UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Axiomatic Set Theory

(Axiomatische Verzamelingentheorie)

2002/2003; 3rd Trimester
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Homework Set # 6 [Jech, § 7 and § 8] Deadline: Thursday, June 19th, 2003

Exercise 6.1 (Ordinal Topologies)

Let α be an ordinal, and the ordinal topology be the topology induced by $\{L_\gamma\,;\,\gamma \in \alpha\} \cup \{R_\gamma\,;\,\gamma \in \alpha\}$ ($L_\gamma := \{\beta\,;\,\beta < \gamma\}$ and $R_\gamma := \{\beta\,;\,\beta > \gamma\}$).
Let $A \subseteq \alpha$. We said that A is **closed** if for each $\lambda \in \alpha$, we have that $\sup(A \cap \lambda) \in A$. Show that A is closed in this sense if and only if A is topologically closed in the ordinal topology.

Exercise 6.2 (Normal Sequences)

Let $\kappa > \aleph_0$ be regular and $F : \kappa \to \kappa$ be a normal sequence (increasing and continuous). Show that the set of fixed points of F (*i.e.*, $F(\alpha) = \alpha$) is closed unbounded in κ.

Exercise 6.3 (Inaccessible and Mahlo cardinals)

Assume that for each $\alpha \in \mathrm{Ord}$, there is an inaccessible cardinal $\kappa > \alpha$ (*i.e.*, a regular cardinal κ such that for all $\lambda < \kappa$ we have $2^\lambda < \kappa$).
Show that there are cardinals κ with the following property:

$\Phi(\kappa) \doteq$ "κ is inaccessible, and the set of inaccessible cardinals below κ has cardinality κ."

Take κ_0 to be the least cardinal such that $\Phi(\kappa_0)$. Show that κ_0 is not a Mahlo cardinal.

Hint. For the first claim, use the function $F(\alpha) \mapsto$ the αth inaccessible cardinal. For the second claim, look at $F^{-1}{\upharpoonright}\kappa_0$ (which is a regressive function on the set of inaccessible cardinals below κ_0).

Exercise 6.4 ($\aleph_0$ is not Ramsey)

Show that $\aleph_0 \not\to (\aleph_0)^{<\omega}_2$.

Hint. For $s \in [\aleph_0]^{<\omega}$, let $F(s) = 1$ if the number of elements of s is in s. Show that F cannot be constant on $[H]^n$ where H is infinite and $n \in H$.