CS:ST Lecture XIV

THE WADGE HIERARCHY

<u>Definition</u> $A, B \subseteq \omega^\omega$.

We say $\boxed{A \leq_W B}$ iff

there is cts fn $f: \omega^\omega \to \omega^\omega$ s.t. $A = f^{-1}[B]$.

<u>Observe</u> Γ is boldface : Γ is closed under $\leq_W$.

<u>Remark</u>. $\leq_W$ can be defined for arbitrary top. spaces, but the game connection uses that we're looking at ω^ω.

$\leq_W \rightsquigarrow A \equiv_W B$: $A \leq_W B$ and $B \leq_W A$.

[Wadge-equivalent]

not antisymmetric, so it's possible to have $A \neq B$ and $A \equiv_W B$.

$A <_W B$ $:\Longleftrightarrow$ $A \leq_W B$ & $B \not\leq_W A$.

Degree structure:

$[A]_{\equiv_W}$

Other connections

If Γ is a b.f. pointclass with a universal set $U \subseteq \omega^\omega \times \omega^\omega$: universal for $\Gamma \restriction \omega^\omega$.

Then for each $A \in \Gamma \restriction \omega^\omega$, there is a cts fn $f : \omega^\omega \times \omega^\omega \longrightarrow \omega^\omega$ s.t. $x \in U \longleftrightarrow f(x) \in A$

So $\Gamma \restriction \omega^\omega := \{ A ;\ A \leq_W U \}$

NOT LITERALLY CORRECT
Lecture II : $\omega^\omega \cong \omega^\omega \times \omega^\omega$

This provides a "degree-theoretic", almost algebraic view of what b.f. pointclasses are.

Now return to games:

Let $A, B \subseteq \omega^\omega$. Define $G_W(A, B)$ WADGE game for A & B

$M = \mathbb{N} \cup \{P\}$ where P is an extra move "PASS".

$$\begin{array}{lccc} \mathrm{I} & \overbrace{x_0 \qquad x_1}^{x} & \cdots \\ \mathrm{II} & \underbrace{y_0 \qquad y_1}_{y} & \end{array}$$

- I may never play P
- II may play play P

$y \in M^\omega$.

I x_0 x_1 x_2 ... $x \in \omega^\omega$

II y_0 y_1 y_2 ... $y \in M^\omega$

$\hat{y}$ is the sequence y with all of the P's removed.

→ If $\hat{y}$ is finite, player II loses.

If $\hat{y}$ is infinite, then II wins iff

$$\boxed{x \in A \longleftrightarrow \hat{y} \in B.}$$

Trivial examples

$\boxed{A = \emptyset}$ B is anything

if $B \neq \omega^\omega$, then a w.s. would be "play $y \notin B$".

If $B \neq \omega^\omega$, II will not be able to win.

$\boxed{A = \omega^\omega}$ B is anything

if $B \neq \emptyset$, ~~then~~ "play $x \in B$" is a w.s.

if $B = \emptyset$, then II can't win.

PROP. For any $A, B \subseteq \omega^\omega$.
Player II has a w.s. in $G_W(A,B)$
iff
$A \leq_W B$.

Proof. Remember HW sheet #3 (1.)
Characterisation of CONTINUITY in terms of functions
$$g : \omega^{<\omega} \longrightarrow \omega^{<\omega}.$$
This is precisely what happens in this game:
$$f(x) = \bigcup_{n \in \mathbb{N}} g(x \upharpoonright n)$$
$$\Downarrow$$
w.s. τ for player II.
Suppose $\tau(s)$ is already defined and fits with g. Now $\tau(s\, y_n\, x_{n+1})$
$[s = (x_0\, y_0, \dots x_n)]$
is either P [if g does not give you further info beyond $\tau(s)$]
or n [if n is the next value given by g]

Conversely: if τ is a winning strategy, define $g(s)$ to be $\hat{t}$ where t is the response of τ to s and the $\wedge$-operation is removing P moves.

OBSERVATION $\qquad x \in A \longleftrightarrow \hat{y} \in B$

1. If I/II has a w.s. in $G_W(A,B)$, the same strategy is winning in $G_W(\omega^\omega \setminus A, \omega^\omega \setminus B)$.

1a. $A \leq_W \omega^\omega \setminus B \iff \omega^\omega \setminus A \leq_W B$.

2. What if player I has a w.s. in $G_W(A,B)$?

W.s. for player I also give us cts functions (even better): so get $f: \omega^\omega \longrightarrow \omega^\omega$ s.t.

$$f(y) \in A \longleftrightarrow y \notin B.$$

$$f^{-1}[A] = \{y ; f(y) \in A\} = \omega^\omega \setminus B$$

$$\omega^\omega \setminus B \leq_W A$$

Theorem (Wadge's Lemma)

If $G_W(A,B)$ is determined, then either

$$\boxed{\begin{array}{l} A \leq_W B \text{ or} \\ B \leq_W \omega^\omega \setminus A \end{array}}$$

$[\Leftrightarrow \omega^\omega \setminus B \leq_W A]$

SLO: SEMI-LINEAR ORDERING PRINCIPLE

Looks a bit like linearity but not really.

Let's prove that if A, B are not $\leq_W$ comparable, then $A \equiv_W \omega^\omega \setminus B$.

$A \not\leq_W B$ and $B \not\leq_W A$.

WL $\downarrow$ $\qquad$ $\downarrow$ WL

$\omega^\omega \setminus B \leq_W A \qquad \omega^\omega \setminus A \leq_W B$

$\Updownarrow$

$A \leq_W \omega^\omega \setminus B$

$\Rightarrow A \equiv_W \omega^\omega \setminus B$.

So, if A, B are not $\leq_W$-comparable, then $A \equiv_W \omega^\omega \setminus B$.

[I.e., antichains have length at most 2.]

<u>Remark</u> If $A, B \subseteq \omega^\omega$, then there is some $X \subseteq \omega^\omega$

$$\underline{X} := \{ z \,;\, z = x * y \text{ with } \hat{y} \text{ is finite or } x \in \underline{A} \not\leftrightarrow \hat{y} \in \underline{B} \}$$

s.t. I wins $G(X)$ iff I wins $G_W(A, B)$

If $\underset{\sim}{\Gamma}$ is a boldface pointclass, closed under Boolean operations and contains $\underset{\sim}{\Sigma}^0_2$ sets, then $\mathrm{Det}(\underset{\sim}{\Gamma})$ implies Wadge-Determinacy for all $A, B \in \underset{\sim}{\Gamma}$.

In particular: ZFC ($\Rightarrow$ Borel determinacy) $\Longrightarrow$ for all Borel sets A, B

$$A \leq_W B$$

or $\omega^\omega \setminus B \leq_W A$.

Consider the bottom of the Wadge hierarchy.

$$\emptyset \le_W B \quad \text{if } B \neq \omega^\omega$$

$$\omega^\omega \le_W B \quad \text{if } B \neq \emptyset$$

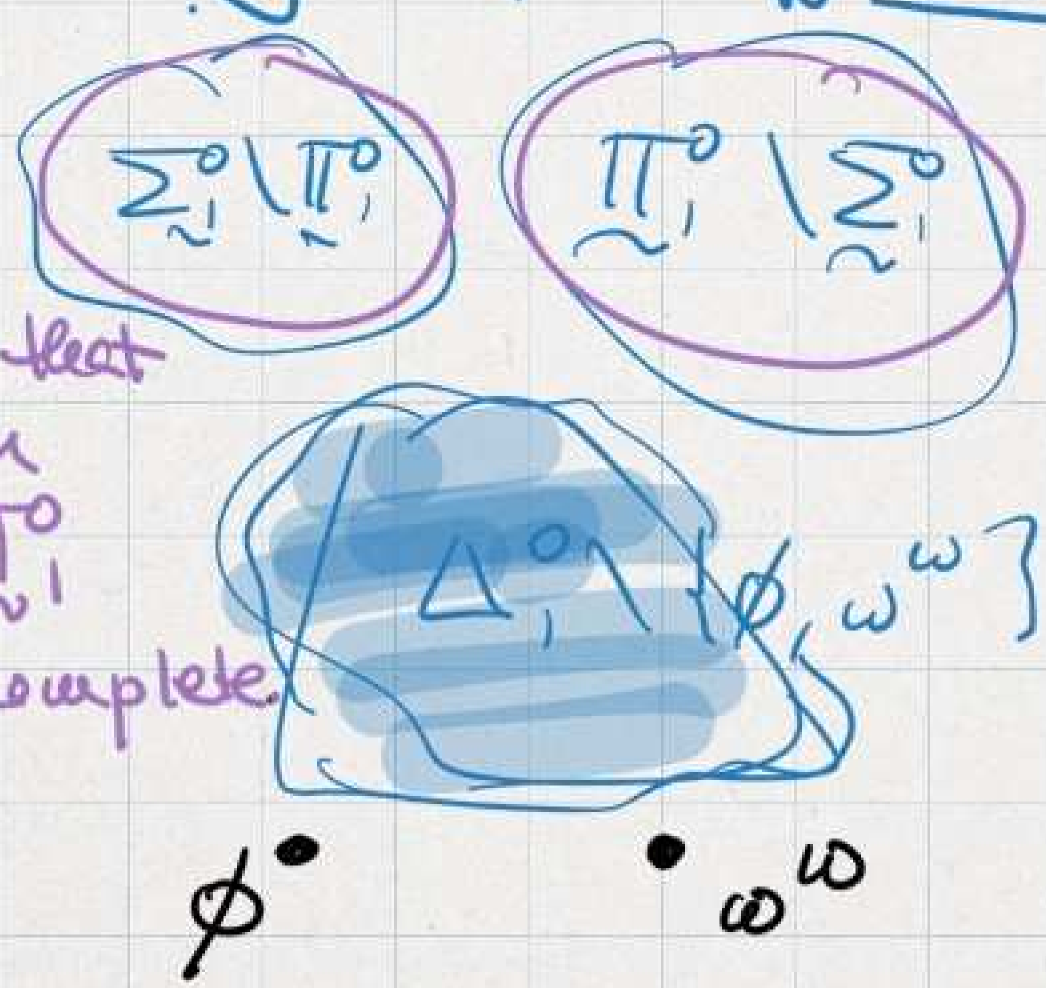
So $\emptyset, \omega^\omega$ are minimal in $\le_W$ and $\emptyset \not\equiv_W \omega^\omega$.

If $A \in \boldsymbol{\Sigma}^0_1 \setminus \boldsymbol{\Pi}^0_1$. If $B \in \boldsymbol{\Pi}^0_1$, then $A \not\le_W B$; also if $B \le_W A$, then $B \in \boldsymbol{\Delta}^0_1$.

CLAIM Then every $B \in \boldsymbol{\Sigma}^0_1$ has the property $B \le_W A$.

[Suppose not: By WL, $A \le_W \omega^\omega \setminus B$. Then $A \in \boldsymbol{\Pi}^0_1$. Contradiction]

$\boldsymbol{\Sigma}^0_1$-hardness



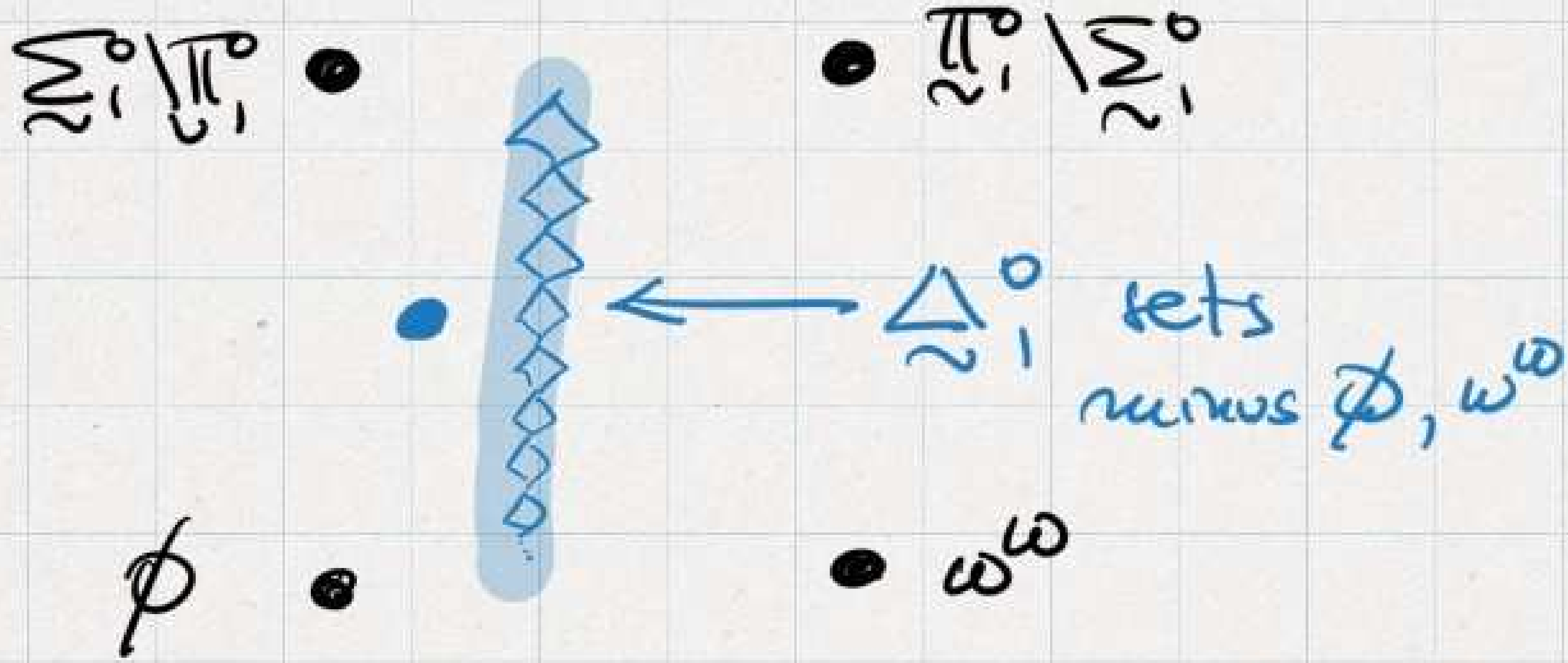
We proved that every set in $\boldsymbol{\Sigma}^0_1 \setminus \boldsymbol{\Pi}^0_1$ is $\boldsymbol{\Sigma}^0_1$-complete.

So far $\underset{\sim}{\Sigma}^0_1 \setminus \underset{\sim}{\Pi}^0_1$

and therefore $\underset{\sim}{\Pi}^0_1 \setminus \underset{\sim}{\Sigma}^0_1$

are a single Wadge-degree [$A, B \in \underset{\sim}{\Sigma}^0_1 \setminus \underset{\sim}{\Pi}^0_1$ then $A \equiv_W B$]

so the picture looks correctly like this:



<u>Claim</u> If A, B are $\underset{\sim}{\Delta}^0_1$, $A, B \notin \{\emptyset, \omega^\omega\}$, then $A \equiv_W B$.

Suppose there is A, B $A \not\leq_W B$.

So $A \equiv_W \omega^\omega \setminus B$.

Define $C := \{\underline{0x} ; x \in A\} \cup \{\underline{1y} ; y \in B\}$.

Then $A \leq_W C$ $\qquad$ $B \leq_W C$.

$x \longmapsto 0x$ $\qquad$ $y \longmapsto 1y$.

But $C \leq_W A$ and
$C \leq_W B$

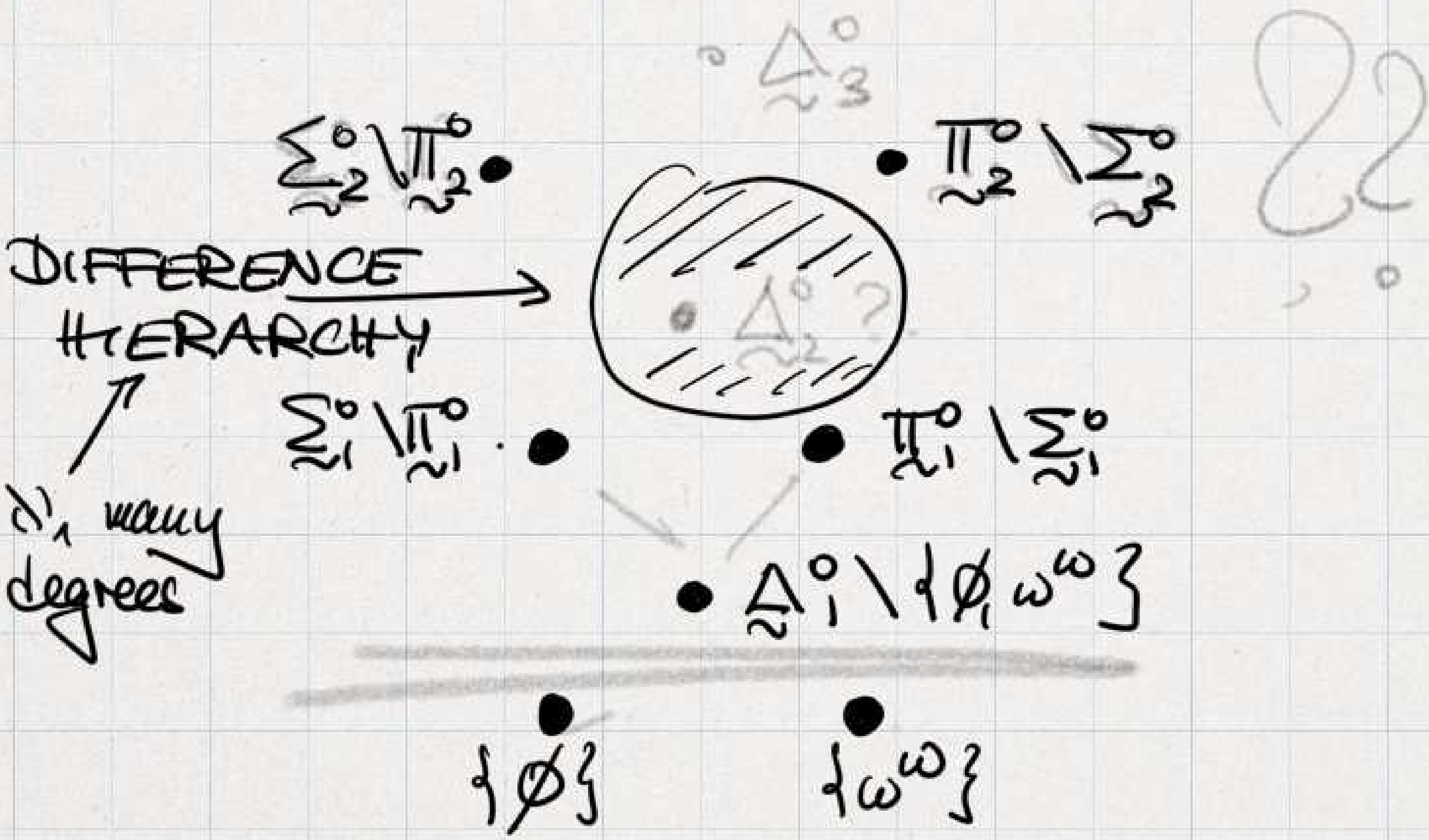
$C = \{0x ; x \in A\} \cup \{1y ; y \in B\}$.

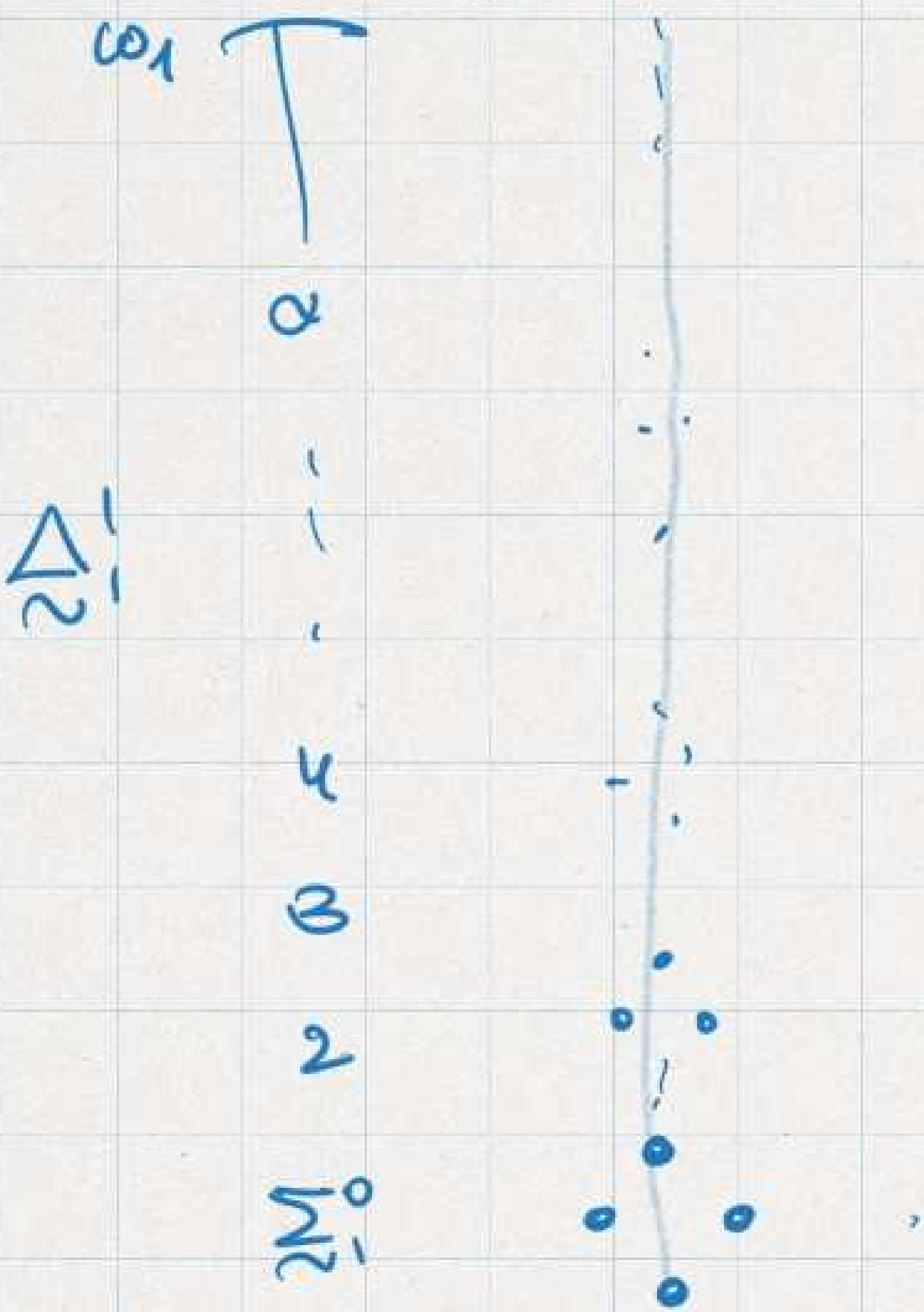
Construct winning strategies concretely.

— $A, B \leq C \leq A, B$

So everything is Wadge equivalent.

$A \leq C \leq \omega^\omega \setminus B \leq C$





The Borel hierarchy in Wadge degrees is a long semi-linear hierarchy where the Wadge degrees refine the Borel classes.

The Wadge hierarchy is in fact not just (semi) linearly ordered by a (semi) well ordering.

Last theorem of CS:ST 2020 is thus wellfoundedness!

We're are going to show that as the

MARTIN – MONK Theorem

Assume AD. Then there is no strictly decreasing sequence of Wadge degrees.

[Observe that the proof is "LOCAL": If $\underset{\sim}{\Gamma}$ is a bf pcl and you assume $Det(\underset{\sim}{\Gamma})$, then the result holds for seq. with all sets in $\underset{\sim}{\Gamma}$.

E.g., ZFC $\Longrightarrow$ there is no strictly decreasing seq. of Borel Wadge degrees.]

LEMMA We call a set $X \subseteq 2^\omega$ a FLIP SET if for all $z, z' \in 2^\omega$ s.t. z, z' differ in precise one digit

$$z \in X \longleftrightarrow z' \notin X.$$

Consider the game $G^{**}(X)$:

$G^{**}(X)$

$$\begin{array}{lllll} \mathrm{I} & s_0 & & s_2 & & \cdots \\ \mathrm{II} & & s_1 & & s_3 & \cdots \end{array}$$

$s_i \in 2^{<\omega} \setminus \{\emptyset\}$

$x := s_0 s_1 s_2 \cdots \in 2^\omega$

I wins if $x \in X$.

If $G^{**}(X)$ is determined, then X is not a flip set.

[In particular: AD $\Longrightarrow$ there are no flip sets.]

Khomskii, Infinite Games § 5.4.

Proof of Martin-Monk

Towards a contradiction

$A_0 >_W A_1 >_W A_2 >_W \cdots$

$\boxed{A_{i+1} <_W A_i} \longleftrightarrow A_{i+1} \leq_W A_i$

$* \ \underline{A_i \not\leq_W A_{i+1}}$

Claim $\omega^\omega \setminus A_i \not\leq_W A_{i+1}$.

[If $\underline{\omega^\omega \setminus A_i} \leq_W A_{i+1} \leq \underline{A_i} \Rightarrow A_i \equiv_W \omega^\omega \setminus A_i$

$\Rightarrow A_i \leq_W A_{i+1}$ Contradiction w/ $*$]

$G_{i0} \quad G_W(A_i, A_{i+1})$

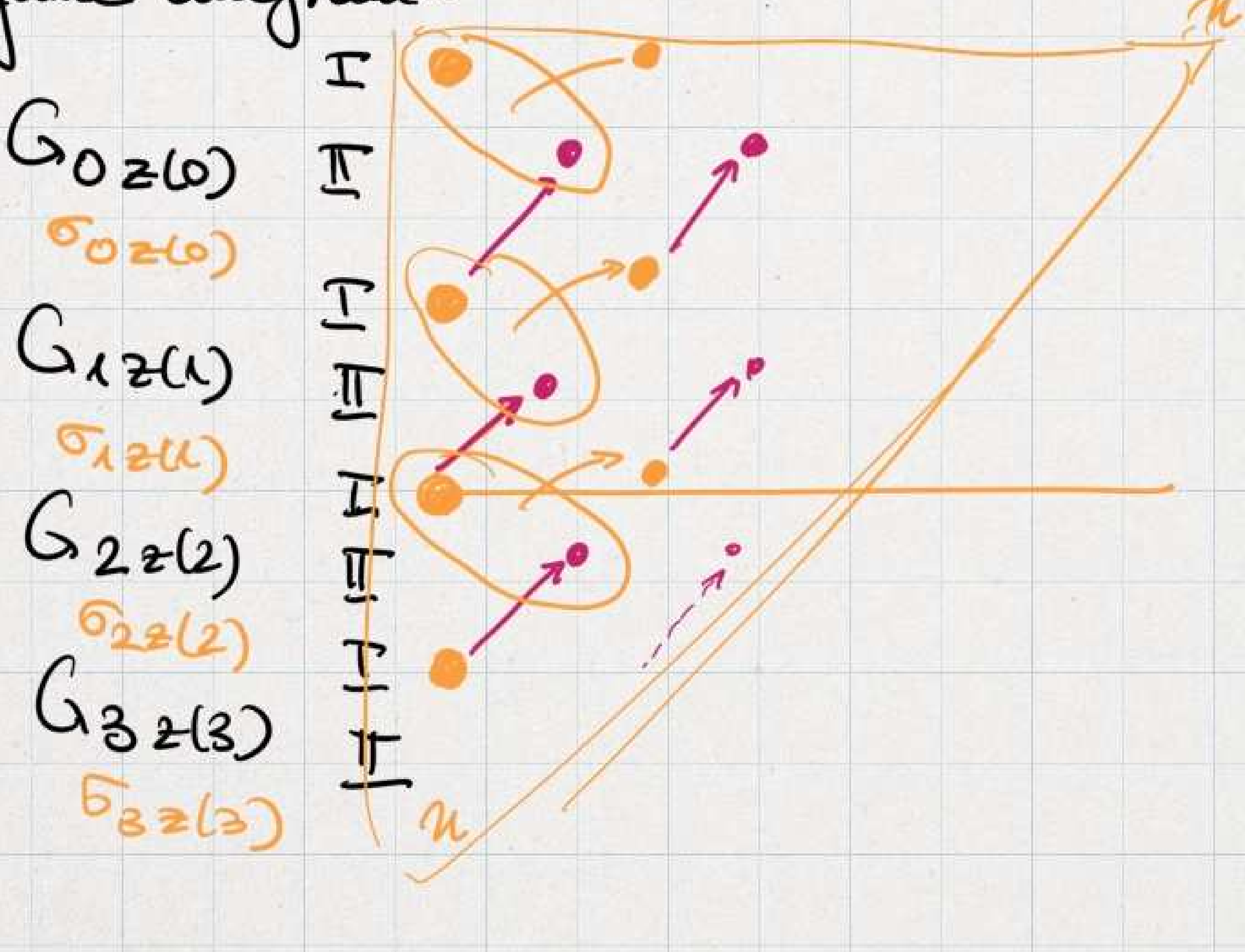
$G_{i1} \quad G_W(A_i, \omega^\omega \setminus A_{i+1})$

neither is won by II, so for each we have w.s.

$\left.\begin{array}{l}\sigma_{i0}\\ \sigma_{i1}\end{array}\right.$ for player I.

$[AC_\omega(\mathbb{R})]$

Let $z \in 2^\omega$ and define the following game diagram:



Since $\sigma_{0z(0)}$ was winning in $G_{0z(0)}$.

$G_{0z(0)}$ $\quad a_{00} := \sigma_{0z(0)}(\emptyset)$ $\quad a_{01} := \sigma_{0z(0)}(a_{00}a_{10})$

$\sigma_{0z(0)}$ $\quad a_{10}$

$a_0^z \in A_0 \longleftrightarrow$ $\quad a_0^z = (a_{00}, a_{01}, a_{02}, \dots)$

$a_1^z \in A_1$ if $z(0)=1$ $\quad a_1^z = (a_{10}, a_{11}, a_{12}, \dots)$

$G_{1z(1)}$ $\quad a_{10} := \sigma_{1z(1)}(\emptyset)$ $\quad a_{11} := \sigma_{1z(1)}(a_{10}a_{20})$

$\sigma_{1z(1)}$ $\quad a_{20}$

$a_1^z = (a_{10}, a_{11}, a_{12}, \dots)$

$a_2^z = (a_{20}, a_{21}, a_{22} \dots)$

$G_{2z(2)}$ $\quad a_{20} := \sigma_{2z(2)}(\emptyset)$ $\quad a_{21} := \sigma_{2z(2)}(a_{20}a_{30})$

$\sigma_{2z(2)}$ $\quad a_{30}$

a_2^z

a_3^z

$G_{3z(3)}$ $\quad a_{30} := \sigma_{3z(3)}(\emptyset)$

$\sigma_{3z(3)}$ $\quad a_{40}$

a_3^z:

a_4^z:

By construction, if $z(i) = 0$

$$G_{iz(i)} = G_{i0} = G_W(A_i, A_{i+1})$$

then $a_i^z \in A_i \longleftrightarrow a_{i+1}^z \notin A_{i+1}$

if $z(i) = 1$

$$G_{iz(i)} = G_{i1} = G_W(A_i, \omega^\omega \setminus A_{i+1})$$

then $a_i^z \in A_i \longleftrightarrow a_{i+1}^z \in A_{i+1}$.

<u>Observe</u> if z, z' are eventually the same, i.e., $\exists n \, \forall k$

$$z(k) = z'(k),$$

then for all $k \geq n$

$$a_k^z = a_k^{z'}.$$

Thus if z, z' differ in precisely one digit, then $a_0^z \in A_0 \longleftrightarrow a_0^{z'} \notin A_0$.

$X := \{z \in 2^\omega ; \, a_0^z \in A_0\}$ By construction, this is a flip set

The technique used here is the
MARTIN - MONK technique
filling in an infinite diagram
of games with information
from w.s. (obtained by
determinacy) and copying
their responses upwards
so that the strategies play
against each other.

Other applications of the technique:

First Periodicity Theorem
Steel - Van Wesep Theorem.