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# Symplectic Geometry

## Problem Set 8

**1.** Consider a Kähler manifold $(M, \omega, J)$ and suppose that $\varphi : M \to M$ is an isometric involution ($\varphi^2 = \mathrm{id}$) of the corresponding Kähler metric $g_J = \omega(\,.\,, J\,.\,)$ which is antiholomorphic, i.e. such that $\varphi_* \circ J = -J \circ \varphi_*$.

**a)** Prove that $\varphi$ is antisymplectic, i.e. $\varphi^*\omega = -\omega$.

**b)** Prove that the fixed point set of $\varphi$ is a totally geodesic submanifold for the metric $g_J$.

**c)** Prove that the fixed point set is a Lagrangian submanifold of $(M, \omega)$.

**d)** What is the fixed point set of $\varphi : \mathbb{C}P^n \to \mathbb{C}P^n$, given in homogeneous coordinates as complex conjugation

$$\varphi([z_0 : \ldots : z_n]) = [\bar{z}_0 : \ldots : \bar{z}_n]?$$

*Remark: Note that if $X \subset \mathbb{C}P^n$ is a smooth complex submanifold given as the zero set of finitely many homogeneous polynomials with real coefficients, then $\varphi$ also induces an antiholomorphic and antisymplectic involution on $X$. This gives many interesting examples.*

**2.** Prove that $(\mathbb{R}^4, \omega_{\mathrm{st}})$ can be embedded symplectically into its subset

$$Z_L(1) := \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4 \,:\, x_1^2 + x_2^2 < 1\} \subseteq \mathbb{R}^4.$$

*This shows that in the statement of Gromov's Nonsqueezing theorem it is important to use the* symplectic *cylinder, defined as the product of a* symplectic *disk with $\mathbb{R}^{2n-2}$.*

**Bitte wenden!**

**3.** Let $(M, \omega)$ be a closed symplectic manifold. In class, we introduced the Hofer metric on $\mathrm{Ham}(M, \omega)$ by defining the length of a path $\varrho_t$ in $\mathrm{Ham}(M, \omega)$ starting at the identity and generated by the time-dependent Hamiltonian $H : M \times [0, 1] \to \mathbb{R}$ as

$$\mathrm{length}(\varrho_t) = \int_0^1 \|H_t\|_\infty \, \mathrm{d}t, \quad \text{where } \|H_t\|_\infty = \max_M H_t - \min_M H_t,$$

and then setting

$$d(\varphi, \psi) := \inf\{\mathrm{length}(\varrho_t) \ : \ \varrho : [0, 1] \to \mathrm{Ham}(M, \omega), \varrho_0 = \mathrm{id}_M, \varrho_1 = \varphi^{-1}\psi\}.$$

**a)** Prove the assertions made in class that this metric is biinvariant, i.e.

$$d(\varphi, \psi) = d(\theta\varphi, \theta\psi) = d(\varphi\theta, \psi\theta)$$

for any $\theta \in \mathrm{Ham}(M, \omega)$.

**b)** Consider $M = S^2 \subseteq \mathbb{R}^3$ with its standard volume form $\sigma$ and compute the length of the path $\varphi : [0, 1] \to \mathrm{Ham}(S^2, \sigma)$ where $\varphi_t$ is rotation by $2\pi t$ around the $z$-axis.