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Differential Topology

Problem Set 10

1. A diffeomorphism $f : U \to V$ of open subsets $U, V \subseteq \mathbb{R}^n$ gives rise to a diffeomorphism

$$\Phi : T^*V \to T^*U\,, \quad \Phi(x, \alpha) = (f^{-1}(x), f^*\alpha).$$

Suppose $(x_1, \dots, x_n)$ are coordinates on $\mathbb{R}^n$ and $(y_1, \dots, y_n)$ are dual coordinates on the fibers of $T^*\mathbb{R}^n$, i.e. those obtained from writing a linear form in the basis $dx_1, \dots, dx_n$.

a) Prove that the form $\alpha \in \Omega^1(T^*\mathbb{R}^n)$, given as

$$\alpha := \sum_{i=1}^{n} y_i dx_i, \tag{1}$$

has the property that

$$\Phi^*(\alpha|_{T^*U}) = \alpha|_{T^*V}.$$

b) Deduce that on the cotangent bundle of any smooth manifold M there exists a canonically defined 1-form $\alpha \in \Omega^1(T^*M)$, which in every local coordinate chart takes the form (1).

c) Prove that the form $\omega := d\alpha$ has the property that $\omega \wedge \dots \wedge \omega$ ($\dim M$ factors) is a volume form on T^*M.

2. The aim of this exercise is to prove that integration provides an isomorphism from the top-dimensional de Rham cohomology group $H^n_{\mathrm{dR}}(M)$ of a closed, connected and oriented n-dimensional manifold to $\mathbb{R}$.

a) Prove by induction on n that if $f : \mathbb{R}^n \to \mathbb{R}$ is a function with compact support and $\int_{\mathbb{R}^n} f(x)\, dx_1 \dots dx_n = 0$, then there exist functions $u_i : \mathbb{R}^n \to \mathbb{R}$, $i \in \{1, \dots, n\}$ with compact support such that $f = \sum_i \frac{\partial u_i}{\partial x_i}$.
Hint: The case $n = 1$ is an easy consequence of the fundamental theorem of calculus. For the induction step consider the auxiliary function

$$g(x_2, \dots, x_n) := \int_{\mathbb{R}} f(x_1, x_2, \dots, x_n)\, dx_1,$$

and observe that by Fubini's theorem one can apply the induction hypothesis to obtain $u_2 \dots, u_n$. To get the remaining function u_1, adjust

$$w_1(x_1, \dots, x_n) := \int_{-\infty}^{x_1} f(t, x_2, \dots, x_n)\, dt$$

by subtracting a suitably cut off version of g.

b) Deduce from this that every compactly supported form $\omega \in \Omega^n(\mathbb{R}^n)$ with vanishing integral is the differential of a compactly supported form $\eta \in \Omega^{n-1}(\mathbb{R}^n)$.

Bitte wenden!

c) Now prove that for a manifold M satisfying the above assumptions, there are finitely many open sets $U_0, U_1, \dots, U_r$ diffeomorphic to balls and covering M and diffeomorphisms $\varphi_i : M \to M$ isotopic to the identity with $\varphi_i(U_0) = U_i$.

d) Prove that for any closed form $\alpha \in \Omega^n(M)$ with compact support in some U_i, $\varphi_i^* \alpha$ and α are cohomologous.
Hint: Consider an isotopy $\Phi_t : M \to M$, $t \in [0,1]$ with $\Phi_0 = id_M$ and $\Phi_1 = \varphi_i$. Now argue that for $t, t' \in [0,1]$ sufficiently close, $\Phi_t^* \alpha$ and $\Phi_{t'}^* \alpha$ will both have support in $\Phi_t^{-1}(U_i)$ and have the same integral, and so by part **b)** *they must be cohomologous. Finish with a standard open-and-closed argument.*

e) Now complete the proof of the original claim by using a partition of unity subordinate to the cover $\{U_i\}_{i=0,\dots,r}$ of M from part **c)** to break up a given form $\omega \in \Omega^n(M)$ whose integral over M vanishes into components ω_i with support in U_i and applying the result of part **b)** to the form

$$\tilde{\omega} = \sum_{i=0}^{n} \varphi_i^* \omega_i$$

with support in U_0, which by part **d)** is cohomologous to ω.

3. Let M and N be two closed oriented manifolds of dimension n. By the previous exercise, $H^n_{\mathrm{dR}}(M) \cong \mathbb{R} \cong H^n_{\mathrm{dR}}(N)$. So, given any smooth map $f : M \to N$, the induced map

$$f^* : H^n_{\mathrm{dR}}(N) \longrightarrow H^n_{\mathrm{dR}}(M)$$

can be viewed as a linear map $\mathbb{R} \to \mathbb{R}$. Such a map is necessarily multiplication by some constant $d \in \mathbb{R}$. Prove that in fact $d = \deg f$, the degree of the map f.
Hint: Find an open disk $V \subset N$ consisting of regular values such that $f : f^{-1}(V) \to V$ is a covering map, and use a form with support in V to compute d.

4. Prove that there is no map of nonzero degree from S^2 to T^2.
Hint: One strategy uses the multiplication on de Rham cohomology induced from the wedge product.