Modal Fixpoint Logics

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Example

- Add connective $\langle * \rangle$ to the language ML of modal logic
- $\langle * \rangle p := \bigvee_{n \in \omega} \Diamond^n p$

  $s \Vdash \langle * \rangle p$ iff there is a finite path from s to some p-state

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- **Fact** $\langle * \rangle p$ is the least fixpoint of the 'equation' $x \leftrightarrow p \vee \Diamond x$

  (a fixpoint of a map $f : C \to C$ is an $a \in C$ with $fa = a$)

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  (a fixpoint of a map $f : C \to C$ is an $a \in C$ with $fa = a$)
- Notation: $\langle * \rangle p \equiv \mu x.p \vee \Diamond x$.

Other Examples

- common knowledge: $C_G p \equiv \nu x.p \wedge \bigwedge_{a \in G} K_a x$
- until: $Upq \equiv \mu x.p \vee (q \wedge \Diamond x)$
- no infinite paths: $\mathsf{F} \equiv \mu x.\Box x$.

Modal Fixpoint Logics

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  - new fixpoint connectives such as $\langle * \rangle$, U, C_G, F, . . .
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- Many applications in process theory, epistemic logic, . . .
- Rich theory:
  - game-theoretical semantics
  - connections with theory of automata on infinite objects
  - connections with theory of (complete) lattice expansions

General Program

Achieve a better understanding of modal fixpoint logics by studying the interaction between

- combinatorial
- model-theoretic and
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Here: consider simple, ‘flat’ modal fixpoint logics, in full generality

Overview

- Introduction
- Flat Modal Fixpoint Logics
- Constructiveness & continuity

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For simplification assume ML has only one diamond $\Diamond$, and Γ is singleton.

Modal Logic: Kripke Semantics

- Kripke frame $\mathbb{S} = \langle S, R \rangle$ with $R \subseteq S \times S$.
- Complex algebra: $\mathbb{S}^+ := \langle \wp(S), \varnothing, S, \sim_S, \cup, \cap, \langle R \rangle \rangle$,

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  Want: $\sharp^{\mathbb{S}}(\boldsymbol{P})$ is the least fixpoint of the map $\gamma^{\mathbb{S}}_{\boldsymbol{P}} = \lambda X.\gamma^{\mathbb{S}}(X, \boldsymbol{P})$.

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Hence $q = fq$ and so $\bigwedge \mathsf{PRE}(f)$ is the least fixpoint of f.

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Definition LFP.f is constructive if LFP.$f = f^\omega(\bot) = \bigvee_{n\in\omega} f^n(\bot)$.

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This definition applies to non-complete lattices too!

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- By Knaster-Tarski we may define $\sharp^{\mathbb{S}} : \wp(S)^n \to \wp(S)$ by

$$\sharp^{\mathbb{S}}(\boldsymbol{P}) := \mathsf{LFP}.\gamma^{\mathbb{S}}_{\boldsymbol{P}}.$$

Questions

- When are fixpoint connectives constructive?
- How to axiomatize flat fixpoint logics?

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- Introduction
- Flat modal fixpoint logics
- Constructiveness & continuity

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with EF_x x-existential fragment given by

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Proof
In both cases, γ is continuous in x.

Continuity

Definition
Let S be an LTS. A formula γ is continuous in x on S if

$$\gamma^S(X, \boldsymbol{P}) = \bigcup_{F \subseteq_\omega X} \gamma^S(F, \boldsymbol{P}).$$

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Let $\gamma(x, \boldsymbol{p})$ be a modal formula.
If γ is continuous in x then it is 'uniformly continuous': $\exists k < \omega$, $\forall S$ LTS:

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- Is it decidable whether a formula γ is continuous/constructive in x?