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Differential Topology

Problem Set 3

Here is a third set of problems related to the material of the course. If you want to get feedback on your solution to a particular exercise or have questions, please contact me by mail.

1. Suppose Σ_1 and Σ_2 are two closed oriented surfaces of genus g_1 and g_2, respectively.

a) Let $i_j : S^0 \to \Sigma_j$ be embeddings with images $S_j \subset \Sigma_j$. Describe the result of gluing Σ_1 and Σ_2 along these two submanifolds. Does the result depend on the embeddings?

b) Now consider embeddings $i_j : S^1 \to \Sigma_j$ with images $S_j \subset \Sigma_j$ instead, and answer the same questions.

2. Let M be a manifold of dimension n, and let $D \subseteq M$ be an embedded closed ball. Removing the interior of D we obtain a manifold M' whose boundary is identified with $S^{n-1} = \partial D$. Let P be the space obtained from M' by identifying antipodal points of $\partial M'$. Prove that

$$P = M \# \mathbb{R}P^n.$$

3. **a)** Prove that surgery on S^3 along the unknot $U \cong S^1 = S^3 \cap (\mathbb{R}^2 \times \{0\}) \subseteq S^3 \subseteq \mathbb{R}^4$ with framing ± 1 results in a manifold diffeomorphic to S^3.

b) Prove more generally that surgery along the same unknot U with a framing of linking number $\pm p$ with U yields a lens space $L(1,p)$.

4. Let $M \subseteq \mathbb{R}^{n+1}$ be a smooth closed submanifold. For each $v \in S^n \subseteq \mathbb{R}^{n+1}$ we define a function

$$f_v : M \to \mathbb{R}, \quad f_v(p) := \langle v, p \rangle,$$

where $\langle\, .\, , \,.\, \rangle$ is the standard Euclidean scalar product on $\mathbb{R}^{n+1}$. Prove that the set of $v \in \mathrm{Syl}^n$ such that f_v is a Morse function is open and dense.

5. Suppose M is a smooth closed n-dimensional manifold which admits a smooth Morse function $f : M \to \mathbb{R}$ with only two critical points. Prove that M is homeomorphic to S^n.
Remark: A remarkable theorem of Milnor says that such manifolds are not always diffeomorphic *to S^n. His original construction gave examples of this phenomenon in dimension 7.*

6. Consider real projective space as the quotient $\mathbb{R}P^n = (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}_+$, where the multiplicative group $\mathbb{R}_+$ acts by scaling. Write $[x_0 : \ldots : x_n]$ for the point in $\mathbb{R}P^{n+1}$ represented by $x = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$. Now consider the function $f : \mathbb{R}P^n \to \mathbb{R}$,

$$f([x_0 : \ldots : x_n]) := \frac{1}{\|x\|^2} \sum_{j=1}^{n} j x_j^2,$$

where $\|x\| = \sqrt{\sum_j x_j^2}$ is the standard norm on $\mathbb{R}^{n+1}$.

a) Prove that f is a Morse function, and determine its critical points (there are $n+1$ of them) as well as their Morse indices.

b) Let $p : S^n \to \mathbb{R}P^n$, $p(x) = [x]$ be the double cover of $\mathbb{R}P^n$ by S^n. Sketch the critical points and the qualitative behavious of the gradient flow with respect to the round metric on S^n of the lift $\tilde{f} = f \circ p : S^n \to \mathbb{R}$ of the function f for small values of n.

7. Let $f : M \to \mathbb{R}$ be a function and let X be the gradient vector field of f with respect to some Riemannian metrix g on M. Clearly zeroes of X correspond to critical points of f.

a) Prove that f is a Morse function if and only if X, viewed as a section of TM, is transverse to the zero section.

b) Prove that at a Morse critical point of f the index of X as a vector field and the Morse index of p as a critical point of f are related by

$$\mathrm{ind}_p(X) = (-1)^{\mathrm{ind}_f(p)}.$$