Universität Hamburg
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Symplectic Homology

Problem Set 2

1. Let (M, ω) be a closed symplectic manifold and let $\varphi : M \to M$ be a Hamiltonian diffeomorphism. Prove that there exists a smooth, time-periodic family of Hamiltonians $H : \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$ such that φ is the corresponding time-one-map.

2. Let $H : \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$ be a time-periodic Hamiltonian and let X_t be the corresponding family of Hamiltonian vector fields.

a) Prove that the one-form

$$\Psi_H : T\mathcal{L}M \to \mathbb{R}$$
$$\xi \in T_x\mathcal{L}M \mapsto (\Psi_H)_x(\xi) := \int_0^1 \omega(X_t(x(t)) - \dot{x}(t), \xi(t))\, dt$$

is closed.

b) Prove that if $\omega|_{\pi_2(M)} = 0$, then the action functional $\mathcal{A}_H : \mathcal{L}^0M \to \mathbb{R}$ on the component $\mathcal{L}^0M$ of contractible loops, defined as

$$\mathcal{A}_H(x) := \int_{D^2} u^*\omega - \int_0^1 H(t, x(t))\, dt,$$

where $u : D^2 \to M$ is any smooth map satisfying $u(e^{2\pi i t}) = x(t)$ is a primitive of Ψ_H, i.e. we have

$$d\mathcal{A}_H = \Psi_H.$$

3. Prove that if all fixpoints of a Hamiltonian diffeomorphism $\varphi : (M, \omega) \to (M, \omega)$ of a closed symplectic manifold are nondegenerate, then there are only finitely many of them.

4. Prove the assertion in the rescaling argument during the proof of Proposition 1 in Wednesday's lecture that the uniform energy bound on the sequence gives a uniform energy bound on the limiting map $v : \mathbb{C} \to M$.

5. Find holomorphic maps $v : \mathbb{C} \to \mathbb{C}$ for which the inequality

$$(\ell(r))^2 \leq 2\pi r A'(r)$$

between the square of the length of $v(\partial B(0,r))$ and the radial derivative of the area of $v(B(0,r))$ is sharp.