Universität Hamburg
Janko Latschev

Winter 2023/24

Geometric Topology

Problem Set 3

1. **a)** Let $f : M \to N$ and $g : N \to P$ be smooth maps between closed connected oriented manifolds of the same dimension. Prove that

$$\deg(g \circ f) = \deg(f) \cdot \deg(g).$$

b) Prove that a smooth homotopy equivalence between closed connected oriented manifolds must have degree ± 1.

2. **a)** Interpret $S^2 = \mathbb{C} \cup \{\infty\}$ and consider a rational map $f : S^2 \to S^2$ given by $f(z) = \frac{p(z)}{q(z)}$, where p and q are polynomials of degrees n and m without common roots. What is $\deg(f)$?

b) Now consider the maps $f_k : S^2 \to S^2$ which are defined by sending $\exp_N(v)$ to $\exp_N(kv)$, where $N \in S^2$ is the north pole and $\exp_N : T_N S^2 \to S^2$ is the exponential map. What is $\deg(f_k)$?

c) Does the answer change if one replaces S^2 by S^3?

3. Prove the following statements:

a) Every map $S^n \to S^n$ with degree different from $(-1)^{n+1}$ has a fixed point.

b) Every map $S^n \to S^n$ of odd degree maps some pair of antipodal points onto a pair of antipodal points.

4. Let $f : M \to N$ be a map between closed connected manifolds of the same dimension n.

a) Prove that every regular value $q \in N$ has a neighborhood $V \subseteq N$ which is evenly covered, i.e. the preimage

$$f^{-1}(V) = U_1 \sqcup U_2 \cdots \sqcup U_r$$

is a disjoint union of open sets $U_j \subseteq M$ such that $f|_{U_j} : U_j \to V$ is a diffeomorphisms.

b) Now suppose M and N are oriented and $\eta \in \Omega^n(N)$ is a form with support in V. Prove that

$$\int_M f^*\eta = \deg(f) \cdot \int_N \eta.$$

Remark: One can prove that for closed connected manifolds P, integration induces an isomorphism $H_{\mathrm{dR}}^{\dim P}(P) \cong \mathbb{R}$. So this exercise relates our definition of degree to one you might have seen in algebraic topology.