Universität Hamburg
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Geometric Topology

Problem Set 5

1. Suppose S_1 and S_2 are closed oriented submanifolds of the closed oriented manifold M with $\dim S_1 + \dim S_2 = \dim M$.

a) Prove that in this situation we have

$$S_1 \bullet S_2 = (-1)^{\dim S_2} (S_1 \times S_2) \bullet \Delta,$$

where on the right hand side we consider the intersection number of the oriented submanifold $S_1 \times S_2$ with the diagonal $\Delta \subseteq M \times M$ (with its induced orientation from the projection to either factor).

b) Prove that if $S_1 = S_2 = S$ (so that $\dim S = \frac{1}{2} \dim M$), the formula can be simplified to

$$S \bullet S = (S \times S) \bullet \Delta.$$

2. Suppose that M and N are connected closed oriented manifolds of the same dimension, and let $f : M \to N$ be any smooth map.

a) Prove that

$$\deg(f) = (M \times \{q\}) \bullet \operatorname{graph}(f),$$

where $q \in N$ is any point, and

$$\operatorname{graph}(f) = \{(x, f(x)) \,|\, x \in M\} \subseteq M \times N$$

is the graph of f.

Now suppose $M = N$, so that $f : M \to M$ is a self-map. In this case the Lefschetz number $L(f) \in \mathbb{Z}$ of f is defined as

$$L(f) = \operatorname{graph}(f) \bullet \Delta.$$

Prove that

b) $L(f) \in \mathbb{Z}$ makes sense even if M is not assumed to be orientable.

c) Homotopic maps have the same Lefschetz number.

d) $L(\mathrm{id}_M) = \chi(M)$.

e) Any map $f : M \to M$ with $L(f) \neq 0$ must have a fixed point.

What is the Lefschetz number of a constant map?

Please turn!

3. The intersection number readily generalizes to the case where $S_2 \subset M$ is a closed cooriented submanifold and $f : S_1 \to M$ is any smooth map of a closed oriented manifold S_1 into M, where $\dim S_1 + \dim S_2 = \dim M$. We denote the resulting integer by $f \bullet S_2$.

a) Prove that if f and g are smoothly homotopic maps, then $f \bullet S_2 = g \bullet S_2$.

b) Assume that $Z \subseteq N$ is a cooriented closed submanifold and S is a closed oriented manifold such that $\dim S + \dim Z = \dim N$. Prove that if $f : S \to M$ and $g : M \to N$ are smooth maps, then

$$(g \circ f) \bullet Z = f \bullet (g^{-1}Z),$$

where the first intersection number is taken in N and the second one in M.

c) Give an explicit example where the resulting integer is nonzero, and $\dim M > \dim N$.

4. The goal of this exercise is to show that every diffeomorphism $h : \mathbb{C}P^2 \to \mathbb{C}P^2$ preserves the orientation.

a) Use the fact that $\mathbb{C}P^2 \cong S^5/S^1$, where we think of $S^5 \subseteq \mathbb{R}^6 \cong \mathbb{C}^3$ and $S^1 \subseteq \mathbb{C}$ acts on it by multiplication in each of the coordinates and the long exact sequence in homotopy for this fibration to deduce that $\pi_2(\mathbb{C}P^2) \cong \mathbb{Z}$, with a generator given by the natural inclusion $\iota : S^2 \cong \mathbb{C}P^1 \subseteq \mathbb{C}P^2$.[1]

b) Argue that for any diffeomorphism $h : \mathbb{C}P^2 \to \mathbb{C}P^2$ we must have $[h\iota] = \pm[\iota] \in \pi_2(\mathbb{C}P^2)$, and deduce that

$$(h(\mathbb{C}P^1)) \bullet (h(\mathbb{C}P^1)) = \mathbb{C}P^1 \bullet \mathbb{C}P^1 = 1.$$

c) Prove that, in general, for embeddings $f : S^2 \to \mathbb{C}P^2$ and $g : S^2 \to \mathbb{C}P^2$ we have

$$(h \circ f(S^2)) \bullet (h \circ g(S^2)) = (\deg h) \cdot (f(S^2)) \bullet (g(S^2)).$$

d) Deduce that $\deg h = 1$.

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[1] If you are unfamiliar with the long exact sequence of a fibration, you may skip this step and assume the result in the sequel.