Universität Hamburg
Janko Latschev
Andreas Gerstenberger Winter 2012/13

Symplectic Geometry

Problem Set 5

1. We consider the two Lagrangian submanifolds of $(\mathbb{R}^{2n}, \omega_{\text{st}})$ that were discussed in the lecture.

a) Verify directly that for $n \geq 1$ and any fixed $0 < \epsilon < 1$ the map

$$\begin{aligned} \varphi_0 : S^1 \times S^{n-1} &\to \mathbb{C}^n \cong \mathbb{R}^{2n} \\ (e^{it}, x) &\mapsto (1 + \epsilon e^{it}) \cdot x \end{aligned}$$

is a Lagrangian embedding, and compute the Maslov index of the loop $\gamma_0 : \mathbb{R}/\mathbb{Z} \to L$, given by $\gamma_0(t) = \varphi_0(e^{2\pi i t}, (1, 0, \ldots, 0))$.

b) For $n \geq 2$, consider the Lagrangian submanifold $L \subset \mathbb{R}^{2n}$ given as the image of the immersion

$$\begin{aligned} \varphi_1 : S^1 \times S^{n-1} &\to \mathbb{C}^n \cong \mathbb{R}^{2n} \\ (\lambda, x) &\mapsto \lambda \cdot x, \end{aligned}$$

and compute the Maslov index of the loop $\gamma_1 : \mathbb{R}/\mathbb{Z} \to L$, given by $\gamma_1(t) = \varphi_1(e^{i\pi t}, (\cos(\pi t), \sin(\pi t), 0, \ldots, 0))$.

2. Consider a closed coisotropic submanifold $W \subset (M, \omega)$ of a symplectic manifold, and suppose that $W = H^{-1}(c)$ for a smooth function $H : M \to \mathbb{R}^k$ which has $c \in \mathbb{R}^k$ as a regular value. We denote by $H_1, \ldots, H_k$ the components of H.

a) Prove that the subbundle $\ker \omega|_W \subset TW$ is spanned pointwise by the Hamiltonian vector fields $X_{H_1}, \ldots, X_{H_k}$.

b) Prove that $[X_{H_i}, X_{H_j}]|_W = 0$ for all $1 \leq i, j \leq k$. By Frobenius' Theorem, this implies that $\ker \omega|_W$ is integrable, i.e. tangent to a foliation.

c) Suppose that the Hamiltonian flows $\varphi^{X_{H_i}}$ are 1-periodic, so that together these flows combine to a free action of the torus T^k on W. Prove that the quotient space W/T^k inherits a symplectic structure from M.

Bitte wenden!

d) Prove that for $W = S^{2n-1} \subset (\mathbb{R}^{2n}, \omega_{\mathrm{st}})$, viewed as a level set of the function $H(z) = \frac{1}{2}\|z\|^2$, part **c)** applies with $k = 1$, and the quotient space is $\mathbb{C}P^{n-1}$.

3. **a)** Show that if L_1 and L_2 are two Lagrangian submanifolds passing through $p = (x, y) \in (\mathbb{R}^{2n}, \omega_{\mathrm{st}})$ such that $T_pL_0 \cap T_pL_1 = \{0\}$, there exists a symplectomorphism $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that for a sufficiently small $\epsilon > 0$ one has $\varphi(L_0 \cup L_1) \cap B(p, \epsilon) = (\mathbb{R}^n \times \{y\} \cup \{x\} \times \mathbb{R}^n) \cap B(p, \epsilon)$.

b) Construct a Lagrangian submanifold $L \subset (\mathbb{R}^{2n}, \omega_{\mathrm{st}})$ diffeomorphic to $\mathbb{R} \times S^{n-1}$, such that

$$L \cap (\mathbb{R}^{2n} \setminus B^{2n}(0,1)) = (\mathbb{R}^n \times \{0\} \cup \{0\} \times \mathbb{R}^n) \cap (\mathbb{R}^{2n} \setminus B^{2n}(0,1)).$$