0.1 Embeddings are open

In class I only sketched the proof of the following result:

Theorem 0.1.1. *For sufficiently differentiable manifolds M and N the subset*

$$\mathrm{Emb}^r(M,N) := \{f \in C^r(M,N) \mid f \text{ is an embedding } \} \subseteq C^r(M,N)$$

is open in the strong C^r topology for every $r \geq 1$.

Here, I will give more details on the proof. We will use the following auxillary result, which was also discussed in class:

Lemma 0.1.2. *Let $U \subseteq \mathbb{R}^n$ and $W \subseteq U$ be open such that $\overline{W} \subseteq U$ is compact, and let $f : U \to \mathbb{R}^m$ be a C^1 embedding. Then there exists $\varepsilon > 0$ such that if $g : U \to \mathbb{R}^m$ is a C^1 map with*

$$\sup_{x\in\overline{W}} \|g(x) - f(x)\| < \varepsilon \quad \textit{and} \quad \sup_{x\in\overline{W}} \|Dg_x - Df_x\| < \varepsilon,$$

then the restriction $g|_W : W \to \mathbb{R}^m$ is an embedding. □

Proof of the Theorem. Since $f : M \to N$ is an embedding, we can find a locally finite covering of $f(M) \subseteq N$ by embedding charts[1] $\{(V_\alpha, \psi_\alpha)\}_{\alpha\in A}$ whose images are open balls $B(0, r_\alpha) \subseteq \mathbb{R}^m$ of finite radius and such that the preimages $\widetilde{V}_\alpha := \psi_\alpha^{-1}(B(0, \frac{1}{2}r_\alpha))$ of the balls of half the radius still cover all of $f(M)$. In particular, the sets $W_\alpha := f^{-1}(\widetilde{V}_\alpha) \subset M$ will cover all of M and have compact closure in M. Since the embedding charts (V_α, ψ_α) give rise to charts (U_α, ϕ_α) on M with $U_\alpha = f^{-1}(V_\alpha)$ and $\phi_\alpha := pr_{\mathbb{R}^n} \circ \psi_\alpha \circ f$, we can apply the lemma in local coordinates and find positive real constants $\varepsilon_\alpha > 0$ such that every C^r map $g : M \to N$ in the neighborhood $\mathfrak{N}_1$ of f defined by these choices has the property that $g|_{W_\alpha}$ is an embedding from W_α into N for each $\alpha \in A$. Note that every map in $\mathfrak{N}_1$ will automatically be an immersion (since this is a local condition satisfied by embeddings).

Lemma 0.1.3. *In this situation, there is a countable locally finite covering $\{B_i\}_{i\geq 1}$ of M by open sets with compact closure $K_i = \overline{B}_i$ with the following properties:*

- *each K_i is the inverse image of some closed ball in N under f.*
- *for each $i \geq 1$ there exists an index $\alpha \in A$ such that the union of all B_j with $K_i \cap B_j \neq \varnothing$ is contained in W_α.*

Proof of Lemma. Fix a distance function d on M.
Since the original covering $\{V_\alpha\}_{\alpha\in A}$ of $f(M)$ is locally finite, so is the covering $\{W_\alpha\}_{\alpha\in A}$ of M. Since for each $\alpha \in A$ the set $\overline{W}_\alpha$ is compact, the cover of this

[1] This means that ψ_α maps $V_\alpha \cap f(M)$ onto $f(V_\alpha) \cap \mathbb{R}^n \times \{0\} \subseteq \mathbb{R}^m$.

set by the W_β has a Lebesgue number $\eta_\alpha > 0$ with respect to our chosen distance function. Now we can cover M by preimages of coordinate balls in the charts (V_α, ψ_α) under $\psi_\alpha \circ f$ such that all balls intersecting W_α have diameter less than $\varrho_\alpha := \frac{1}{3}\min(\eta_\beta \mid W_\alpha \cap W_\beta \neq \varnothing)$. After taking a locally finite refinement if necessary, we obtain a covering as described in the lemma.

Indeed, if some ball B_i intersects W_α, then by construction the union

$$Q_i := \bigcup_{K_i \cap B_j \neq \varnothing} B_j$$

has diameter less than η_α, and so it must be contained in one of the W_β intersecting $\overline{W}_\alpha$. $\square$

We return to our proof of the Theorem. For each $i \geq 1$ the clsoure $K_i = \overline{B}_i$ is compact in M, and $M \setminus Q_i$ is a closed subset of M. Choose a distance d' on N. Then the distance $d_i := d'(f(K_i), f(M \setminus Q_i))$ is positive for each $i \geq 1$, because $K_i \subseteq Q_i$ by construction.

Now define $\varepsilon_1 := \frac{1}{3}d_1$, and then for $r > 1$ inductively define

$$\varepsilon_r := \min(\varepsilon_1, \ldots, \varepsilon_{r-1}, \frac{1}{3}d_r).$$

For each $i \geq 1$ we also choose an index $\alpha(i)$ satisfying the conclusion of the last lemma, so in particular $K_i \subset W_{\alpha(i)}$ and $f(K_i) \subset V_{\alpha(i)}$. Now let $\mathfrak{N}_2$ be the neighborhood of f determined by the data $\{(W_{\alpha(i)}, V_{\alpha(i)}, K_i, \varepsilon_i)\}_{i \geq 1}$. We claim that every map in the intersection $\mathfrak{N}_1 \cap \mathfrak{N}_2$ is a global embedding of M into N.

First, we prove injectivity of any $g \in \mathfrak{N}_1 \cap \mathfrak{N}_2$. For that, let $x \neq y \in M$ be given such that $x \in B_i$ and $y \in B_j$ for some $i \leq j$. There are two possibilities:

- $y \in Q_i$, and so $g(x) \neq g(y)$ since $Q_i \subset W_{\alpha(i)}$ and the restriction of g to $W_{\alpha(i)}$ is an embedding, or
- $y \notin Q_i$, and so $g(x) \neq g(y)$ since then $d(f(x), f(y)) \geq d_i$ and we arranged that $\varepsilon_j \leq \varepsilon_i \leq \frac{1}{3}d_i$. So by the reverse triangle inequality

$$d'(g(x), g(y)) \geq d'(f(x), f(y)) - d'(f(x), g(x)) - d'(f(y), g(y)) \geq \frac{1}{3}d_i > 0.$$

A similar argument proves that any $g \in \mathfrak{N}_1 \cap \mathfrak{N}_2$ is a homeomorphism onto its image. Indeed, suppose $x_n \in M$ is a sequence such that the points $y_n = g(x_n)$ converge to $y = g(x)$ in N. Assume $x \in K_i$ and let $X_i \subseteq N$ be the open neighborhood $V_{\alpha(i)} \cap \{z \in N \mid d(z, f(x)) < \varepsilon_i\}$. Note that $g(x) \in X_i$ by construction of $\mathfrak{N}_2$, and so all but finitely many of the y_n must also lie in X_i. On the other hand all points $w \in M \setminus Q_i$ satisfy $g(w) \notin X_i$. Therefore all but finitely many of the x_n must lie in Q_i, and since g is an embedding on this set, we must in fact have convergence $x_n \to x$. $\square$