

# Intuitionism

*Brouwer, Heyting and  
Dummett*

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# Motivation

*The long belief in the principle of the excluded third in mathematics is considered by Intuitionism as a phenomenon of history of civilisation.*

*Intuitionism tries to explain the long persistence of this dogma by the practical validity of classical logic for an extensive group of simple everyday phenomena.*

*(excerpt from Brouwer 1948:94,96)*

*TND (tertium non datur) traced to RIO and/or RITV.*

- *Metaphysics!*

# General comparison

Classical mathematics

■ Classical logic

$$\begin{aligned} & \Rightarrow \quad \varphi \vee \neg \varphi \\ & \varphi \quad \Leftrightarrow \quad \neg \neg \varphi \\ \exists x \neg P(x) \quad \Leftrightarrow \quad \neg \forall x P(x) \end{aligned}$$

Intuitionistic mathematics

■ Intuitionistic logic

$$\begin{aligned} & \Rightarrow \quad \neg \neg \varphi \\ \exists x \neg P(x) \quad \Rightarrow \quad \neg \forall x P(x) \end{aligned}$$

# Brouwer's Mathematics

- Mathematics is synthetic.
- Truth is a priori.
- Mathematical knowledge is mind dependent.
  - Anti Realist in Ontology
  - Anti Realist in Truth Value
  - But: non-empiricist
- Found mathematics on Kantian view of time.
  - Sequence of moments gives natural numbers
  - Infinite divisibility of moments gives linear continuum
- Found geometry on reals using Cartesian techniques.

# Brouwer's Mathematical Method

- Mathematics is about idealized mental constructions.
- Existential proofs require a construction procedure.
- No unknowable truths.
  - Knowledge is provability. Truths are provable statements.
- Impredicative definitions are circular.
- Impossible to consider infinite collections of mathematical objects as completed totalities.
  - Construction of the elements can never be completed.
  - Example of a meaningless sentence:  
The set of continuous univariate functions has property P.

# Example: reals

## Classical reals

- Any decimal expansion
- Digit  $n$  is fixed from the beginning for each  $n$
- Actual infinity

## Intuitionistic reals

- Early Brouwer
  - rule-governed decimal expansions (Cauchy sequences)
- Later Brouwer
  - Free-choice sequences.
  - Digits are iterated on demand by a *creative subject*.
- Potential infinity

# Properties of reals

- Theorems of real numbers must follow from finite amounts of information
  - The rule
  - An initial segment of the free-choice sequence
- Method of weak counterexamples
  - An object is not well-defined if it implies TND.
- Any function from reals to reals is continuous.

# Heyting

- Logic
  - Codification of the rules of communicating mathematics via language
  - Dependent on mathematics. New methods of reasoning require extensions of logical system
  - Not truth conditions, proof conditions.

# Heyting Calculus

- A proof of  $e \wedge \Box$  consists of a proof of  $e$  and a proof of  $\Box$
- A proof of  $e \vee \Box$  consists of a proof of  $e$  or a proof of  $\Box$
- A proof of  $e \rightarrow \Box$  consists of a method of transforming any proof of  $e$  into a proof of  $\Box$
- A proof of  $\neg e$  consists of a method of transforming any proof of  $e$  into an absurdity
- A proof of  $\exists x e(x)$  consists of a procedure that, given  $n$ , produces a proof of  $e(n)$
- A proof of  $\forall x e(x)$  consists of a construction of an item  $n$  and a proof of  $e(n)$

# Problem and conclusion

- Discontinuous functions are essential to physics
  - Brouwer: Physics holds less “mathematical truth”. Divorce.
- Which mathematics to pick?
  - Brouwer: Intuitionistic mathematics.
  - Heyting: Both, but Intuitionistic mathematics preferred.
  - Bishop: Common core of both

# Sir Michael A.E. Dummett

1925 -



# Biography

Born 1925

1943-1947 Military Service

1944 Received into Roman Catholic Church

1947 Studies at Christ Church College, Oxford

1950 Graduates with First Class Honours in Philosophy, Politics and Economics, fellowship at All Souls College

1962-1974 Reader Philosophy Of Math. at Oxford

1979-1992 Professorship of Logic at Oxford

1999 Received Knighthood

# Publications

- 1973 *Frege: Philosophy of Language*
- 1977 *Elements of Intuition*
- 1981 *The Interpretation of Frege's Philosophy*
- 1991 *The Logical Basis of Metaphysics, Frege: Philosophy of Mathematics*
- Collected papers:  
*Truth and Other Enigmas*  
*Frege and Other Philosophers*  
*Origins of Analytical Philosophy*

# Philosophy of Language

- “Why prefer Intuitionism?”
- Correctness of logic turns on questions of *meaning*.
- “Meaning consists solely in its role as instrument of communication” (connection to language).
- Meaning = Use (Manifestation Requirement).
- What is use?



# Humpty Dumpty

*"There's glory for you!"*

*'I don't know what you mean by "glory,"' Alice said. Humpty Dumpty smiled contemptuously. 'Of course you don't--till I tell you. I meant "there's a nice knock-down argument for you!"'*

*'But "glory" doesn't mean "a nice knock-down argument,"' Alice objected.*

*'When I use a word,' Humpty Dumpty said in rather a scornful tone, 'it means just what I choose it to mean --neither more nor less.'*

# Holism

- Web of Belief.
- Reject holism and insist that each statement must have determinate individual content.
- Alternative theory: *Molecular semantics* .

# Molecular Semantics

- “...*individual sentences carry a content which belongs to them in accordance with the way they are compounded out of their own constituents*, independent of other sentences of the language not involving those constituents...”
- Logic can be understood independently.
- LEM doesn't follow from  $\forall$  and  $\exists$  separately.
- Hence: classical mathematics is wrong!

# Truth / Verifiability

Classical mathematics:

- Central notion is *truth*,
- Truth cannot be determined in general,
- Problem with Manifestation Requirement,
- Hence: realism is wrong!

Dummett:

- Replace truth by *verifiability*.
- In mathematics: verifiability is *proof*.
- Heyting semantics.

# Reply to Dummett

- All truths are knowable, or aren't they?
- Intuitionism needs idealizations ( $\exists x \neg(x \rightarrow \neg x) \rightarrow \neg \exists x (x \rightarrow \neg x)$ ).
- But not too much idealization.
- Accept Dummett?
- Reject Dummett?

# Dummett's Reply

Dummett tries to particularize the theory to mathematics:

- Hard-headed finitism
- Indefinite extensibility

An intuitionist:

- Rejects holism?
- Idealizes, but not too much?
- Is in denial?