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Symplectic Geometry

Problem Set 8

1. For a function $a : \mathbb{R}^4 \to \mathbb{R}$, we consider the almost complex structure J_a on the manifold $M = \mathbb{R}^4$ which in the global coordinates (x_1, x_2, y_1, y_2) has the form

$$J_a(p) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ a(p) & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -a(p) & 0 \end{pmatrix}, \text{ i.e. } J_a\left(\frac{\partial}{\partial x_1}\right) = a(p)\frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_1} \quad \text{etc.}$$

a) Prove that if $|a(p)| \leq 1$ for all $p \in \mathbb{R}^4$, then J_a is tamed by the standard symplectic form $\omega_{\mathrm{st}} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$!
Hint: Recall that the taming condition means that $\omega(v, Jv) > 0$ for all non-zero v, but ω need not be J-invariant, so that the bilinear form $\omega(\,.\,, J\,.\,)$ need not be symmetric.

b) Under which conditions on the function a is the almost complex structure J_a on $\mathbb{R}^4$ integrable?
Hint: Argue that in order to determine N_{J_a} on any two vectors $v, w \in T_p\mathbb{R}^4$, it suffices to know $N_{J_a}\left(\frac{\partial}{\partial x_1}(p), \frac{\partial}{\partial x_2}(p)\right)$, and then compute this.

2. Consider an almost complex structure J on an open set $U \subset \mathbb{R}^{2n}$. Prove:

a) If $f : U \to \mathbb{C}$ is J-holomorphic, meaning that $df \circ J = i \circ df$, then at each point $p \in U$ the rank of the differential df_p is either 0 or 2.

b) The inverse image of a regular value $z \in \mathbb{C}$ is a J-complex submanifold (of codimension 2), i.e. its tangent bundle is invariant under J.

c) The image of the Nijenhuis tensor N_J is contained in $\ker df$.

d) Consider the case $n = 2$, i.e. a subset $U \subset \mathbb{R}^4$ and find an almost complex structure on a suitable U for which there do not exists nonconstant J-holomorphic functions $f : U \to \mathbb{C}$.

Bitte wenden!

3. Consider a Kähler manifold (M, ω, J) and suppose that $\varphi : M \to M$ is an isometric involution ($\varphi^2 = \mathrm{id}$) of the corresponding Kähler metric $g_J = \omega(\,.\,, J\,.\,)$ which is antiholomorphic, i.e. such that $\varphi_* \circ J = -J \circ \varphi_*$.

a) Prove that φ is antisymplectic, i.e. $\varphi^*\omega = -\omega$.

b) Prove that the fixed point set of φ is a totally geodesic submanifold for the metric g_J.

c) Prove that the fixed point set is a Lagrangian submanifold of (M, ω).

d) What is the fixed point set of $\varphi : \mathbb{C}P^n \to \mathbb{C}P^n$, given in homogeneous coordinates as complex conjugation

$$\varphi([z_0 : \ldots : z_n]) = [\bar{z}_0 : \ldots : \bar{z}_n]?$$

Remark: Note that if $X \subset \mathbb{C}P^n$ is a smooth complex submanifold given as the zero set of finitely many homogeneous polynomials with real coefficients, then φ also induces an antiholomorphic and antisymplectic involution on X. This gives many interesting examples.