Universität Hamburg
Janko Latschev
Fabian Kirchner

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Symplectic Geometry

Problem Set 5

1. *(Poisson structure of a symplectic manifold)*
Let (M, ω) be a symplectic manifold. Given two functions $F, G \in C^\infty(M)$, define their *Poisson bracket* to be the new function

$$\{F, G\} := -\omega(X_F, X_G).$$

a) Prove that $X_{\{F,G\}} = [X_F, X_G]$.

b) Prove that $\{\,.\,,\,.\,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ is a Lie bracket, i.e. it satisfies

$$\{G, F\} = -\{F, G\}$$
$$\{F, \{G, H\}\} = \{\{F, G\}, H\} + \{G, \{F, H\}\}.$$

c) Prove that $\{FG, H\} = F\{G, H\} + G\{F, H\}$.

d) Prove that the time evolution of a function F along the Hamiltonian flow φ_t^H of an autonomous Hamiltonian function H satisfies the equation

$$\frac{d}{dt}(F \circ \varphi_t^H) = \{F, H\} \circ \varphi_t^H.$$

e) What is the local expression for $\{F, G\}$ in Darboux coordinates?

2. *(Lagrangian surgery)*

a) Show that if L_1 and L_2 are two Lagrangian submanifolds passing through $p = (x, y) \in (\mathbb{R}^{2n}, \omega_{\mathrm{st}})$ such that $T_pL_0 \cap T_pL_1 = \{0\}$, there exists a symplectomorphism $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that for a sufficiently small $\epsilon > 0$ one has $\varphi(L_0 \cup L_1) \cap B(p, \epsilon) = (\mathbb{R}^n \times \{y\} \cup \{x\} \times \mathbb{R}^n) \cap B(p, \epsilon)$.

b) Construct a Lagrangian submanifold $L \subset (\mathbb{R}^{2n}, \omega_{\mathrm{st}})$ diffeomorphic to $\mathbb{R} \times S^{n-1}$, such that

$$L \cap (\mathbb{R}^{2n} \setminus B^{2n}(0, 1)) = (\mathbb{R}^n \times \{0\} \cup \{0\} \times \mathbb{R}^n) \cap (\mathbb{R}^{2n} \setminus B^{2n}(0, 1)).$$

Together with Darboux' theorem, **a)** *and* **b)** *show that one can form the connected sum of two Lagrangian submanifolds which intersect transversely at one point such that the result is a new Lagrangian submanifold.*

c)* Formulate and prove a similar result for Lagrangian submanifolds that intersect cleanly, i.e. such that $L_1 \cap L_2$ is a submanifold and for each point $x \in L_1 \cap L_2$ one has $T_xL_1 \cap T_xL_2 = T_x(L_1 \cap L_2)$.