Universität Hamburg
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Differential Topology

Problem Set 13

1. Let M be an oriented manifold of dimension n, and let P and Q be two closed oriented submanifolds which intersect transversely.

a) Prove that if $\dim P + \dim Q = \dim M$, then

$$Q \cdot P = \int_P \tau_Q = \int_M \tau_Q \wedge \tau_P,$$

where $\tau_Q \in \Omega_c^{n-\dim Q}(M)$ and $\tau_P \in \Omega_c^{n-\dim P}(M)$ are closed forms representing the Poincaré dual cohomology class of Q and P, respectively.

b) What can you say when $\dim P + \dim Q \neq \dim M$?

2. Let $f : M \to \mathbb{R}$ be a C^2-function and let $p \in M$ be a critical point of f.

a) Prove that the Hessian of f at p has the following alternative description:
Given tangent vectors $v, w \in T_pM$, choose (arbitrary) vector fields X and Y defined locally near p with $X_p = v$ and $Y_p = w$. Then

$$\mathrm{Hess}_p f(v, w) = X(Y(f))(p).$$

b) Deduce that $\mathrm{Hess}_p f$ is symmetric.

c) Prove that f is Morse if and only if df is transverse to the zero section in T^*M.

d) Prove that the index $\mathrm{ind}_p\, df$ of df at p as a section of T^*M and the index $\mathrm{ind}\, p$ of p as a critical point of f are related by

$$\mathrm{ind}_p\, df = (-1)^{\mathrm{ind}\, p}.$$

3. Prove that on a smooth manifold M Morse functions are open and dense in $C^\infty(M)$ with the strong topology.

4. Find a Morse function on $\mathbb{C}P^n$ with exactly $n + 1$ critical points. What are their indices?

5. Prove that if M is a closed manifold of dimension n and there is a Morse function $f : M \to \mathbb{R}$ with exactly two critical points, then M must be *homeomorphic* to S^n.
Remark: In general, the M in question will not be diffeomorphic *to the sphere. When Milnor proved the existence of exotic smooth structures on the 7-sphere, he used this result to prove that his examples were indeed homeomorphic to S^7.*