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Symplectic Geometry

Problem Set 3

1. Show that if $\gamma : M \to M$ is any symplectomorphism of (M, ω) and $H : M \to \mathbb{R}$ is smooth, then the Hamiltonian vector fields of the functions H and $H \circ \gamma^{-1}$ are related by

$$X_{H\circ\gamma^{-1}}(\gamma(x)) = \gamma_*(X_H(x))$$

where $\gamma_* : TM \to TM$ is the differential of γ.

2. Consider a Hamiltonian function $H : B^2(0,1) \to \mathbb{R}$ of the form $H = y \cdot \rho(r)$, where $\rho : B^2(0,1) \to [0,1]$ is a smooth function of the radius $r = \sqrt{x^2+y^2}$ which equals 1 for $0 \leq r \leq \frac{1}{2}$ and equals 0 for $\frac{3}{4} \leq r \leq 1$.
Describe the image of the ball $B^2(0, \frac{1}{100})$ under the time-t-map φ_t of the Hamiltonian flow of H for $t = 1$, $t = 10^2$ and $t = 10^5$ qualitatively!

3. *(Hamiltonian diffeomorphisms)*
Let $\varphi_t : (M, \omega) \to (M, \omega)$ be the family of diffeomorphisms determined by the time-dependent Hamiltonian function $H : [0,1] \times M \to \mathbb{R}$ via

$$\dot{\varphi}_t = X_{H_t} \circ \varphi_t.$$

a) For each $t \in (0,1)$, write φ_t as time one map of a family of diffeomorphisms determined by a new Hamiltonian function built from H.

b) Find a time-dependent Hamiltonian function whose time one map is $(\varphi_1)^{-1}$.

c) Now suppose ψ is the time one map of a second family ψ_t determined by $F : [0,1] \times M \to \mathbb{R}$. Find a time-dependent Hamiltonian function with time one map $\psi \circ \varphi$.

In summary, you have shown that Hamiltonian diffeomorphisms form a group.

Bitte wenden!

4. *(Poisson structure of a symplectic manifold)*
Let (M, ω) be a symplectic manifold. Given two functions $F, G \in C^\infty(M)$, define their *Poisson bracket* to be the new function

$$\{F, G\} := -\omega(X_F, X_G).$$

a) Prove that $\{\,.\,,\,.\,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ is a Lie bracket, i.e. it satisfies

$$\begin{aligned} \{G, F\} &= -\{F, G\} \\ \{F, \{G, H\}\} &= \{\{F, G\}, H\} + \{G, \{F, H\}\}. \end{aligned}$$

b) Prove that $\{FG, H\} = F\{G, H\} + G\{F, H\}$.

c) Prove that $X_{\{F,G\}} = [X_F, X_G]$.