Universität Hamburg
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Differential Topology

Problem Set 12

1. Prove that if V is a finite-dimensional real vector space that admits a nondegenerate skew-symmetric bilinear form, then V must have even dimension.

2. **a)** Prove that if $f : M \to N$ is a map of nonzero degree between closed oriented manifolds of equal dimension, then the induced map

$$f^* : H^*(N) \to H^*(M)$$

on de Rham cohomology is injective.

b) Prove that if Σ_g and Σ_h are closed oriented surfaces with genus g and h respectively, then there exists a smooth map $f : \Sigma_g \to \Sigma_h$ of nonzero degree if and only if $g \geq h$.

3. Let $\pi : E \to M$ be an oriented vector bundle of rank k over a closed oriented manifold of dimension n. We define the (real) *Euler class* of E as

$$e(E) := \iota^*([\tau]) \in H^k(M),$$

where $\iota : M \to E$ is the embedding as the zero section and $[\tau] \in H^k_c(E)$ is the Thom class. Prove the following assertions:

a) If E admits a nonvanishing section, then $e(E) = 0$.

b) If E is the pull-back of some other bundle $F \to N$ by a map $f : M \to N$, then $e(E) = f^*(e(F))$.

c) If $E = E_1 \oplus E_2$, then $e(E) = e(E_1) \cup e(E_2)$, where $\cup : H^*(M) \times H^*(M) \to H^*(M)$ is the cup product, i.e. the multiplication induced from the wedge product of forms.

d) If $\eta \in \Omega^n(M)$ is any representative of $e(TM)$, then

$$\int_M \eta = \chi(M),$$

the Euler characteristic of M.