

GEOMETRIC TOPOLOGY

Problem Set 8

1. Prove that the function $f : \mathbb{R}P^n \rightarrow \mathbb{R}$ given in homogeneous coordinates by

$$f([x_0 : \dots : x_n]) := \frac{1}{\|x\|^2} \sum_{k=0}^n kx_k^2$$

is a Morse function. What are the indices of its critical points?

Can you find an analogous Morse function on $\mathbb{C}P^n$? What are the indices of its critical points?

2. Let K^2 denote the Klein bottle. Prove that

- a) $\mathbb{R}P^2 \# \mathbb{R}P^2 \cong K^2$
- b) $\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2 \cong T^2 \# \mathbb{R}P^2$

In particular, this shows that for non-orientable closed 2-manifolds there is no unique “prime decomposition” with respect to connected sum.

3. a) Prove that if M is a closed oriented manifold of positive dimension, then $M \# (-M)$ is the boundary of a compact oriented manifold.
- b) Prove that if M_1 and M_2 are closed oriented n -dimensional manifolds, then there is a compact oriented manifold of dimension $n + 1$ with boundary

$$M_1 \# M_2 - M_1 - M_2.$$

- c) Prove that the connected sum of two homotopy spheres is again a homotopy sphere.
4. Let p and q be two relatively prime integers with $p \geq 2$, and consider the diffeomorphism of $S^3 \subseteq \mathbb{C}^2$ given as

$$\begin{aligned} \sigma_{p,q} : S^3 &\rightarrow S^3 \\ \sigma_{p,q}(z, w) &:= (e^{\frac{2\pi i}{p}} z, e^{\frac{2\pi i q}{p}} w). \end{aligned}$$

- a) Prove that $\sigma_{p,q}$ generates an action of \mathbb{Z}_p on S^3 without fixpoints. The quotient space is called *the lens space of type (p, q)* and denoted by $L(p, q)$.
- b) Prove that $L(p, q)$ is a closed 3-manifold which inherits a natural orientation from S^3 .
- c) Note that for $q \equiv q' \pmod{p}$ the spaces $L(p, q)$ and $L(p, q')$ are naturally diffeomorphic.
- d) Prove that there is an orientation reversing diffeomorphism between $L(p, q)$ and $L(p, -q)$.
- e) Prove that $L(p, q)$ can be obtained from gluing two solid tori $S^1 \times D^2$ along a diffeomorphism of their boundary tori. Can you describe that diffeomorphism in terms of p and q ?
- f) Can you describe a Morse function on $L(p, q)$ with one critical point for each index $k \in \{0, 1, 2, 3\}$?