+1} \cup \{a_j^0 : j \leqslant n+1\}) \text{ past } t) \Vdash \dot{x}_{s^\frown t \restriction \min(\ell(s^\frown t))}^n = x_{s^\frown t \restriction \min(\ell(s^\frown t))} \restriction |\dot{x}_{s^\frown t}^n|.$$

The set A_{n+1} is actually constructed inductively. Let's say we enumerate $\mathrm{FU}\{a_j^0 \ : \ j \leqslant n\}$ as $(t_i)_{i \in m}$, we set $B_0 = A_n$ past a_n^0. We form a sequence $(B_i)_{i \in m}$, such that $B_{i+1} \sqsubseteq B_i$, $b_i^0 \subseteq b_{i+1}^0$, and for all $i \in m$,

$$(s^\frown t_i^\frown b_i^0, B_i \text{ past } b_i^0) \Vdash \dot{x}_{s^\frown t_i^\frown b_i^0}^{B_i \text{past} b_i^0} \neq x_{s^\frown t_i} \restriction |\dot{x}_{s^\frown t_i^\frown b_i^0}^{B_i \text{past} b_i^0}|$$

and

$$(s^\frown t_i, B_i \text{ past } b_i^0) \Vdash \dot{x}_{s^\frown t_i}^{B_i \text{past} b_i^0} = x_{s^\frown t_i} \restriction |\dot{x}_{s^\frown t_i^\frown b_i^0}^{B_i \text{past} b_i^0}|.$$

here, $\dot{x}^{B_i \text{past} b_i^0}_{s^\frown t_i^\frown b_i^0}$ denotes the initial segment decided by $(s^\frown t_i, B_i \text{ past } b_i^0)$.

This is possible because all the guiding reals are ground model and (s, A) forces that $\dot{x}$ is not ground model. Finally, $A_{n+1} = B_m$.

Now, we just let $B = \{a_n^0 : n \in \omega\}$. Then, we have that the function $f : [(s, B)] \to 2^\omega$ defined as

$$f(x) = \bigcup_{k \in \omega} \dot{x}_{s^\frown \frown_{k \in \omega} b^{n_k}}$$

where, $x = s^\frown \frown_{k \in \omega} b^{n_k}$, to be a continous injective ground model one, such that $(s, B) \Vdash f(x_G) = \dot{x}$. □

Note that the function f in the proof of Theorem 5.2.5 is a continuous function that lives in the ground model.

Matet forcing and graphs. We shall show how Matet forcing avoids quasigenerics of closed locally countable graphs. If G is a closed locally countable graph on 2^ω, we shall show that for any real say r added by the Matet forcing, it is contained in a ground model Borel set B, such that B is G-independent, i.e., any two elements of B do not form a G edge. As said earlier, this will also be a fusion argument. It is easy to observe that

$$T_{(s,B)}(\dot{x}) = \{\dot{x}_{(t,C)} \ : \ (t, C) \leqslant (s, B)\}$$

is a perfect tree.

Theorem 5.2.6. *Matet forcing does not add quasigenerics of closed locally countable graphs.*

Proof. Now, we look forward to create once again a sequence A_n as before, starting with $A_{-1} = A$ such that $A_{n+1} \sqsubseteq A_n$ past a_n^0 and for every $t \in \mathrm{FU}(\{a_j^0 : j \leqslant n\})$ we have

$$(s^\frown t, (A_{n+1} \cup \{a_j^0 : j \leqslant n+1\}) \text{ past } t) \Vdash ([\dot{x}^n_{s^\frown t}] \times [x_{s^\frown t \restriction \min(\ell(s^\frown t))} \restriction |\dot{x}^n_{s^\frown t}|]) \cap G = \varnothing.$$

Like in the proofs of Theorems 5.2.5 & 5.2.3, given A_n, we enumerate $\mathrm{FU}(\{a_j^0 \ : \ j \leqslant n\})$ as $(t_i)_{i \in m}$ and set $B_0 = A_n$ past a_n^0 and we define a sequence $(B_i)_{i \in m}$ such that, $B_{i+1} \sqsubseteq B_i$, $b_i^0 \subseteq b_{i+1}^0$, and

$$(s^\frown t_i^\frown b_i^0, B_i \text{ past } b_i^0) \Vdash ([\dot{x}^{B_i \text{past} b_i^0}_{s^\frown t_i^\frown b_i^0}] \times [x_{s^\frown t_i} \restriction |\dot{x}^{B_i \text{past} b_i^0}_{s^\frown t_i^\frown b_i^0}|]) \cap G = \varnothing$$

This is possible, due to the fact that $T_{(s,B)}$ is a perfect tree and choosing b_{i+1}^0, long enough, we shall have $(x_{s^\frown t_i^\frown b_{i+1}^0}, x_{s^\frown t_i}) \notin G$, and B_{i+1} is then obtained by deleting sufficiently many elements of B_i past b_{i+1}^0 in an increasing order, since for sufficiently long initial segments σ and τ of $x_{s^\frown t_i^\frown b_{i+1}^0}$ and $x_{s^\frown t_i}$ respectively, we have $([\sigma] \times [\tau]) \cap G = \varnothing$, due to closedness of G. $B_m = A_{n+1}$ and we define C to be $\{a_n^0 \ : \ n \in \omega\}$. Then, (s, C) is the required condition for which $[T_{(s,C)}(\dot{x})]$ is G independent that is for any two elements x and y of it, $(x, y) \notin G$. It is also a ground model closed set. Moreover $(s, C) \Vdash \dot{x} \in [T_{(s,C)}(\dot{x})]$. This completes the proof. □

5.3 Silver regularity in the Matet model

As in Chapter 3, the argument of Theorem 5.2.6 can be generalised to the iteration case. In essence, we are now going to adapt the general definitions of § 2.9 to the case of Matet forcing.

Definition 5.3.1. *Let α be an ordinal such that $\alpha < \omega_2$. Then, if $(s(\xi), A(\xi))_{\xi\in\alpha} \in \mathbb{T}_\alpha$, and $F \subseteq \operatorname{supp}(s(\xi), A(\xi))_{\xi\in\alpha}$, finite and $k : F \to \omega$. We say that $(t(\xi), B(\xi))_{\xi\in\alpha} \leqslant_{F,k} (s(\xi), A(\xi))_{\xi\in\alpha}$ iff for all $\gamma \in F$ $(t(\xi), B(\xi))_{\xi\in\alpha}\restriction\gamma \Vdash t(\gamma) = s(\gamma)$ and $(b_i(\gamma) = a_i(\gamma))$ for all $i \leqslant k(\gamma)$.*

Definition 5.3.2. *A fusion sequence consists of sequences F_n, k_n, $s_n(\xi)$, and $A_n(\xi)$, for $n \in \omega$ and $\xi \in \alpha$ such that*

1. *$F_n \subseteq \operatorname{supp}(s(\xi), A(\xi))_{\xi\in\alpha}$ is a finite set,*
2. *the sequence $(F_n\,;\, n \in \omega)$ is $\subseteq$-increasing,*
3. *$k_n : F_n \to \omega$,*
4. *$\bigcup_{n\in\omega} F_n = \operatorname{supp}(s(\xi), A(\xi))_{\xi\in\alpha}$,*
5. *for every $\gamma \in \operatorname{supp}(s(\xi), A(\xi))_{\xi\in\alpha}$ and every $n \in \omega$, there exists $m \in \omega$ such that $\gamma \in F_m$ and $\eta_m(\gamma) \geqslant n$,*
6. *$k_{n+1}(\gamma) \geqslant k_n(\gamma)$, for all $\gamma \in F_n$, and*
7. *$(s_{n+1}(\xi), A_{n+1}(\xi))_{\xi\in\alpha} \leqslant_{F_n,k_n} (s_n(\xi), A_n(\xi))_{\xi\in\alpha}$.*

Then $(t(\xi), B(\xi))_{\xi\in\alpha}$ is the fusion *of $(s_n(\xi), A_n(\xi))_{\xi\in\alpha}$ if and only if $(t(\xi), B(\xi))_{\xi\in\alpha}\restriction\gamma \Vdash t(\gamma) = s(\gamma)$ and $B(\gamma) = \bigcap_{n\in\omega} A_n(\gamma)$.*

We aim to build a fusion sequence $(s_n(\xi), A_n(\xi))_{\xi\in\alpha}, F_n, k_n$ such that if $(t(\xi), B(\xi))_{\xi\in\alpha}$ is the fusion of $(s_n(\xi), A_n(\xi))_{\xi\in\alpha}$, then $T_{(t(\xi),B(\xi))_{\xi\in\alpha}}(\dot{x})$ is G-independent.

For a Matet condition (s, A) we shall denote by $T_{(s,A)}$ the tree on ω^ω defined by $\operatorname{st}(T_{(s,A)}) = s$ and for $t \in T_{(s,A)}$, $\operatorname{succ}(t) = \{\ell \in A : t < \ell\}$. For the sake of notational convenience we shall be identifying the nodes of a Matet tree with $\omega^{<\omega}$ with the help of the natural order preserving-bijection. We define recursively $(s, A) *_0 \varnothing = \operatorname{st}(s, A)$ and let $(s, A) *_0 (\sigma^\frown n)$ be the nth immediate successor of $(s, A) *_0 \sigma$ according to the lexicographic ordering on $T_{(s,A)}$. We shall be using the notations $(s, A) *_0 \sigma$ and $T_{(s,A)} *_0 \sigma$ interchangeably. For a condition $(s(\xi), A(\xi))_{\xi\in\alpha} \in \mathbb{T}_\alpha$, we shall denote the initial segment of $\dot{x}$ decided by $(s(\xi), A(\xi))_{\xi\in\alpha}$ as $\dot{x}_{(s(\xi),A(\xi))_{\xi\in\alpha}}$.

Definition 5.3.3. *We say that a condition $(s(\xi), A(\xi))_{\xi\in\alpha} \in \mathbb{T}_\alpha$ is an (F_n, k_n)-faithful condition if and only if for all $\sigma, \sigma' \in \prod_{\gamma\in F_n} k(\gamma)^{k(\gamma)}$, such that $\sigma \neq \sigma'$, $([\dot{x}_{(s(\xi),A(\xi))_{\xi\in\alpha}*_0\sigma}] \times [\dot{x}_{(s(\xi),A(\xi))_{\xi\in\alpha}*_0\sigma'}]) \cap G = \varnothing$.*

Lemma 5.3.4. *Suppose that $(s(\xi), A(\xi))_{\xi\in\alpha}$ is (F_n, k_n)-faithful and k'_n is such that $k'_n(\gamma) = k_n(\gamma)+1$ and for all $\beta \in F_n\backslash\{\gamma\}$, $k'_n(\beta) = k_n(\beta)$. Then one can find $(s(\xi), B(\xi))_{\xi\in\alpha} \leqslant_{F_n,k_n} (s(\xi), A(\xi))_{\xi\in\alpha}$, such that $(s(\xi), B(\xi))_{\xi\in\alpha}$ is (F_n, k'_n)-faithful.*

Proof. Let $\{\sigma_0, \sigma_1, ..., \sigma_k\}$ be an enumeration of $\prod_{\gamma\in F_n} k(\gamma)^{k(\gamma)}$ and write $\gamma_{\max} := \max(F_n)$. We inductively define a $\leqslant_{F_n,k_n}$ decreasing sequence $(s(\xi), B_m(\xi))_{\xi\in\alpha}$, such that for all natural numbers n, n' with $n \neq n'$:

$$([\dot{x}_{(s(\xi),B_m(\xi))_{\xi\in\alpha}*_0(\sigma_m(\gamma_{\max})^\frown n))}] \times [\dot{x}_{(s(\xi),B_m(\xi))*_0(\sigma_m(\gamma_{\max})^\frown n')}]) \cap G = \varnothing.$$

Suppose that $(s(\xi), B_{m-1}(\xi))_{\xi\in\alpha}$ has already been defined. Then, due to the fact that G is closed and locally countable, just as in the single step proof (Theorem 5.2.6), $(s(\xi), B_{m-1}(\xi))_{\xi\in\alpha}*_0(\sigma_m)\restriction\gamma_{\max}$ forces that for every $n \in \omega$, there is a tail $t_n \leqslant (s(\xi), B_{m-1}(\xi))*_0 (\sigma_m(\gamma_{\max})^\frown n)[\gamma_{\max}, \alpha)$, such that for any two natural numbers n, n' with $n \neq n'$, we have

$$([\dot{x}_{t_n}] \times [\dot{x}_{t_{n'}}]) \cap G = \varnothing.$$

Therefore, one can find $(s(\xi), B_m(\xi))_{\xi\in\alpha} \leqslant_{(F_n,k_n)} (s(\xi), B_{m-1}(\xi))_{\xi\in\alpha}$, such that the condition

$$(s(\xi), B_m(\xi))_{\xi\in\alpha} *_0 ((\sigma_m))\restriction\gamma_{\max}$$

forces that for all $n \in \omega$ there exists some $p_n \in \omega$ such that $(s(\xi), B_m(\xi))_{\xi\in\alpha} *_0 (\sigma_m(\gamma)^\frown n) = t_{p_n}$ and therefore we have that for all $n, n' \in \omega$ such that $n \neq n'$:

$$([\dot{x}_{(s(\xi),B_m(\xi))_{\xi\in\alpha}*_0(\sigma_m(\gamma_{\max})^\frown n)}] \times [\dot{x}_{(s(\xi),B_m(\xi))_{\xi\in\alpha}*_0(\sigma_m(\gamma_{\max})^\frown n')}]) \cap G = \varnothing.$$

We let $(s(\xi), B(\xi))_{\xi\in\alpha}$ to be $(s(\xi), B_k(\xi))_{\xi\in\alpha}$. This completes the proof. □

Theorem 5.3.5. *The Matet model and the ω_2-Matet model have no quasigenerics of closed locally countable graphs.*

Proof. Using Lemma 5.3.4, for every $\alpha \in \omega_1$ for the Matet model and $\alpha \in \omega_2$ for the ω_2-Matet model, and $(s(\xi), A(\xi))_{\xi\in\alpha} \in \mathbb{T}_\alpha$, one can construct a fusion sequence as $(s_n(\xi), A_n(\xi))_{\xi\in\alpha}$ as above and define the fusion of it as $(s(\xi), B(\xi))_{\xi\in\alpha}$ such that

$$\forall\gamma \in \alpha((s(\xi), B(\xi))_{\xi\in\alpha}\restriction\gamma \Vdash \forall n \in \omega(B(\gamma) \sqsubseteq A_n(\gamma))).$$

We now define a function $f : (\omega^\omega)^{\mathrm{supp}(s(\xi),B(\xi))_{\xi\in\alpha}} \to 2^\omega$ with

$$f(x(\gamma)_{\gamma\in\mathrm{supp}(s(\xi),B(\xi))_{\xi\in\alpha}}) := \bigcup_{n\in\omega} \dot{x}_{(s(\xi),B(\xi))_{\xi\in\alpha}*(x(\gamma)\restriction k_n(\gamma))_{\gamma\in F_n}}.$$

Notice that this is a ground model Borel injective map and it maps the generic to $\dot{x}$. □

Corollary 5.3.6. *If G is a locally countable graph then $\chi_{\mathrm{B}}(G) = \aleph_1$ in the ω_2-Matet model.*

Corollary 5.3.7. *The Matet model does not satisfy $\mathbf{\Delta}^1_2(\mathbb{V})$.*

Proof. There is a graph G_1 that is closed locally countable with the property that G_1-quasigenerics are precisely the Silver quasigenerics (cf. [36, Claim 2.3.39]). Now the claim follows from Theorem 2.6.1. □

Corollary 5.3.8. *The statement $\mathbf{\Delta}^1_2(\mathbb{T})$ does not imply the statement $\mathbf{\Delta}^1_2(\mathbb{V})$.*

Proof. Directly from Corollary 5.3.7 with Proposition 2.4.4. □

5.4 Willowtree regularity in the Sacks Model

In this section, we prove a weaker version of condition (c) in Observation 5.1.1, viz. that the Sacks model does not satisfy $\boldsymbol{\Delta}^1_2(\mathbb{W})$.

Theorem 5.4.1. *Countable support iteration of length ω_1 of Sacks forcing does not add Willowtree quasigenerics.*

Fix $\alpha \in \omega_1$, a name $\dot{x}$ for a real not added at a proper initial stage of the iteration; we ensure that for every condition $p \in \mathbb{S}_\alpha$, one can find an extension r of p, such that $T_r(\dot{x})$ has all its splitting levels at different heights. The fact that $[T_r(\dot{x})]$ is Borel will ensure that it is Willow regular, but at the same time for any willow tree T, $[T] \nsubseteq [T_r(\dot{x})]$. We shall here be assuming that there is a ground model homeomorphism $h : (2^\omega)^{supt(q)} \to T_p(\dot{x})$ as outlined in [16, Lemma 78].

Given a finite set $F \subseteq \text{supt}(p)$ and $\eta : F \to \omega$, we say that a condition $q \leqslant p$ is (F,η)-faithful if for any two elements σ and τ of $\prod_{\gamma\in F} 2^{\eta(\gamma)}$, $|\dot{x}_{q*_0\sigma}| \neq |\dot{x}_{q*_0\tau}|$. Here $\dot{x}_p$ denotes the initial segment decided by p. For any two conditions q and p in $\mathbb{S}_\alpha$, we say that $q \leqslant_{(F,\eta)} p$, if for all $\sigma \in \prod_{\gamma\in F} 2^{\eta(\gamma)}$, $q *_0 \sigma = p *_0 \sigma$.

Our goal is to build a sequence (p_n, F_n, η_n) which satisfies the following properties:

(i) $p_{n+1} \leqslant_{(F_n,\eta_n)} p_n$,

(ii) p_n is (F_n,η_n)-faithful,

(iii) $F_n \subseteq F_{n+1}$,

(iv) for every $n \in \omega$ and $\gamma \in \text{supp}(p_n)$ there exists $m \in \omega$ such that $\gamma \in F_m$ and $\eta_m(\gamma) \geqslant n$,

(v) $\bigcup_{n\in\omega} F_n = \text{supt}(p)$, and

(vi) $\eta_n(m) \leqslant \eta_{n+1}(m)$ for all $m \in F_n$.

To this end, the following lemma plays a crucial role.

Lemma 5.4.2. *Suppose that $\alpha < \omega_1$ is an ordinal, p an $\mathbb{S}_\alpha$ condition, $F \subseteq \alpha$ is finite, $\eta : F \to \omega$, $\eta' : F \to \omega$ are such that $\eta\restriction F\setminus\{\beta\} = \eta'\restriction F\setminus\{\beta\}$ and $\eta'(\beta) = \eta(\beta)+1$. Moreover let p be (F,η)-faithful. Then, there exists $q \leqslant_{(F,\eta)} p$ such that for all $\sigma, \tau \in \prod_{\gamma\in F} 2^{\eta'(\gamma)}$, $|\dot{x}_{q*_0\sigma}| \neq |\dot{x}_{q*_0\tau}|$.*

Proof. Suppose we have an enumeration $\{\sigma_1, ..., \sigma_m\}$ of $\prod_{\gamma\in F} 2^{\eta(\gamma)}$. Then we shall inductively build a $\leqslant_{(F,\eta)}$ decreasing sequence q_i.

Suppose that we have already found q_{i-1}. Then, we take $q_{\sigma_i,0}$ and $q_{\sigma_i,1}$ to be such that and $q_{i-1} *_0 \sigma_i\restriction\delta$ forces the following:

(i) $q_{\sigma_i,k} \leqslant q(\delta) *_0 q(\sigma_i^\frown k)^\frown q\restriction(\delta,\alpha)$,

(ii) $|\dot{x}_{q_{\sigma_i,k}}| > |\dot{x}_{q_{\sigma_i}}|$, and

(iii) $|\dot{x}_{q_{\sigma_i,0}}| < |\dot{x}_{q_{\sigma_i,1}}|$.

One can now choose a condition $q_j \leqslant_{(F,\eta)} q_{j-1}$ such that

$$q_j *_0 \sigma_j \restriction \delta \Vdash q_j(\delta) *_0 \sigma_j(\delta)^\frown k^\frown q_j^\frown q(\delta, \alpha) = q_{\sigma_i,k}$$

Then our required q is simply q_m. □

Using Lemma 5.4.2, one can construct a fusion sequence (p_n, F_n, η_n), such that it's fusion say r, is such that $T_r(\dot{x})$ is a tree with splitting levels all at different heights. This completes the proof of Theorem 5.4.1.

Corollary 5.4.3. The statement $\mathbf{\Delta}^1_2(\mathbb{S})$ does not imply $\mathbf{\Delta}^1_2(\mathbb{W})$.

Proof. By Proposition 2.4.4, the Sacks model satisfies $\mathbf{\Delta}^1_2(\mathbb{S})$. Since Willowtree forcing has the Ikegami property and the Sacks model does not contain any Willowtree quasigenerics, it does not satisfy $\mathbf{\Delta}^1_2(\mathbb{W})$. □

Bibliography

[1] Banerjee, R. & Gaspar, M. Borel chromatic numbers of locally countable F_σ graphs and forcing with superperfect trees. Submitted, 2023.

[2] Banerjee, R., Löwe, B., & Wansner, L. Amoebas and their regularities. In preparation.

[3] Bartoszyński, T. & Judah, H. *Set Theory. On the Structure of the Real Line.* A K Peters, 1995.

[4] Brendle, J. Strolling through paradise. Fundamenta Mathematicae 148:1 (1995): 1-25.

[5] Brendle, J. How small can the set of generics be? In: Buss, S. R. Hájek, P., & Pudlák, P., editors. *Logic Colloquium '98. Proceedings of the annual European summer meeting of the Association for Symbolic Logic, Prague, Czech Republic, August 9-15, 1998.* Lecture Notes in Logic, 13. A K Peters, 2000, pp. 109–126.

[6] Brendle, J., Halbeisen, L. & Löwe, B. Silver measurability and its relation to other regularity properties. Mathematical Proceedings of the Cambridge Philosophical Society 138 (2005), 135–149.

[7] Brendle, J. & Khomskii, Y. Polarized partitions on the second level of the projective hierarchy. Annals of Pure and Applied Logic 163 (2012), 1345–1357.

[8] Brendle, J., Khomskii, Y., & Wohofsky, W. Cofinalities of Marczewski-like ideals. Colloquium Mathematicum 150:2 (2017), 269–279.

[9] Brendle, J. & Löwe, B. Solovay-type characterizations for forcing-algebras. Journal of Symbolic Logic 64 (1999), 1307–1323.

[10] Brendle, J. & Löwe, B. Eventually different functions and inaccessible cardinals. Journal of the Mathematical Society of Japan 63 (2011), 137–151.

[11] Eisworth, T. Forcing and Stable Ordered-Union Ultrafilters. Journal of Symbolic Logic 67:1 (2002), 449–464.

[12] Fischer, V., Friedman, S. D., & Khomskii, Y. Cichoń's diagram, regularity properties and $\boldsymbol{\Delta}^1_3$ sets of reals. Archive for Mathematical Logic 53 (2014), 695-729.

[13] Gaspar, M. *Borel chromatic numbers in models of set theory.* Ph.D. Thesis. Universität Hamburg, 2022.

[14] Gaspar, M. & Geschke, S. Borel chromatic numbers of closed graphs and forcing with uniform trees. Preprint, 2022 (arXiv:2208.06914).

[15] Geschke, S. Weak Borel chromatic numbers. Mathematical Logic Quarterly 57 (2011), 5–13.

[16] Geschke, S. & Quickert, S. On Sacks forcing and the Sacks property. In: Löwe, B., Piwinger, B., & Räsch, T., editors, *Classical and New Paradigms of Computation and their Complexity. Papers of the conference "Foundations of the Formal Sciences III".* Trends in Logic, 23. Springer-Verlag, 2004, pp. 95–139.

[17] Gray, C. *Iterated forcing from the strategic point of view.* Ph.D. Thesis. University of California, Berkeley. 1980.

[18] Halbeisen, L. J. *Combinatorial Set Theory. With a Gentle Introduction to Forcing.* Springer Monographs in Mathematics. Springer-Verlag, 2nd edition, 2017.

[19] Harrington, L. A., Kechris, A. S. & Louveau, A. A Glimm-Effros dichotomy for Borel equivalence relations. Journal of the American Mathematical Society 3:4 (1990), 903–928.

[20] Ikegami, D. *Games in Set Theory and Logic.* Ph.D. Thesis. Universiteit van Amsterdam, 2010.

[21] Jech, T. *Set theory. The third millennium edition, revised and expanded.* Springer Monographs in Mathematics. Springer-Verlag, 2003.

[22] Judah, H. & Repický, M. Amoeba reals. Journal of Symbolic Logic 60:4 (1995), 1168–1185.

[23] Kanamori, A. *The higher infinite. Large cardinals in set theory from their beginnings.* Springer Monographs in Mathematics. Springer-Verlag, 2nd edition, 2003.

[24] Kechris, A., Solecki, S., & Todorčević, S. Borel chromatic numbers. Advances in Mathematics 141 (1999), 1–44.

[25] Khomskii, Y. *Regularity Properties and Definability in the Real Number Continuum.* Ph.D. Thesis. Universiteit van Amsterdam, 2012.

[26] Łabędzki, G. A topology generated by eventually different functions. Acta Universitatis Carolinae, Mathematica et Physica 37:2 (1996), 37–53.

[27] Łabędzki, G. & Repický, M. Hechler reals. Journal of Symbolic Logic 60:2 (1995), 444–458.

[28] Martin, D. A. & Solovay, R. A. Internal Cohen extensions. Annals of Mathematical Logic 2:2 (1970), 143–178.

[29] Matet, P. Some filters of partitions. Journal of Symbolic Logic 53 (1988), 540–553.

[30] Moschovakis, Y. N. *Descriptive set theory.* Mathematical Surveys and Monographs, 155. American Mathematical Society, 2nd edition, 2009.

[31] Shelah, S. Can you take Solovay's inaccessible away? Israel Journal of Mathematics 48:1 (1984), 1–47.

[32] Truss, J. K. Connections between different amoeba algebras. Fundamenta Mathematicae 130:2 (1988), 137–155.

[33] Wagon, S. *The Banach-Tarski paradox. With a foreword by Jan Mycielski.* Encyclopedia of Mathematics and its Applications, 24. Cambridge University Press, 1985.

[34] Wansner, L. *Generalizing Ikegami's Theorem.* M.Sc. Thesis. Universität Hamburg, 2019.

[35] Wansner, L. *Aspects of Forcing in Descriptive Set Theory and Computability Theory.* Ph.D. Thesis. Universität Hamburg, 2023.

[36] Zapletal, J. *Descriptive set theory and definable forcing*. Memoirs of the American Mathematical Society, 167. American Mathematical Society, 2004.

[37] Zapletal, J. *Forcing idealized*. Cambridge Tracts in Mathematics, 174. Cambridge University Press, 2008.

English summary

This thesis studies implications between regularity properties at the second level of the projective hierarchy. The results of Chapters 3 and 5 show non-implications between certain statements of the form "all $\mathbf{\Delta}^1_2$ sets are regular"; the results of Chapter 4 show that certain statements of the form "all $\mathbf{\Sigma}^1_2$ sets are regular" are equivalent to "$\aleph_1$ is inaccessible by reals", the strongest regularity property. The following theorems are the main contributions of this thesis:

1. In the Laver model, $\mathbf{\Sigma}^1_2(\mathbb{L})$ holds and $\mathbf{\Delta}^1_2(\mathbb{E}_0)$ and $\mathbf{\Delta}^1_2(\mathbb{V})$ fail (Corollary 3.4.2).
2. In the Laver model, $\mathbf{\Delta}^1_2(\mathbb{V})$ fails, but for every real r, there is a splitting real over $\mathbf{L}[r]$ (Corollary 3.4.4).
3. The statement $\mathbf{\Sigma}^1_2(\mathbb{A})$ is equivalent to the statement "$\aleph_1$ is inaccessible by reals" (Corollary 4.3.5).
4. The statement $\mathbf{\Sigma}^1_2(\mathbb{UM})$ is equivalent to the statement "$\aleph_1$ is inaccessible by reals" (Theorem 4.4.2).
5. The statement $\mathbf{\Sigma}^1_2(\mathbb{LOC})$ is equivalent to the statement "$\aleph_1$ is inaccessible by reals" (Theorem 4.5.2).
6. In the Matet model, $\mathbf{\Delta}^1_2(\mathbb{V})$ fails (Corollary 5.3.8).
7. In the Sacks model, $\mathbf{\Delta}^1_2(\mathbb{W})$ fails (Corollary 5.4.3).

Result 1. solves an open question mentioned three times in the literature; result 2. solves a question asked by Brendle, Halbeisen, and Löwe.

Deutsche Zusammenfassung

Diese Dissertation untersucht Implikationen zwischen Regularitätseigenschaften auf der zweiten Ebene der projektiven Hierarchie. Die Ergebnisse in Kapitel 3 und 5 liefern Nicht-Implikationen zwischen bestimmten Aussagen der Form "alle $\mathbf{\Delta}^1_2$-Mengen sind regulär"; die Ergebnisse in Kapitel 4 zeigen, daß bestimmte Aussagen der Form "alle $\mathbf{\Sigma}^1_2$-Mengen sind regulär" äquivalent zu "$\aleph_1$ ist durch reelle Zahlen unerreichbar" ist, der stärksten aller Regularitätseigenschaften. Die folgenden Theoreme sind die Hauptresultate der Dissertation:

1. Im Laver-Modell gilt $\mathbf{\Sigma}^1_2(\mathbb{L})$ und $\mathbf{\Delta}^1_2(\mathbb{E}_0)$ sowie $\mathbf{\Delta}^1_2(\mathbb{V})$ gelten nicht (Korollar 3.4.2).

2. Im Laver-Modell gilt $\mathbf{\Delta}^1_2(\mathbb{V})$ nicht, aber für jede reelle Zahl r gibt es eine spaltende Zahl über $\mathbf{L}[r]$ (Korollar 3.4.4).

3. Die Aussage $\mathbf{\Sigma}^1_2(\mathbb{A})$ ist äquivalent zu "$\aleph_1$ ist durch reelle Zahlen unerreichbar" (Korollar 4.3.5).

4. Die Aussage $\mathbf{\Sigma}^1_2(\mathbb{UM})$ ist äquivalent zu "$\aleph_1$ ist durch reelle Zahlen unerreichbar" (Theorem 4.4.2).

5. Die Aussage $\mathbf{\Sigma}^1_2(\mathbb{LOC})$ ist äquivalent zu "$\aleph_1$ ist durch reelle Zahlen unerreichbar" (Theorem 4.5.2).

6. Im Matet-Modell gilt $\mathbf{\Delta}^1_2(\mathbb{V})$ nicht. (Korollar 5.3.8).

7. Im Sacks-Modell gilt $\mathbf{\Delta}^1_2(\mathbb{W})$ nicht. (Korollar 5.4.3).

Resultat 1. löst eine offene Frage, die dreifach in der Literatur erwähnt war; Resultat 2. löst eine Frage, die von Brendle, Halbeisen und Löwe gestellt wurde.