# Homework Set #3

**Capita Selecta: Set Theory**
*2016/17: 1st Semester; block a*
Universiteit van Amsterdam

**Homework.** There will be six homework sheets; handed in by each student individually. Homework has to contain the name and student ID of the student. If your homework is handwritten, make sure that it is legible. You submit your solutions either by e-mail to `h(dot)nobrega(at)uva(dot)nl` or in person before the Tuesday lecture or by placing them in Hugo's mailbox at the ILLC at Science Park 107.

**Deadline.** This homework set is due on **Tuesday, 4 October 2016** before the lecture.

In all of the exercises, work in a sufficiently strong metatheory.

1. The following is an excerpt of Kunen's book (Chapter V, § 1) proving the absoluteness of the definability operator. The proof is rather sketchy: do the details.

   1.4. Definition. By recursion on $m \in \omega$, $\mathrm{En}(m, A, n)$ is defined (for all $n$ simultaneously) by the following clauses:
   (a) If $m = 2^i \cdot 3^j$ and $i, j < n$, then $\mathrm{En}(m, A, n) = \mathrm{Diag}_\in(A, n, i, j)$.
   (b) If $m = 2^i \cdot 3^j \cdot 5$ and $i, j < n$, then $\mathrm{En}(m, A, n) = \mathrm{Diag}_=(A, n, i, j)$.
   (c) If $m = 2^i \cdot 3^j \cdot 5^2$, then $\mathrm{En}(m, A, n) = A^n \setminus \mathrm{En}(i, A, n)$.
   (d) If $m = 2^i \cdot 3^j \cdot 5^3$, then $\mathrm{En}(m, A, n) = \mathrm{En}(i, A, n) \cap \mathrm{En}(j, A, n)$.
   (e) If $m = 2^i \cdot 3^j \cdot 5^4$, then $\mathrm{En}(m, A, n) = \mathrm{Proj}(A, \mathrm{En}(i, A, n+1), n)$.
   (f) If $m$ is not of the form specified in one of (a)–(e), then $\mathrm{En}(m, A, n) = 0$. □

   1.5. Lemma. *For any $n$ and $A$,* $\mathrm{Df}(A, n) = \{\mathrm{En}(m, A, n) : m \in \omega\}$.

   Proof. First, by induction on $m$, $\forall n\, (\mathrm{En}(m, A, n) \in \mathrm{Df}(A, n))$; observe (for clause (f)) that $0 \in \mathrm{Df}(A, n)$ since $\mathrm{Df}(A, n)$ is closed under intersections and complements. Next, by induction on $k$,
   $$\forall n\, (\mathrm{Df}'(k, A, n) \subset \{\mathrm{En}(m, A, n) : m \in \omega\}). \quad \square$$

   1.6. Corollary. $|\mathrm{Df}(A, n)| \leq \omega$. □

   1.7. Lemma. *The defined functions* Df *and* En *are absolute for transitive models of* ZF − P.

   Proof. The absoluteness of Proj, $\mathrm{Diag}_\in$, $\mathrm{Diag}_=$, Df′, Df, and En are easily checked successively by the methods of IV § 5. The fact that functions defined recursively using absolute notions are absolute (IV 5.6) is applied several times. For En, we also use the absoluteness of ordinal exponentiation (IV 5.7). □

2. Recall the recursive definition of the *Beth sequence*: $\beth_0 = \aleph_0$; $\beth_{\alpha+1} = 2^{\beth_\alpha}$ (cardinal exponentiation); and $\beth_\lambda = \bigcup_{\alpha<\lambda} \beth_\alpha$ for limit $\lambda$. Show that for all ordinals $\alpha > \omega$, the following are equivalent:

   (a) $|\mathbf{L}_\alpha| = |\mathbf{V}_\alpha|$ and
   (b) $\alpha = \beth_\alpha$.

3. Calculate the $\mathbf{L}$-rank of $\mathbb{Z}$ and $\mathbb{Q}$. (Note that you first need to say what exactly these objects are.) Let $\varrho_\mathbf{L}(x) = \alpha$ and $\varrho_\mathbf{L}(y) = \beta$. Calculate the $\mathbf{L}$-rank of $\{x\}$, $x \times y$, $x \cup y$, and $\langle x, y \rangle$.

4. Prove the following basic properties of $\mathbf{L}$:

(a) For all $\alpha$, $\mathbf{L}_\alpha = \{x \in \mathbf{L}\,;\ \varrho_{\mathbf{L}}(x) < \alpha\}$.

(b) For all $\alpha$, $\mathbf{L}_\alpha \subseteq \mathbf{V}_\alpha$.

(c) For all $n \in \mathbb{N}$, $\mathbf{L}_n = \mathbf{V}_n$ (therefore $L_\omega = V_\omega$ as well).