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Symplectic Geometry

Problem Set 6

1. We consider the two Lagrangian submanifolds of $(\mathbb{R}^{2n}, \omega_{\text{st}})$ that were discussed in the lecture.

a) Consider the Lagrangian embedding

$$\begin{aligned} \varphi_1 : S^{n-1} \times S^1 &\to \mathbb{C}^n \cong \mathbb{R}^{2n} \\ (\xi, e^{it}) &\mapsto (1 + \epsilon e^{it}) \cdot \xi \end{aligned}$$

and compute the Maslov index of the loop $\gamma_1 : \mathbb{R}/\mathbb{Z} \to \varphi_1(S^{n-1} \times S^1)$, given by $\gamma_1(t) = \varphi_1((1, 0, \ldots, 0), e^{2\pi i t})$.

b) For $n \geq 2$, consider the Lagrangian submanifold $Q \subseteq \mathbb{R}^{2n}$ given as the image of the immersion

$$\begin{aligned} \varphi_2 : S^{n-1} \times S^1 &\to \mathbb{C}^n \cong \mathbb{R}^{2n} \\ (\xi, \lambda) &\mapsto \lambda \cdot \xi, \end{aligned}$$

and compute the Maslov index of the loop $\gamma_2 : \mathbb{R}/\mathbb{Z} \to Q$, given by $\gamma_2(t) = \varphi_1((\cos(\pi t), \sin(\pi t), 0, \ldots, 0), e^{i\pi t})$.

2. Suppose $Q \subseteq (\mathbb{R}^{2n}, \omega_{\text{st}})$ is a Lagrangian submanifold.

a) Prove that if $u : D^2 \times [0,1] \to \mathbb{R}^{2n}$ is a family of maps connecting $u_0 = u(\,.\,, 0)$ to $u_1(\,.\,, 1)$ such that $u(x,t) \in Q$ for all $(x,t) \in S^1 \times [0,1]$, then

$$\int_{D^2} u_0^* \omega_{\text{st}} = \int_{D^2} u_1^* \omega_{\text{st}},$$

so that the symplectic area is indeed well-defined on $\pi_2(\mathbb{R}^{2n}, Q)$.

b) Prove that if $\varphi : (\mathbb{R}^{2n}, \omega_{\text{st}}) \to (\mathbb{R}^{2n}, \omega_{\text{st}})$ is a Hamiltonian diffeomorphism, then for any map $u : (D^2, S^1) \to (R^{2n}, Q)$ we have

$$\int_{D^2} (\varphi \circ u)^* \omega_{\text{st}} = \int_{D^2} u^* \omega_{\text{st}}.$$

So if $\varphi_* : \pi_2(\mathbb{R}^{2n}, Q) \to \pi_2(\mathbb{R}^{2n}, \varphi(Q))$ is the map induced by φ, we have $A = A \circ \varphi_* : \pi_2(\mathbb{R}^{2n}, Q) \to \mathbb{R}$.
Hint: Compute the derivative of $A(\varphi_t \circ u)$ along a Hamiltonian isoptopy φ_t.

Please turn!

3. Prove that a contact manifold of dimension $2n+1$ with n odd (i.e. of dimension $4m-1$ for some $m \in \mathbb{N}$) has a preferred orientation determined by the contact structure.

4. Consider the following three contact forms on $\mathbb{R}^3$:

- $\lambda_1 = dz - ydx$, where (x, y, z) are cartesian coordinates,
- $\lambda_2 = dz + xdy$, where (x, y, z) are cartesian coordinates,
- $\lambda_3 = dz + r^2 d\varphi$, where (r, φ) are polar coordinates in $\mathbb{R}^2$, and z is the third coordinate.

a) Picture these contact structures and their Reeb vector fields (these will be defined on Tuesday).

b) Prove that $(\mathbb{R}^3, \text{Ker } \lambda_i)$ are pairwise contactomorphic, i.e. there are diffeomorphisms $\Phi_{ij} : \mathbb{R}^3 \to \mathbb{R}^3$ and functions $\rho_{ij} : \mathbb{R}^3 \to \mathbb{R}$ such that $\Phi_{ij}^*(\lambda_i) = \rho_{ij}\lambda_j$.

c) Prove that for each $i \in \{1, 2, 3\}$ there is a contactomorphism of $(\mathbb{R}^3, \text{Ker } \lambda_i)$ with a bounded subset $B \subset (\mathbb{R}^3, \text{Ker } \lambda_i)$.

5. Let (M^{2n}, ω) be symplectic and let $W \subset M$ be a smooth hypersurface.

a) Prove that every point $x \in W$ has a neighborhood $U \subset M$ such that $W' = W \cap U$ is a hypersurface of contact type, i.e. there exists a vector field Y defined on a neighborhood $U' \subseteq U$ of W' such that Y is transverse to W' and $L_Y\omega = \omega$.

b) In fact, if U is sufficiently small, the normal bundle of $W' = W \cap U$ is trivial, and one can find such a vector field Y giving the normal bundle either of the two possible orientations.

6. Let (M^{2n}, ω) be symplectic and let $H : M \to \mathbb{R}$ be a function. Suppose $W := H^{-1}(0) \subset M$ is a smooth **closed** oriented hypersurface of contact type, i.e. there is a vector field Y defined near W and transverse to W such that $L_Y\omega = \omega$. As we have seen in class, this means that $\alpha := (\iota(Y)\omega)|_W$ is a contact form on W.

a) Assuming $n > 1$, prove that there is no *closed* 1-form β on W such that $\beta(X_H) > 0$ at all points of W. *Hint: You may want to use Stokes' Theorem.*

b) Use this to prove that, if $n > 1$, any other vector field Z also transverse to W and satisfying $L_Z\omega = \omega$ defines the same normal orientation of W as Y.

c) What happens for $n = 1$?