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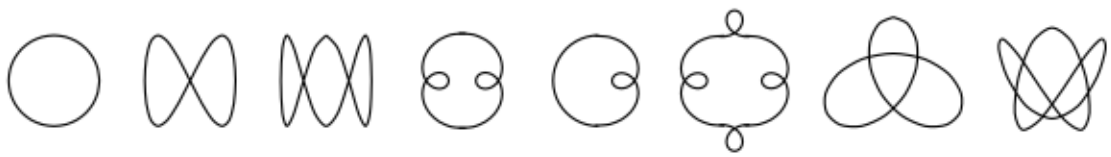
Geometric Topology

Problem Set 7

1. Find and prove a criterion for two immersions of S^1 into $\mathbb{R}^2$ to be homotopic through immersions. Apply your criterion to decide which of the following immersions (with either of their orientations) are homotopic to each other through immersions.



2. Let $p \in M$ be a critical point of the smooth function $f : M \to \mathbb{R}$. Prove that

a) $\mathrm{Hess}_p(f)(v,w) = X(Y(f))(p)$, where X and Y are local vector fields with $X(p) = v$ and $Y(p) = w$.

b) $\mathrm{Hess}_p(f)$ is a symmetric bilinear form on T_pM.

c) in local coordinates $\mathrm{Hess}_p(f)$ is represented by the matrix of second derivatives of f at p.

d) p is a nondegenerate critical point if and only if $\mathrm{Hess}_p(f)$ has trivial kernel.

3. Prove that if a closed connected manifold M admits a smooth function $f : M \to \mathbb{R}$ with exactly two critical points, then M must be homeomorphic to a sphere.

4. Suppose $M \subseteq \mathbb{R}^d$ is a smooth closed submanifold. For each $v \in S^{d-1}$ let $f_v : M \to \mathbb{R}$ be the function $f_v(x) := \langle v, x \rangle$, where $\langle\, .\, ,\, .\, \rangle$ is the standard euclidean metric on $\mathbb{R}^d$. Prove that the subset of $v \in S^{d-1}$ such that f_v is a Morse function on M is open and dense.

5. Let M be a closed manifold, $f : M \to \mathbb{R}$ a smooth function and $X = \operatorname{grad} f$ the gradient of f with respect to some Riemannian metric on M.

a) Suppose p is a nondegenerate critical point of f, so that p is an isolated zero of X. Prove that the index of p as a critical point of f and the index of p as an isolated zero of X are related by

$$\mathrm{ind}_p X = (-1)^{\mathrm{ind}_p(f)}.$$

b) Conclude that if f is a Morse function, then

$$\chi(M) = \sum_{i=0}^{\dim M} (-1)^{c_i(f)},$$

where $c_i(f)$ is the number of critical points of f of index i.