Seminar on the h-Cobordism Theorem

Definitions and Key Facts - Homology Theory

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November 11 and November 18, 2014

1. Homology

Let $(C_i)_{i\in\mathbb{Z}}$ be abelian groups and let $\partial_i\colon C_i \to C_{i-1}$ be homomorphisms such that $\partial_i \circ \partial_{i+1} = 0$ for all $i \in \mathbb{Z}$. The sequence

$$\ldots \longrightarrow C_{n+1} \stackrel{\partial_{n+1}}{\longrightarrow} C_n \stackrel{\partial_n}{\longrightarrow} C_{n-1} \stackrel{\partial_{n-1}}{\longrightarrow} C_{n-2} \longrightarrow \ldots$$

is called **chain complex** and the maps $\partial_n\colon C_n \to C_{n-1}$ are called **boundary maps**.

Elements in the kernel $\ker \partial_n$ of a boundary map ∂_n are called n-**cycles** and elements in the image of ∂_n are called **boundaries**.

The $n^{\mathbf{th}}$-**homology group** with respect to the given chain complex is defined as the quotient

$$H_n := \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

A collection of homomorphisms $(f_i : A_i \longrightarrow B_i)$ between chain complexes A and B which commute with the boundary maps, i.e. $\partial_i^A f_{i-1} = f_i \partial_i^B$, is called **chain map**.

2. Singular Homology

Let X be a topological space.

a. Definitions

Let $v_0, \ldots, v_n$ be pairwise distinct points in $\mathbb{R}^n$ such that $v_1 - v_0, \ldots, v_n - v_0$ are linearly independent. The convex hull of $v_0, \ldots, v_n$ is denoted by $[v_0, v_1, \ldots, v_n]$ and is called n-**simplex**. If $e_1, \ldots, e_{n+1}$ is the standard basis of $\mathbb{R}^{n+1}$, the hull $[e_1, \ldots, e_{n+1}]$ is called **standard** n-**simplex**. It is denoted by Δ^n. Subsets of the form $[v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n]$ are identified with Δ^{n-1}. If $n < 0$, then $\Delta^n = \emptyset$.

A continuous map $\sigma\colon \Delta^n \to X$ is called **singular-n-simplex**. A finite formal sum $\sum_\alpha n_\alpha \sigma_\alpha$ of singular-n-simplices σ_α is called **singular-n-chain**.

By $\Delta_n(X)$ one denotes the free abelian group consisting of all singular-n-chains. For $n \geq 1$ a group homomorphism

$$\partial_n\colon \Delta_n(x) \to \Delta_{n-1}(X)$$

is defined by

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^n \sigma_{|[v_0,\ldots,v_{i-1},v_{i+1},\ldots,v_n]}.$$

For $n \leq 0$ let $\partial_n = 0$. The sequence

$$\ldots \longrightarrow \Delta_2(X) \xrightarrow{\partial_2} \Delta_1(X) \xrightarrow{\partial_1} \Delta_0(X) \xrightarrow{0} 0 \longrightarrow 0 \longrightarrow \ldots$$

is called **(singular) chain complex** and the quotient

$$H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$$

is called the $n^{\mathbf{th}}$**-(singular) homology group of** X.

b. Key Facts

Proposition. If $(X_\beta)_\beta$ are the path-components of X, then for all $n \in \mathbb{Z}$

$$H_n(X) \cong \bigoplus_\beta H_n(X_\beta).$$

Proposition. If X is non-empty and path-connected, then $H_0(X) \cong \mathbb{Z}$.

Proposition. Let X be path-connected and let $\pi_1(X)$ denote the fundamental group of X. Moreover let $[\pi_1(X), \pi_1(X)]$ be the commutator subgroup of $\pi_1(X)$. Then

$$H_1(X) \cong \pi_1(X) / \left[\pi_1(X), \pi_1(X)\right].$$

Theorem. Let $n \in \mathbb{Z}$. Every continuous map $f\colon X \to Y$ induces a homomorphism $f_*\colon H_n(X) \to H_n(Y)$. If $g\colon X \to Y$ is another continuous map homotopic to f, then $f_* = g_*$.

Corollary. A homotopy equivalence $f\colon X \to Y$ induces an isomorphism $f_*\colon H_n(X) \to H_n(Y)$.

3. Homological Algebra

a. Definitions

Let $(G_n)_{n\in\mathbb{Z}}$ be groups and $f_n\colon G_n \to G_{n-1}$ be homomorphisms for all $n \in \mathbb{Z}$. The sequence

$$\ldots \longrightarrow G_{n+1} \xrightarrow{f_{n+1}} G_n \xrightarrow{f_n} G_{n-1} \xrightarrow{f_{n-1}} G_{n-2} \longrightarrow \ldots$$

is called **exact** if and only if the identity $\operatorname{im} f_{n+1} = \ker f_n$ holds for all $n \in \mathbb{Z}$.

An exact sequence of the form

$$0 \longrightarrow G_3 \xrightarrow{f_3} G_2 \xrightarrow{f_2} G_1 \longrightarrow 0$$

is called a **short exact sequence**.

Let

$$\begin{array}{c} \ldots \xrightarrow{\partial} A_{n+1} \xrightarrow{\partial} A_n \xrightarrow{\partial} A_{n-1} \xrightarrow{\partial} \ldots \\ \ldots \xrightarrow{\partial} B_{n+1} \xrightarrow{\partial} B_n \xrightarrow{\partial} B_{n-1} \xrightarrow{\partial} \ldots \\ \ldots \xrightarrow{\partial} C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \ldots \end{array}$$

be chain complexes with homology groups denoted by $H_n(A)$, $H_n(B)$ and $H_n(C)$. For $k \in \mathbb{Z}$ let i_k and j_k be homomorphisms such that the sequences

$$0 \longrightarrow A_k \xrightarrow{i_k} B_k \xrightarrow{j_k} C_k \longrightarrow 0$$

are exact for all $k \in \mathbb{Z}$. If the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \xrightarrow{\partial} & A_{n+1} & \xrightarrow{\partial} & A_n & \xrightarrow{\partial} & A_{n-1} & \xrightarrow{\partial} & \dots \\
 & & \downarrow i & & \downarrow i & & \downarrow i & & \\
\dots & \xrightarrow{\partial} & B_{n+1} & \xrightarrow{\partial} & B_n & \xrightarrow{\partial} & B_{n-1} & \xrightarrow{\partial} & \dots \\
 & & \downarrow j & & \downarrow j & & \downarrow j & & \\
\dots & \xrightarrow{\partial} & C_{n+1} & \xrightarrow{\partial} & C_n & \xrightarrow{\partial} & C_{n-1} & \xrightarrow{\partial} & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

commutes, i.e. i and j are chain maps, then the chain complexes together with the chain maps are called **short exact sequence of chain complexes**.

b. Key Fact

Theorem. A short exact sequence of chain complexes induces a **long exact sequence** of homology groups

$$\dots \longrightarrow H_{n+1}(A) \xrightarrow{i_*} H_{n+1}(B) \xrightarrow{j_*} H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow \dots$$

4. Relative and Reduced Homology

a. Definitions

For a subset $A \subset X$ we have $\Delta_n(A) \subset \Delta_n(X)$. The boundary map of X induces a boundary map on the quotient $\Delta_n(X)/\Delta_n(A)$. The associated homology groups are called **relative homology groups** and are denoted by $H_n(X, A)$. Note that by definition $H_n(X, \emptyset) = H_n(X)$.

The chain complex given by the sequence

$$\dots \longrightarrow \Delta_2(X) \xrightarrow{\partial_2} \Delta_1(X) \xrightarrow{\partial_1} \Delta_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0 \longrightarrow \dots$$

is called **augmented (singular) chain complex** and it's homology groups $\tilde{H}_n(X)$ are called **reduced (singular) homology groups**, where ϵ is defined as $\epsilon(\sum_\alpha n_\alpha \sigma_\alpha) = \sum_\alpha n_\alpha$.

Reduced homology can be understood as subtracting one $\mathbb{Z}$-factor in dimension 0. If X is non-empty, then $\tilde{H}_n(X) = H_n(X)$ for $n \neq 0$ and $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$.

For $\emptyset \neq A \subset X$ one defines $\tilde{H}_n(X, A) := H_n(X, A)$.

The pair (X, A) is called **good**, if A is closed in X and there exists a neighborhood of A which deformation retracts to A.

b. Key Fact

Theorem. We have a **long exact sequence of a pair** (X, A) in singular homology.

$$... \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow ...$$

Here $i : A \hookrightarrow X$ is the inclusion and $j : \Delta_n(X) \longrightarrow \Delta_n(X)/\Delta_n(A)$ the quotient map.

Proposition. The long exact sequence of a pair (X, A) in singular homology also holds for reduced homology.

Proposition. If (X, A) is a good pair, the quotient map $q : X \longrightarrow X/A$ induces an isomorphism

$$q_* : H_n(X,A) \xrightarrow{\sim} H_n(X/A, A/A) \cong \tilde{H}_n(X/A)$$

Theorem. For good pairs, there is an exact sequence

$$... \longrightarrow \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{q_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \longrightarrow ...$$

Corollary. The reduced homology of the n-sphere S^n is given by

$$\tilde{H}_k(S^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n \\ 0, & \text{if } k \neq n \end{cases}$$

Corollary. Given topological spaces X_α with base points $x_\alpha \in X_\alpha$, such that $(X_\alpha, \{x_\alpha\})$ are good pairs, the inclusions $i_\alpha : X_\alpha \hookrightarrow \bigvee_\beta X_\beta$ induce an isomorphism

$$\oplus_\alpha (i_\alpha)_* : \bigoplus_\alpha \tilde{H}_n(X_\alpha) \xrightarrow{\sim} \tilde{H}_n\left(\bigvee_\alpha X_\alpha\right)$$

Theorem (Excision). Let $Z \subset A \subset X$ such that $\overline{Z} \subset \text{int}(A)$. Assuming this, the inclusion $(X \backslash Z, A \backslash Z) \hookrightarrow (X, A)$ induces an isomorphism

$$H_n(X \backslash Z, A \backslash Z) \xrightarrow{\sim} H_n(X, A)$$

Theorem (Mayer-Vietoris Sequence). Let $A, B \subset X$ such that $\text{int}(A) \cup \text{int}(B) = X$. Then we have a long exact sequence with homomorphisms induced by the inclusions

$$... \longrightarrow H_n(A \cap B) \xrightarrow{(i_A)_* \oplus (i_B)_*} H_n(A) \oplus H_n(B) \xrightarrow{(j_A)_* - (j_B)_*} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \longrightarrow ...$$

5. Cellular Homology

a. Definitions

In this section X always is a CW-complex. We denote its n-skeleton by X^n.

Consider the boundary map $\partial_{n+1} : H_{n+1}(X^{n+1}, X^n) \longrightarrow H_n(X^n)$ from the long exact sequence of the pair (X^{n+1}, X^n) and the map $j_n : H_n(X^n) \longrightarrow H_n(X^n, X^{n-1})$, induced from the quotient map, in the long exact sequence of the pair (X^n, X^{n-1}). Let $d_{n+1} := j_n \partial_{n+1}$ be the composition. We get a chain complex

$$... \longrightarrow H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \longrightarrow ...$$

called **cellular chain complex**. It's homology is called **cellular homology**.

b. Key Facts

Lemma. The following statements hold true for singular homology

a) $H_k(X^n, X^{n-1})$ is 0 for $k \neq n$ and free abelian with a basis in one-to-one correspondence with the n-cells of X for $k = n$.

b) For $k > n$ we have $H_k(X^n) = 0$.

c) For $k < n$ the inclusion $i : X^n \hookrightarrow X$ induces an isomorphism $i_* : H_k(X^n) \xrightarrow{\sim} H_k(X)$.

Proposition. The cellular chain complex as defined above is indeed a chain complex, i.e. $d_n d_{n+1} = 0$.

Theorem. Cellular homology is isomorphic to singular homology.

Corollary. Cellular homology is independent of the choice of a particular CW-structure for X.

Corollary.

i) $H_n(X) = 0$ if X has no n-cells.

ii) The number of generators of $H_n(X)$ is at most the number of n-cells in X.

iii) If X has no two cells in adjacent dimensions, then $H_n(X) \cong H_n(X^n, X^{n-1})$.

Using the identification of $H_n(X^n, X^{n-1})$ with the free abelian group generated by the n-cells e^n_α, we can compute the boundary map d_n differently.

Proposition. Let $d_{\alpha\beta}$ be the degree of the map $S^{n-1}_\alpha \longrightarrow X^{n-1} \longrightarrow S^{n-1}_\beta$ that is the composition of the attaching map of e^n_α restricted to $\partial e^n_\alpha = S^{n-1}_\alpha$ with the quotient map collapsing $X^{n-1} \backslash \mathrm{int}(e^{n-1}_\beta)$ to a point. Then the boundary map can be computed using the formula $d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta$.

While our presentation mostly stems from Hatchers book, we found Bredon taking a more general approach to the topic, which has some nice results on how to compute degrees of maps $S^n \longrightarrow S^n$.

References

(1) G. Bredon, *Topology and Geometry*, Springer GTM 139, 1993

(2) A. Hatcher, *Algebraic topology*, http://www.math.cornell.edu/~hatcher/AT/ATpage.html