Weakly mixing diffeomorphisms preserving a measurable Riemannian metric with prescribed Liouvillean rotation behavior

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Abstract

We show that on any smooth compact connected manifold of dimension $m \geq 2$ admitting a smooth non-trivial circle action $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}$, $S_{t+1} = S_t$, the set of weakly mixing C^{∞}-diffeomorphisms which preserve both a smooth volume ν and a measurable Riemannian metric is dense in $\mathcal{A}_{\alpha}(M) = \overline{\{h \circ S_{\alpha} \circ h^{-1} : h \in \text{Diff}^{\infty}(M, \nu)\}}^{C^{\infty}}$ for every Liouvillean number α. The proof is based on a quantitative version of the approximation by conjugation-method with explicitly constructed conjugation maps and partitions.

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1 Introduction

To begin, recall that a dynamical system (X, T, ν) is ergodic if and only if every measurable complex-valued function h on (X, ν) which is invariant (i.e. such that $h(Tx) = h(x)$ for every $x \in X$) must necessarily be constant. We define (X, T, ν) to be weakly mixing if it satisfies the stronger condition that there is no non-constant measurable complex valued function h on (X, ν) such that $h(Tx) = \lambda \cdot h(x)$ for some $\lambda \in \mathbb{C}$. Equivalently there is an increasing sequence $(m_n)_{n\in\mathbb{N}}$ of natural numbers such that $\lim_{n\to\infty} |\nu(B \cap T^{-m_n}(A)) - \nu(A) \cdot \nu(B)| = 0$ for every pair of measurable sets $A, B \subseteq X$ (see [Skl67] or [AK70, Theorem 5.1]). We call a circle action $\{S_t\}_{t\in\mathbb{R}}$ on a manifold M non-trivial if there exists $t \in \mathbb{R}$ and $x \in M$ with $S_t(x) \neq x$; in other words, not all orbits are fixed points (even though some may be).

Until 1970 it was an open question if there exists an ergodic area-preserving smooth diffeomorphism on the disc $\mathbb{D}^2$. This problem was solved by the so-called "approximation by conjugation"-method developed by D. Anosov and A. Katok in [AK70]. In fact, on every smooth compact connected manifold M of dimension $m \geq 2$ admitting a non-trivial circle action $\mathcal{S} = \{S_t\}_{t\in\mathbb{S}^1}$ preserving a smooth volume ν this method enables the construction of smooth diffeomorphisms with specific ergodic properties (e. g. weakly mixing ones in [AK70, section 5]) or non-standard smooth realizations of measure-preserving systems (e. g. [AK70, section 6], [Be13] and [FSW07]). These diffeomorphisms are constructed as limits of conjugates $f_n = H_n \circ S_{\alpha_{n+1}} \circ H_n^{-1}$, where $\alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}} = \alpha_n + \frac{1}{k_n \cdot l_n \cdot q_n^2} \in \mathbb{Q}$, where $H_n = H_{n-1} \circ h_n$ and where h_n are measure-preserving diffeomorphisms satisfying $S_{\frac{1}{q_n}} \circ h_n = h_n \circ S_{\frac{1}{q_n}}$. In each step the conjugation map h_n and the parameter k_n are chosen such that the diffeomorphism f_n imitates the desired property with a certain precision. In a final step of the construction, the parameter l_n is chosen large enough to guarantee closeness of f_n to f_{n-1} in the C^∞-topology, and so the convergence of the sequence $(f_n)_{n\in\mathbb{N}}$ to a limit diffeomorphism is provided. It is even possible to keep this limit diffeomorphism within any given C^∞-neighbourhood of the initial element S_{α_1} or, by applying a fixed diffeomorphism g first, of $g \circ S_{\alpha_1} \circ g^{-1}$. So the construction can be carried out in a neighbourhood of any diffeomorphism conjugate to an element of the action. Thus, $\mathcal{A}(M) = \overline{\{h \circ S_t \circ h^{-1} \, : \, t \in \mathbb{S}^1, h \in \mathrm{Diff}^\infty(M,\nu)\}}^{C^\infty}$ is a natural space for the produced diffeomorphisms. Moreover, we will consider the restricted space $\mathcal{A}_\alpha(M) = \overline{\{h \circ S_\alpha \circ h^{-1} \, : h \in \mathrm{Diff}^\infty(M,\nu)\}}^{C^\infty}$ for $\alpha \in \mathbb{S}^1$.

In the following let M be a smooth compact connected manifold of dimension $m \geq 2$ admitting a non-trivial circle action $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}$, $S_{t+1} = S_t$. Note that any such action possesses a smooth invariant volume: Every smooth manifold carries a Riemannian metric and hence a smooth Riemannian volume form $\hat{\nu}$. Any smooth volume form is given by $f \cdot \hat{\nu}$, where f is a positive scalar function. If $\bar{f}$ is the fiberwise average of f, then $\bar{f} \cdot \hat{\nu}$ is a smooth volume form which is invariant under $\mathcal{S}$. In case of a manifold with boundary by a smooth diffeomorphism we mean infinitely differentiable in the interior and such that all the derivatives can be extended to the boundary continuously.

In their influential paper [AK70] Anosov and Katok proved amongst others that in $\mathcal{A}(M)$ the set of weakly mixing diffeomorphisms is generic (i. e. it is a dense G_δ-set) in the $C^\infty(M)$-topology. For this they used the "approximation by conjugation"-method. In [GK00] the conjugation maps are constructed more explicitly such that they can be equipped with the additional structure of being locally very close to an isometry, thus showing that there exists a weakly mixing smooth diffeomorphism preserving a smooth measure and a measurable Riemannian metric on any manifold with non-trivial circle action. Actually, it follows from the respective proofs that both results are true in $\mathcal{A}_\alpha(M)$ for a G_δ-set of $\alpha \in \mathbb{R}$. However, both proofs do not give a full description of the set of $\alpha \in \mathbb{R}$ for which the particular result holds in $\mathcal{A}_\alpha(M)$. Such an investigation is started in [FS05]: B. Fayad and M. Saprykina showed in case of dimension 2 that if $\alpha \in \mathbb{S}^1$ is Liouville, the set of weakly mixing diffeomorphisms is generic in the $C^\infty(M)$-topology in $\mathcal{A}_\alpha(M)$. Here an irrational number α is called Liouville if and only if for every $C \in \mathbb{R}_{>0}$ and for every $n \in \mathbb{N}$ there are infinitely many pairs of coprime integers p, q such that $\left|\alpha - \frac{p}{q}\right| < \frac{C}{q^n}$.

In this article we prove the following theorem generalizing the results of [GK00] as well as [FS05]:

Theorem 1. *Let M be a smooth compact and connected manifold of dimension $m \geq 2$ with a non-trivial circle action $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}$, $S_{t+1} = S_t$. For any $\mathcal{S}$-invariant smooth volume ν the following is true: If $\alpha \in \mathbb{R}$ is Liouville, then the set of volume-preserving diffeomorphisms, that are weakly mixing and preserve a measurable Riemannian metric, is dense in the C^∞-topology in $\mathcal{A}_\alpha(M)$.*

See [GK00, section 3] for a comprehensive consideration of IM-diffeomorphisms (i. e. diffeomorphisms preserving an absolutely continuous probability measure and a measurable Riemannian metric) and IM-group actions. In particular, the existence of a measurable invariant metric for a diffeomorphism is equivalent to the existence of an invariant measure for the projectivized derivative extension which is absolutely continuous in the fibers. It is a natural question to ask about the ergodic properties of the derivative extension with respect to such a measure. While

in our construction the projectivized derivative extension is as non-ergodic as possible (in fact, the derivative cocycle is cohomologous to the identity), it is work in progress to realize ergodic behaviour. Recently, it has been proven that for every $\rho > 0$ and $m \geq 2$ there exists a weakly mixing real-analytic diffeomorphism $f \in \mathrm{Diff}^{\omega}_{\rho}\left(\mathbb{T}^m, \mu\right)$ preserving a measurable Riemannian metric ([Ku]).
We want to point out that Theorem 1 is in some sense the best we can obtain:

- By [FS05, corollary 1.4], whose proof uses Herman's last geometric result ([FKr09]), we have the following dichotomy in case of $M = \mathbb{S}^1 \times [0,1]$: A number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is Diophantine if and only if there is no ergodic diffeomorphism of M whose rotation number (on at least one of the boundaries) is equal to α. Since weakly mixing diffeomorphisms are ergodic, there cannot be a weakly mixing $f \in \mathcal{A}_{\alpha}\left(\mathbb{S}^1 \times [0,1]\right)$ for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ Diophantine.
- By a result of A. Furman (appendix to [GK00]) a weakly mixing diffeomorphism cannot preserve a Riemannian metric with L^2-distortion (i.e. both the norm and its inverse are L^2-functions). Moreover, it is conjectured that a weakly mixing diffeomorphism cannot preserve a Riemannian metric with L^1-distortion (see [GK00, Conjecture 3.7.]).

Using the standard techniques to prove genericity of the weak mixing-property and Theorem 1 we conclude in subsection 2.2:

Corollary 1. *Let M be a smooth compact and connected manifold of dimension $m \geq 2$ with a non-trivial circle action $\mathcal{S} = \{S_t\}_{t \in \mathbb{R}}$, $S_{t+1} = S_t$, preserving a smooth volume ν. If $\alpha \in \mathbb{R}$ is Liouville, the set of volume-preserving weakly mixing diffeomorphisms is a dense G_{δ}-set in the C^{∞}-topology in $\mathcal{A}_{\alpha}(M)$.*

Hereby, we clear up some points in [FS05] by generalizing their 2-dimensional constructions to arbitrary higher dimension.

2 Preliminaries

2.1 Definitions and notations

In this chapter we want to introduce advantageous definitions and notations. Initially we discuss topologies on the space of smooth diffeomorphisms on the manifold $M = \mathbb{S}^1 \times [0,1]^{m-1}$. Note that for diffeomorphisms $f = (f_1, ..., f_m) : \mathbb{S}^1 \times [0,1]^{m-1} \to \mathbb{S}^1 \times [0,1]^{m-1}$ the coordinate function f_1 understood as a map $\mathbb{R} \times [0,1]^{m-1} \to \mathbb{R}$ has to satisfy the condition $f_1(\theta + n, r_1, ..., r_{m-1}) = f_1(\theta, r_1, ..., r_{m-1}) + l$ for $n \in \mathbb{Z}$, where either $l = n$ or $l = -n$. Moreover, for $i \in \{2, ..., m\}$ the coordinate function f_i has to be $\mathbb{Z}$-periodic in the first component, i.e. $f_i(\theta + n, r_1, ..., r_{m-1}) = f_i(\theta, r_1, ..., r_{m-1})$ for every $n \in \mathbb{Z}$.
To define explicit metrics on $\mathrm{Diff}^k\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$ and in the following, the subsequent notations will be useful:

Definition 2.1. 1. For a sufficiently differentiable function $f : \mathbb{R}^m \to \mathbb{R}$ and a multi-index $\vec{a} = (a_1, ..., a_m) \in \mathbb{N}_0^m$

$$D_{\vec{a}} f := \frac{\partial^{|\vec{a}|}}{\partial x_1^{a_1} ... \partial x_m^{a_m}} f,$$

where $|\vec{a}| = \sum_{i=1}^{m} a_i$ is the order of $\vec{a}$.

2. For a continuous function $F : (0,1)^m \to \mathbb{R}$

$$\|F\|_0 := \sup_{z \in (0,1)^m} |F(z)|.$$

Diffeomorphisms on $\mathbb{S}^1 \times [0,1]^{m-1}$ can be regarded as maps from $[0,1]^m$ to $\mathbb{R}^m$. In this spirit the expressions $\|f_i\|_0$ as well as $\|D_{\vec{a}} f_i\|_0$ for any multi-index $\vec{a}$ with $|\vec{a}| \leq k$ have to be understood for $f = (f_1, ..., f_m) \in \operatorname{Diff}^k\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$. Since such a diffeomorphism is a continuous map on the compact manifold and every partial derivative can be extended continuously to the boundary, all these expressions are finite. Thus the subsequent definition makes sense:

Definition 2.2. 1. For $f, g \in \operatorname{Diff}^k\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$ with coordinate functions f_i resp. g_i we define

$$\tilde{d}_0(f,g) = \max_{i=1,..,m} \left\{ \inf_{p \in \mathbb{Z}} \|(f-g)_i + p\|_0 \right\}$$

as well as

$$\tilde{d}_k(f,g) = \max \left\{ \tilde{d}_0(f,g), \|D_{\vec{a}}(f-g)_i\|_0 \ : \ i = 1, ..., m \ , \ 1 \leq |\vec{a}| \leq k \right\}.$$

2. Using the definitions from 1. we define for $f, g \in \operatorname{Diff}^k\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$:

$$d_k(f,g) = \max \left\{ \tilde{d}_k(f,g) \ , \ \tilde{d}_k\left(f^{-1}, g^{-1}\right) \right\}.$$

Obviously d_k describes a metric on $\operatorname{Diff}^k\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$ measuring the distance between the diffeomorphisms as well as their inverses. As in the case of a general compact manifold the following definition connects to it:

Definition 2.3. 1. A sequence of $\operatorname{Diff}^\infty\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$-diffeomorphisms is called convergent in $\operatorname{Diff}^\infty\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$ if it converges in $\operatorname{Diff}^k\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$ for every $k \in \mathbb{N}$.

2. On $\operatorname{Diff}^\infty\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$ we declare the following metric

$$d_\infty(f,g) = \sum_{k=1}^{\infty} \frac{d_k(f,g)}{2^k \cdot (1 + d_k(f,g))}.$$

It is a general fact that $\operatorname{Diff}^\infty\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$ is a complete metric space with respect to this metric d_∞.
Again considering diffeomorphisms on $\mathbb{S}^1 \times [0,1]^{m-1}$ as maps from $[0,1]^m$ to $\mathbb{R}^m$ we add the adjacent notation:

Definition 2.4. Let $f \in \operatorname{Diff}^k\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$ with coordinate functions f_i be given. Then

$$\|Df\|_0 := \max_{i,j \in \{1,...,m\}} \|D_j f_i\|_0$$

and

$$|||f|||_k := \max \left\{ \|D_{\vec{a}} f_i\|_0 , \left\|D_{\vec{a}}\left(f_i^{-1}\right)\right\|_0 \ : \ i = 1, ..., m, \ \vec{a} \text{ multi-index with } 0 \leq |\vec{a}| \leq k \right\}.$$

Remark 2.5. By the above-mentioned observations for every multi-index $\vec{a}$ with $|\vec{a}| \geq 1$ and every $i \in \{1, ..., m\}$ the derivative $D_{\vec{a}} h_i$ is $\mathbb{Z}$-periodic in the first variable. Since in case of a diffeomorphism $g = (g_1, ..., g_m)$ on $\mathbb{S}^1 \times [0,1]^{m-1}$ regarded as a map $[0,1]^m \to \mathbb{R}^m$ the coordinate functions g_j for $j \in \{2, ..., m\}$ satisfy $g_j\left([0,1]^m\right) \subseteq [0,1]$, it holds:

$$\sup_{z \in (0,1)^m} \left|\left(D_{\vec{a}} h_i\right)(g(z))\right| \leq |||h|||_{|\vec{a}|}.$$

Furthermore, we introduce the notion of a partial partition of a compact manifold M, which is a pairwise disjoint countable collection of measurable subsets of M.

Definition 2.6.
- A sequence of partial partitions ν_n converges to the decomposition into points if and only if for a given measurable set A and for every $n \in \mathbb{N}$ there exists a measurable set A_n, which is a union of elements of ν_n, such that $\lim_{n\to\infty} \mu(A \Delta A_n) = 0$. We often denote this by $\nu_n \to \varepsilon$.
- A partial partition ν is the image under a diffeomorphism $F : M \to M$ of a partial partition η if and only if $\nu = \{F(I) \ : \ I \in \eta\}$. We write this as $\nu = F(\eta)$.

2.2 First steps of the proof

First of all we show how constructions on $\mathbb{S}^1 \times [0,1]^{m-1}$ can be transfered to a general compact connected smooth manifold M with a non-trivial circle action $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}$, $S_{t+1} = S_t$. By [AK70, Proposition 2.1.], we can assume that 1 is the smallest positive number satisfying $S_t = \mathrm{id}$. Hence, we can assume $\mathcal{S}$ to be effective. We denote the set of fixed points of $\mathcal{S}$ by F and for $q \in \mathbb{N}$ F_q is the set of fixed points of the map $S_{\frac{1}{q}}$.
On the other hand, we consider $\mathbb{S}^1 \times [0,1]^{m-1}$ with Lebesgue measure μ. Furthermore, let $\mathcal{R} = \{R_\alpha\}_{\alpha\in\mathbb{S}^1}$ be the standard action of $\mathbb{S}^1$ on $\mathbb{S}^1 \times [0,1]^{m-1}$, where the map R_α is given by $R_\alpha(\theta, r_1, ..., r_{m-1}) = (\theta + \alpha, r_1, ..., r_{m-1})$. Hereby, we can formulate the following result (see [FSW07, Proposition 1]):

Proposition 2.7. *Let M be a m-dimensional smooth, compact and connected manifold admitting an effective circle action $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}$, $S_{t+1} = S_t$, preserving a smooth volume ν. Let $B := \partial M \cup F \cup \left(\bigcup_{q\geq 1} F_q\right)$. There exists a continuous surjective map $G : \mathbb{S}^1 \times [0,1]^{m-1} \to M$ with the following properties:*

1. *The restriction of G to $\mathbb{S}^1 \times (0,1)^{m-1}$ is a C^∞-diffeomorphic embedding.*
2. $\nu\left(G\left(\partial\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)\right)\right) = 0$
3. $G\left(\partial\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)\right) \supseteq B$
4. $G_*(\mu) = \nu$
5. $\mathcal{S} \circ G = G \circ \mathcal{R}$

By the same reasoning as in [FSW07, section 2.2.], this proposition allows us to carry a construction from $\left(\mathbb{S}^1 \times [0,1]^{m-1}, \mathcal{R}, \mu\right)$ to the general case $(M, \mathcal{S}, \nu)$:
Suppose $f : \mathbb{S}^1 \times [0,1]^{m-1} \to \mathbb{S}^1 \times [0,1]^{m-1}$ is a weakly mixing diffeomorphism sufficiently close to R_α in the C^∞-topology with f-invariant measurable Riemannian metric ω obtained

by $f = \lim_{n\to\infty} f_n$ with $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$, where $f_n = R_{\alpha_{n+1}}$ in a neighbourhood of the boundary (in Proposition 2.8 we will see that these conditions can be satisfied in the constructions of this article). Then we define a sequence of diffeomorphisms:

$$\tilde{f}_n : M \to M \qquad \tilde{f}_n(x) = \begin{cases} G \circ f_n \circ G^{-1}(x) & \textit{if } x \in G\left(\mathbb{S}^1 \times (0,1)^{m-1}\right) \\ S_{\alpha_{n+1}}(x) & \textit{if } x \in G\left(\partial\left(\mathbb{S}^1 \times (0,1)^{m-1}\right)\right) \end{cases}$$

Constituted in [FK04, section 5.1.], this sequence is convergent in the C^∞-topology to the diffeomorphism

$$\tilde{f} : M \to M \qquad \tilde{f}(x) = \begin{cases} G \circ f \circ G^{-1}(x) & \textit{if } x \in G\left(\mathbb{S}^1 \times (0,1)^{m-1}\right) \\ S_{\alpha}(x) & \textit{if } x \in G\left(\partial\left(\mathbb{S}^1 \times (0,1)^{m-1}\right)\right) \end{cases}$$

provided the closeness from f to R_α in the C^∞-topology.
We observe that f and $\tilde{f}$ are measure-theoretically isomorphic. Then $\tilde{f}$ is weakly mixing because the weak mixing-property is invariant under isomorphisms.
Moreover, we want to show how we can construct a $\tilde{f}$-invariant measurable Riemannian metric $\tilde{\omega}$ out of the f-invariant metric ω. Since $\tilde{\omega}$ only needs to be a measurable metric and $\nu\left(G\left(\partial\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)\right)\right) = 0$, we only have to construct it on $G\left(\mathbb{S}^1 \times (0,1)^{m-1}\right)$. Using the diffeomorphic embedding G we consider $\tilde{\omega}|_{G\left(\mathbb{S}^1\times(0,1)^{m-1}\right)} := \left(G^{-1}\right)^* \omega|_{G\left(\mathbb{S}^1\times(0,1)^{m-1}\right)}$ and show that it is $\tilde{f}$-invariant: On $G\left(\mathbb{S}^1 \times (0,1)^{m-1}\right)$ we have $\tilde{f} = G \circ f \circ G^{-1}$ and thus we can compute:

$$\tilde{f}^*\tilde{\omega} = \left(G \circ f \circ G^{-1}\right)^* \left(\left(G^{-1}\right)^* \omega\right) = \left(G^{-1}\right)^* \circ f^* \circ G^* \circ \left(G^{-1}\right)^* \omega = \left(G^{-1}\right)^* \circ f^* \omega = \left(G^{-1}\right)^* \omega = \tilde{\omega}$$

Altogether the construction done in the case of $\left(\mathbb{S}^1 \times [0,1]^{m-1}, \mathcal{R}, \mu\right)$ is transfered to $(M, \mathcal{S}, \nu)$. Hence, it suffices to consider constructions on $M = \mathbb{S}^1 \times [0,1]^{m-1}$ with circle action $\mathcal{R}$ subsequently. In this case we will prove the following result:

Proposition 2.8. *For every Liouvillean number α there is a sequence $(\alpha_n)_{n\in\mathbb{N}}$ of rational numbers $\alpha_n = \frac{p_n}{q_n}$ satisfying $\lim_{n\to\infty} |\alpha - \alpha_n| = 0$ monotonically, and there are sequences $(g_n)_{n\in\mathbb{N}}$, $(\phi_n)_{n\in\mathbb{N}}$ of measure-preserving diffeomorphisms satisfying $g_n \circ R_{\frac{1}{q_n}} = R_{\frac{1}{q_n}} \circ g_n$ as well as $\phi_n \circ R_{\frac{1}{q_n}} = R_{\frac{1}{q_n}} \circ \phi_n$ such that the diffeomorphisms $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ with $H_n := h_1 \circ h_2 \circ ... \circ h_n$, where $h_n := g_n \circ \phi_n$, coincide with $R_{\alpha_{n+1}}$ in a neighbourhood of the boundary, converge in the Diff$^\infty(M)$-topology, and the diffeomorphism $f = \lim_{n\to\infty} f_n$ is weakly mixing, has an invariant measurable Riemannian metric, and satisfies $f \in \mathcal{A}_\alpha(M)$.*
Furthermore, for every $\varepsilon > 0$ the parameters in the construction can be chosen in such a way that $d_\infty(f, R_\alpha) < \varepsilon$.

By this Proposition weakly mixing diffeomorphisms preserving a measurable Riemannian metric are dense in $\mathcal{A}_\alpha(M)$:
Because of $\mathcal{A}_\alpha(M) = \overline{\{h \circ R_\alpha \circ h^{-1} : h \in \text{Diff}^\infty(M,\mu)\}}^{C^\infty}$ it is enough to show that for every diffeomorphism $h \in \text{Diff}^\infty(M,\mu)$ and every $\epsilon > 0$ there is a weakly mixing diffeomorphism $\tilde{f}$ preserving a measurable Riemannian metric such that $d_\infty\left(\tilde{f}, h \circ R_\alpha \circ h^{-1}\right) < \epsilon$. For this purpose, let $h \in \text{Diff}^\infty(M,\mu)$ and $\epsilon > 0$ be arbitrary. By [Om74, p. 3] and [KM97, Theorem 43.1.], $\text{Diff}^\infty(M)$ is a Lie group. In particular, the conjugating map $g \mapsto h \circ g \circ h^{-1}$ is continuous

with respect to the metric d_∞. Continuity in the point R_α yields the existence of $\delta > 0$, such that $d_\infty(g, R_\alpha) < \delta$ implies $d_\infty\left(h \circ g \circ h^{-1}, h \circ R_\alpha \circ h^{-1}\right) < \epsilon$. By Proposition 2.8 we can find a weakly mixing diffeomorphism f with f-invariant measurable Riemannian metric ω and $d_\infty(f, R_\alpha) < \delta$. Hence $\tilde{f} := h \circ f \circ h^{-1}$ satisfies $d_\infty\left(\tilde{f}, h \circ R_\alpha \circ h^{-1}\right) < \epsilon$. Note that $\tilde{f}$ is weakly mixing and preserves the measurable Riemannian metric $\tilde{\omega} := \left(h^{-1}\right)^* \omega$.
Hence, Theorem 1 is deduced from Proposition 2.8.

Remark 2.9. Moreover we can show that the set of weakly mixing diffeomorphisms is generic in $\mathcal{A}_\alpha(M)$ (i.e. it is a dense G_δ-set) using Proposition 2.8 and the same technique as in [Ha56], section *Category*.
Using Proposition 2.8 we can show that the set of weakly mixing diffeomorphisms is generic in $\mathcal{A}_\alpha(M)$ (i.e. it is a dense G_δ-set). Thereby, we consider a countable dense set $\{\varphi_n\}_{n\in\mathbb{N}}$ in $L^2(M,\mu)$, which is a separable space, and define the sets:

$$O(i,j,k,n) = \left\{T \in \mathcal{A}_\alpha(M) \ : \ \left|(U_T^n \varphi_i, \varphi_j) - (\varphi_i, 1)\cdot(1,\varphi_j)\right| < \frac{1}{k}\right\}$$

Since $(U_T\varphi, \psi)$ depends continuously on T, each $O(i,j,k,n)$ is open. Hence,

$$K := \bigcap_{i\in\mathbb{N}} \bigcap_{j\in\mathbb{N}} \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} O(i,j,k,n)$$

is a G_δ-set.
By another equivalent characterisation a measure-preserving transformation T is weakly mixing if and only if for every $\varphi, \psi \in L^2(M,\mu)$ there is a sequence $(m_n)_{n\in\mathbb{N}}$ of density one such that $\lim_{n\to\infty}(U_T^{m_n}\varphi,\psi) = (\varphi,1)\cdot(1,\psi)$. Thus, every weakly mixing diffeomorphism is contained in K. On the other hand, we show that a transformation, that is not weakly mixing, does not belong to K: If T is not weakly mixing, U_T has a non-trivial eigenfunction. W.l.o.g. we can assume the existence of $f \in L^2(M,\mu)$ and $c \in \mathbb{C}$ of absolute value 1 satisfying $U_T f = c \cdot f$, $\|f\|_{L^2} = 1$ and $(1,f) = 0$. Since $\{\varphi_n\}_{n\in\mathbb{N}}$ is dense in $L^2(M,\mu)$, there is an index i such that $\|f - \varphi_i\|_{L^2} < 0.1$. Obviously $\|\varphi_i\|_{L^2} \leq \|f\|_{L^2} + \|f-\varphi_i\|_{L^2} < 1.1$ and $|(U_T^n f, f) - (f,1)\cdot(1,f)| = |(c^n \cdot f, f)| = |c^n| \cdot \|f\|_{L^2}^2 = 1$. Consequently we can estimate:

$$\begin{aligned}
1 &= |(U_T^n f, f) - (f,1)\cdot(1,f)| \\
&\leq |(U_T^n f, f) - (U_T^n f, \varphi_i)| + |(U_T^n f, \varphi_i) - (U_T^n \varphi_i, \varphi_i)| + |(U_T^n \varphi_i, \varphi_i) - (\varphi_i, 1)\cdot(1,\varphi_i)| \\
&\quad + |(\varphi_i,1)\cdot(1,\varphi_i) - (\varphi_i,1)\cdot(1,f)| + |(\varphi_i,1)\cdot(1,f) - (f,1)\cdot(1,f)| \\
&\leq |c|^n \cdot \|f\|_{L^2} \cdot \|f-\varphi_i\|_{L^2} + \|f-\varphi_i\|_{L^2} \cdot \|\varphi_i\|_{L^2} + |(U_T^n\varphi_i,\varphi_i) - (\varphi_i,1)\cdot(1,\varphi_i)| \\
&\quad + \|\varphi_i\|_{L^2} \cdot \|f - \varphi_i\|_{L^2} \\
&\leq 0.1 + 0.11 + |(U_T^n \varphi_i, \varphi_i) - (\varphi_i,1)\cdot(1,\varphi_i)| + 0.11 \\
&< |(U_T^n \varphi_i, \varphi_i) - (\varphi_i,1)\cdot(1,\varphi_i)| + 0.5
\end{aligned}$$

Thus $|(U_T^n\varphi_i,\varphi_i) - (\varphi_i,1)\cdot(1,\varphi_i)|$ has to be larger than $\frac{1}{2}$. Hence T does not belong to $O(i,i,2,n)$ for any value of n and accordingly does not belong to K. So K coincides with the set of weakly mixing diffeomorphisms in $\mathcal{A}_\alpha(M)$. By the observations above we know that this set is dense. In conclusion the set of weakly mixing diffeomorphisms is a dense G_δ-set in $\mathcal{A}_\alpha(M)$. Thus Corollary 1 is proven.

2.3 Outline of the proof

The constructions are based on the “approximation by conjugation”-method developed by D.V. Anosov and A. Katok in [AK70]. As indicated in the introduction, one constructs successively a

sequence of measure preserving diffeomorphisms $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$, where the conjugation maps $H_n = h_1 \circ ... \circ h_n$ and the rational numbers $\alpha_n = \frac{p_n}{q_n}$ are chosen in such a way that the functions f_n converge to a diffeomorphism f with the desired properties.
First of all we will define two sequences of partial partitions, which converge to the decomposition into points in each case. The first type of partial partition, called η_n, will satisfy the requirements in the proof of the weak mixing-property. On the partition elements of the even more refined second type, called ζ_n, the conjugation map h_n will act as an isometry, and this will enable us to construct an invariant measurable Riemannian metric. Afterwards we will construct these conjugating diffeomorphisms $h_n = g_n \circ \phi_n$, which are composed of two step-by-step defined smooth measure-preserving diffeomorphisms. In this construction the map g_n should introduce shear in the θ-direction as in [FS05]. So $\tilde{g}_{[nq_n^\sigma]}(\theta, r_1, ..., r_{m-1}) = (\theta + [n \cdot q_n^\sigma] \cdot r_1, r_1, ..., r_{m-1})$ might seem an obvious candidate. Unfortunately, that map is not an isometry. Therefore, the map g_n will be constructed in such a way that g_n is an isometry on the image under ϕ_n of any partition element $\check{I}_n \in \zeta_n$, and $g_n\left(\hat{I}_n\right) = \tilde{g}_{[nq_n^\sigma]}\left(\hat{I}_n\right)$ as well as $g_n\left(\Phi_n\left(\hat{I}_n\right)\right) = \tilde{g}_{[nq_n^\sigma]}\left(\Phi_n\left(\hat{I}_n\right)\right)$ for every $\hat{I}_n \in \eta_n$, where $\Phi_n = \phi_n \circ R^{m_n}_{\alpha_{n+1}} \circ \phi_n^{-1}$ with a specific sequence $(m_n)_{n\in\mathbb{N}}$ of natural numbers (see section 4) is important in the proof of the weak mixing property. Likewise the conjugation map ϕ_n will be built such that it acts on the elements of ζ_n as an isometry and on the elements of η_n in such a way that it satisfies the requirements of a criterion for weak mixing similar to the one in [FS05] but modified in many places because of the new conjugation map g_n and the new type of partitions. In particular, Φ_n has to map each element of the partial partition η_n on a set of almost full length in the $r_1, ..., r_{m-1}$-coordinates in an almost uniform way (see Definition 4.1 for the precise requirement).
In the 2-dimensional constructions of [FS05] such a behaviour is obtained by putting ϕ_n equal to the identity on one half of the fundamental domain $\left[0, \frac{1}{q_n}\right] \times [0,1]$ and $\phi_n = \tilde{\phi}_{2q_n}$ on the other one, where $\tilde{\phi}_\lambda = C_\lambda^{-1} \circ \varphi \circ C_\lambda$ with C_λ being a stretching by λ in the first coordinate and φ a "quasi-rotation", i.e. a rotation by $\frac{\pi}{2}$ on large part of the domain. So, such a map $\tilde{\varphi}_\lambda$ as well as its inverse map a horizontal interval of length about λ^{-1} to a vertical interval of almost full length 1. Since $R^{m_n}_{\alpha_{n+1}}$ induces a permutation of the sections by choice of the number m_n, $\Phi_n = \phi_n \circ R^{m_n}_{\alpha_{n+1}} \circ \phi_n^{-1}$ maps small horizontal to long vertical intervals. This procedure relies on the 2-dimensional setting. Modifications of that approach by known quantitative Anosov-Katok-constructions in higher dimensions like in [FSW07] and [Be13] do not work. We had to modify the notion of uniform distribution and our corresponding maps $\tilde{\phi}^{(j)}_{\lambda_j,\mu_j}$ constructed in subsection 3.3 will always involve the θ-coordinate. Additionally, we introduce "inner rotations" in order to guarantee that ϕ_n acts as an isometry on the partition elements $\check{I}_n \in \zeta_n$: A map of the form $C_\lambda^{-1} \circ \varphi \circ C_\lambda$ would cause an expansion by λ in one coordinate and by λ^{-1} in another, so far away from being an isometry. The "inner rotations" will cause that C_λ and C_λ^{-1} act on the same coordinate on the elements $\check{I}_n \in \zeta_n$. Unfortunately, this requires a fairly elaborate and slightly technical construction. With the aid of these maps $\tilde{\phi}^{(j)}_{\lambda_j,\mu_j}$ we define the conjugation map $\phi_n = \tilde{\phi}^{(m)}_{2q_n,q_n} \circ \tilde{\phi}^{(m-1)}_{2q_n^2,q_n} \circ ... \circ \tilde{\phi}^{(2)}_{2q_n^{m-1},q_n}$ on the first half of the fundamental domain. On the second one, ϕ_n is the identity.
In section 5 we will show convergence of the sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{A}_\alpha(M)$ for a given Liouville number α by the same approach as in [FS05]. To do so, we have to estimate the norms $|||H_n|||_k$ very carefully. In section 6 we verify that the obtained diffeomorphism $f = \lim_{n\to\infty} f_n$ is weakly mixing. Finally, we will construct the desired f-invariant measurable Riemannian metric in section 7 exploiting the fact that h_n acts as an isometry on large parts of the manifold.

3 Explicit constructions

3.1 Sequences of partial partitions

In this subsection we define the two announced sequences of partial partitions $(\eta_n)_{n\in\mathbb{N}}$ and $(\zeta_n)_{n\in\mathbb{N}}$ of $M = \mathbb{S}^1 \times [0,1]^{m-1}$.

3.1.1 Partial partition η_n

Remark 3.1. For convenience we will use the notation $\prod_{i=2}^{m} [a_i, b_i]$ for $[a_2, b_2] \times ... \times [a_m, b_m]$.

Initially, η_n will be constructed on the fundamental sector $\left[0, \frac{1}{q_n}\right] \times [0,1]^{m-1}$. For this purpose we divide the fundamental sector into 2 sections:

- On $\left[0, \frac{1}{2\cdot q_n}\right] \times [0,1]^{m-1}$ the partial partition η_n consists of all multidimensional intervals of the following form:

$$\left[\frac{1}{52n^4 \cdot q_n}, \frac{1}{2q_n} - \frac{1}{52n^4 \cdot q_n}\right] \times \prod_{i=2}^{m} \left[\frac{j_i}{q_n} + \frac{1}{26n^4 \cdot q_n}, \frac{j_i+1}{q_n} - \frac{1}{26n^4 \cdot q_n}\right],$$

  where $j_i \in \mathbb{Z}$ and $\left\lceil \frac{q_n}{10n^4} \right\rceil \leq j_i \leq q_n - \left\lceil \frac{q_n}{10n^4} \right\rceil - 1$ for $i = 2, ..., m$. We will call these sets partition elements of the first kind.

- On $\left[\frac{1}{2\cdot q_n}, \frac{1}{q_n}\right] \times [0,1]^{m-1}$ the partial partition η_n consists of all sets of the following form:

$$\bigcup \left[\frac{1}{2q_n} + \frac{j_1^{(1)}}{2q_n^2} + \cdots + \frac{j_1^{(m-1)}}{2q_n^m} + \frac{1}{20n^4 \cdot q_n^m}, \frac{1}{2q_n} + \frac{j_1^{(1)}}{2q_n^2} + \cdots + \frac{j_1^{(m-1)}+1}{2q_n^m} - \frac{1}{20n^4 \cdot q_n^m}\right]$$
$$\times \prod_{i=2}^{m} \left[\frac{j_i}{q_n} + \frac{1}{10n^4 \cdot q_n}, \frac{j_i+1}{q_n} - \frac{1}{10n^4 \cdot q_n}\right],$$

  where $j_i \in \mathbb{Z}$, $\left\lceil \frac{q_n}{10n^4} \right\rceil \leq j_i \leq q_n - \left\lceil \frac{q_n}{10n^4} \right\rceil - 1$, for $i = 2, ..., m$ and the union is taken over all $j_1^{(l)} \in \mathbb{Z}$, $\left\lceil \frac{q_n}{10n^4} \right\rceil \leq j_1^{(l)} \leq q_n - \left\lceil \frac{q_n}{10n^4} \right\rceil - 1$, for $l = 1, ..., m-1$. We will call these sets partition elements of the second kind.

By applying the map R_{l/q_n} with $l \in \mathbb{Z}$, this partial partition of $\left[0, \frac{1}{q_n}\right] \times [0,1]^{m-1}$ is extended to a partial partition of $\mathbb{S}^1 \times [0,1]^{m-1}$.

Remark 3.2. By construction this sequence of partial partitions converges to the decomposition into points.

Remark 3.3. Due to our choice of allowed values for the occurring indices $j_i^{(l)}$ the partition elements are positioned in such a way that the requirements in Proposition 3.11 are satisfied. This will be used in Lemma 4.5 in order to show that the map $\Phi_n = \phi_n \circ R^{m_n}_{\alpha_{n+1}} \circ \phi_n^{-1}$ maps each partition element almost uniformly in the $r_1, \ldots, r_{m-1}$-coordinates.

3.1.2 Partial partition ζ_n

The partial partition ζ_n consists of all multidimensional intervals of the following form:

$$\left[\frac{k}{2q_n}+\frac{j_1^{(1)}}{2q_n^2}+\dots+\frac{j_1^{(m-1)}}{2q_n^m}+\frac{1}{n^4\cdot 2q_n^m},\frac{k}{2q_n}+\frac{j_1^{(1)}}{2q_n^2}+\dots+\frac{j_1^{(m-1)}+1}{2q_n^m}-\frac{1}{n^4\cdot 2q_n^m}\right]$$
$$\times\left[\frac{j_2^{(1)}}{q_n}+\dots+\frac{j_2^{(m)}}{q_n^m}+\frac{j_2^{(m+1)}}{16n^4\cdot q_n^m\cdot[nq_n^\sigma]}+\frac{1}{16n^8\cdot q_n^m\cdot[nq_n^\sigma]},\right.$$
$$\left.\frac{j_2^{(1)}}{q_n}+\dots+\frac{j_2^{(m)}}{q_n^m}+\frac{j_2^{(m+1)}+1}{16n^4\cdot q_n^m\cdot[nq_n^\sigma]}-\frac{1}{16n^8\cdot q_n^m\cdot[nq_n^\sigma]}\right]$$
$$\times\prod_{i=3}^m\left[\frac{j_i}{q_n}+\frac{1}{n^4\cdot q_n},\frac{j_i+1}{q_n}-\frac{1}{n^4\cdot q_n}\right],$$

where $k\in\mathbb{Z}$, $j_1^{(l)}\in\mathbb{Z}$, $\left\lceil\frac{q_n}{n^4}\right\rceil\leq j_1^{(l)}\leq q_n-\left\lceil\frac{q_n}{n^4}\right\rceil-1$, for $l=1,...,m-1$ as well as $j_2^{(l)}\in\mathbb{Z}$, $\left\lceil\frac{q_n}{n^4}\right\rceil\leq j_2^{(l)}\leq q_n-\left\lceil\frac{q_n}{n^4}\right\rceil-1$ for $l=1,...,m$ as well as $j_2^{(m+1)}\in\mathbb{Z}$, $16\cdot[n\cdot q_n^\sigma]\leq j_2^{(m+1)}\leq 16n^4\cdot[n\cdot q_n^\sigma]-16\cdot[n\cdot q_n^\sigma]-1$, as well as $j_i\in\mathbb{Z}$, $\left\lceil\frac{q_n}{n^4}\right\rceil\leq j_i\leq q_n-\left\lceil\frac{q_n}{n^4}\right\rceil-1$, for $i=3,...,m$..

Remark 3.4. For every $n\geq 3$ the partial partition ζ_n consists of disjoint sets, covers a set of measure at least $1-\frac{4\cdot m}{n^2}$, and the sequence $(\zeta_n)_{n\in\mathbb{N}}$ converges to the decomposition into points.

Remark 3.5. Due to the allowed values of the indices $j_i^{(l)}$ each partition element is positioned such that the conjugation maps act as isometries on it (see the requirements in Propositions 3.8, 3., and 3.11, 3.).

3.2 The conjugation map g_n

Let $\sigma\in(0,1)$. As mentioned in the sketch of the proof we aim for a smooth measure-preserving diffeomorphism g_n which satisfies $g_n\left(\hat{I}_n\right)=\tilde{g}_{[nq_n^\sigma]}\left(\hat{I}_n\right)$ as well as $g_n\left(\Phi_n\left(\hat{I}_n\right)\right)=\tilde{g}_{[nq_n^\sigma]}\left(\Phi_n\left(\hat{I}_n\right)\right)$ for every $\hat{I}_n\in\eta_n$ and is an isometry on the image under ϕ_n of any partition element $\check{I}_n\in\zeta_n$. Let $a,b\in\mathbb{Z}$ and $\varepsilon\in\left(0,\frac{1}{16}\right]$ such that $\frac{1}{\varepsilon}\in\mathbb{Z}$. Moreover, we consider $\delta>0$ such that $\frac{1}{\delta}\in\mathbb{Z}$ and $\frac{a\cdot b\cdot\delta}{\varepsilon}\in\mathbb{Z}$. We denote $[0,1]^2$ by Δ and $[\varepsilon,1-\varepsilon]^2$ by $\Delta(\varepsilon)$.

Lemma 3.6. *For every $\varepsilon\in\left(0,\frac{1}{16}\right]$ there exists a smooth measure-preserving diffeomorphism $g_\varepsilon:[0,1]^2\to\{(x+\varepsilon\cdot y,y)\ :\ x,y\in[0,1]\}$ that is the identity on $\Delta(4\varepsilon)$ and coincides with the map $(x,y)\mapsto(x+\varepsilon\cdot y,y)$ on $\Delta\setminus\Delta(\varepsilon)$.*

Proof. First of all let $\psi_\varepsilon:\mathbb{R}^2\to\mathbb{R}^2$ be a smooth diffeomorphism satisfying

$$\psi_\varepsilon(x,y)=\begin{cases}(x,y) & \text{on } \mathbb{R}^2\setminus\Delta(2\varepsilon)\\ \left(\frac{1}{2}+\frac{1}{5}\cdot\left(x-\frac{1}{2}\right),\frac{1}{2}+\frac{1}{5}\cdot\left(y-\frac{1}{2}\right)\right) & \text{on } \Delta(4\varepsilon)\end{cases}$$

Furthermore, let τ_ε be a smooth diffeomorphism with the following properties

$$\tau_\varepsilon(x,y)=\begin{cases}(x+\varepsilon\cdot y,y) & \text{on } \left\{\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2\geq\left(\frac{5}{16}\right)^2\right\}\\ (x,y) & \text{on } \left\{\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2\leq\frac{1}{50}\right\}\end{cases}$$

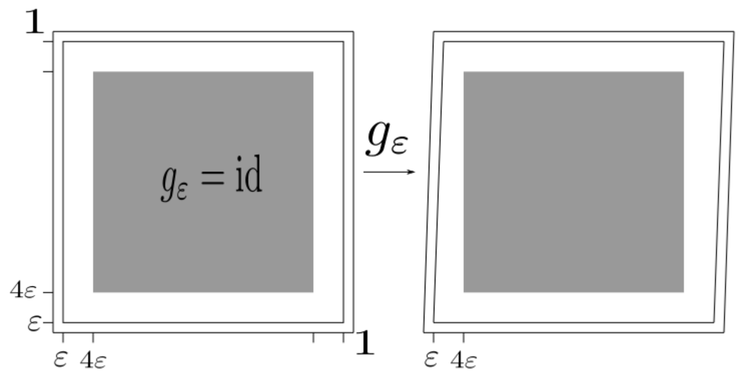

Figure 1: The action of the map g_ε.

We define $\bar{g}_\varepsilon := \psi_\varepsilon^{-1} \circ \tau_\varepsilon \circ \psi_\varepsilon$. Then the diffeomorphism $\bar{g}_\varepsilon$ coincides with the identity on $\Delta(4\varepsilon)$ and with the map $(x, y) \mapsto (x + \varepsilon \cdot y, y)$ on $\mathbb{R}^2 \setminus \Delta(\varepsilon)$. From this we conclude that $\det(D\bar{g}_\varepsilon) > 0$. Moreover, $\bar{g}_\varepsilon$ is measure-preserving on $U_\varepsilon := \left(\mathbb{R}^2 \setminus \Delta(\varepsilon)\right) \cup \Delta(4\varepsilon)$.
With the aid of "Moser's trick" we want to construct a diffeomorphism g_ε which is measure-preserving on the whole $\mathbb{R}^2$ and agrees with $\bar{g}_\varepsilon$ on U_ε. To do so, we consider the canonical volume form Ω_0 on $\mathbb{R}^2$: $\Omega_0 = dx \wedge dy$; in other words, $\Omega_0 = d\omega_0$ using the 1-form $\omega_0 = \frac{1}{2} \cdot (x \cdot dy - y \cdot dx)$. Additionally we introduce the volume form $\Omega_1 := \bar{g}_\varepsilon^* \Omega_0$.
At first we note that $\bar{g}_\varepsilon$ preserves the 1-form ω_0 on U_ε: Clearly this holds on $\Delta(4\varepsilon)$, where $\bar{g}_\varepsilon$ is the identity. On $\mathbb{R}^2 \setminus \Delta(\varepsilon)$ we have $D\bar{g}_\varepsilon(x, y) = (x + \varepsilon y, y)$, and thus we get

$$\bar{g}_\varepsilon^* \omega_0 = \omega_0(x + \varepsilon \cdot y, y) = \frac{1}{2} \cdot ((x + \varepsilon \cdot y)\, dy - y \cdot d(x + \varepsilon \cdot y)) = \frac{1}{2} \cdot (x \cdot dy - y \cdot dx) = \omega_0(x, y).$$

Furthermore, we introduce $\Omega' := \Omega_1 - \Omega_0$. Since the exterior derivative commutes with the pull-back, it holds that $\Omega' = d(\bar{g}_\varepsilon^* \omega_0 - \omega_0)$. In addition we consider the volume form $\Omega_t := \Omega_0 + t \cdot \Omega'$ and note that Ω_t is non-degenerate for $t \in [0, 1]$. Thus, we get a uniquely defined vector field X_t such that $\Omega_t(X_t, \cdot) = (\omega_0 - \bar{g}_\varepsilon^* \omega_0)(\cdot)$. Since Δ is a compact manifold, the non-autonomous differential equation $\frac{d}{dt} u(t) = X_t(u(t))$ with initial values in Δ has a solution defined on $\mathbb{R}$. Hence, we get a one-parameter family of diffeomorphisms $\{\nu_t\}_{t \in [0,1]}$ on Δ satisfying $\dot{\nu}_t = X_t(\nu_t)$, $\nu_0 = \mathrm{id}$.
Referring to [Ber98, Lemma 2.2], it holds that

$$\frac{d}{dt} \nu_t^* \Omega_t = d(\nu_t^*(i(X_t)\Omega_t)) + \nu_t^* \left(\frac{d}{dt} \Omega_t + i(X_t)\, d\Omega_t \right).$$

Because of $d(\nu_t^*(i(X_t)\Omega_t)) = \nu_t^*(d(i(X_t)\Omega_t))$ and $d\Omega_t = d(d\omega_0 + t \cdot (d(\bar{g}_\varepsilon^* \omega_0) - d\omega_0)) = 0$ we compute:

$$\begin{aligned} \frac{d}{dt} \nu_t^* \Omega_t &= \nu_t^*(d(i(X_t)\Omega_t)) + \nu_t^* \left(\frac{d}{dt} \Omega_t \right) = \nu_t^* d(\Omega_t(X_t, \cdot)) + \nu_t^* \Omega' \\ &= \nu_t^* d(\omega_0 - \bar{g}_\varepsilon^* \omega_0) + \nu_t^* \Omega' = \nu_t^*(\Omega_0 - \Omega_1) + \nu_t^*(\Omega_1 - \Omega_0) = 0. \end{aligned}$$

Consequently $\nu_1^* \Omega_1 = \nu_0^* \Omega_0 = \Omega_0$ (using $\nu_0 = \mathrm{id}$ in the last step). As we have seen, it holds that $\bar{g}_\varepsilon^* \omega_0 = \omega_0$ on U_ε. Therefore, on U_ε it holds that $\Omega_t(X_t, \cdot) = 0$. Since Ω_t is non-degenerate, we

conclude that $X_t = 0$ on U_ε and hence $\nu_1 = \nu_0 = \mathrm{id}$ on $U_\varepsilon \cap \Delta$. Now we can extend ν_1 smoothly to $\mathbb{R}^2$ as the identity.
Denote $g_\varepsilon := \bar{g}_\varepsilon \circ \nu_1$. Because of $\nu_1 = \mathrm{id}$ on U_ε, the map g_ε coincides with $\bar{g}_\varepsilon$ on U_ε as announced. Furthermore we have

$$g_\varepsilon^* \Omega_0 = (\bar{g}_\varepsilon \circ \nu_1)^* \Omega_0 = \nu_1^* (\bar{g}_\varepsilon^* \Omega_0) = \nu_1^* \Omega_1 = \Omega_0.$$

Using the transformation formula we compute for an arbitrary measurable set $A \subseteq \mathbb{R}^2$:

$$\mu (g_\varepsilon (A)) = \int_{g_\varepsilon (A)} \Omega_0 = \int_A |\det (D g_\varepsilon)| \cdot \Omega_0.$$

We know $\det (D\nu_1) > 0$ (because $\nu_0 = \mathrm{id}$ and all the maps ν_t are diffeomorphisms) as well as $\det (D\bar{g}_\varepsilon) > 0$, and thus $|\det (Dg_\varepsilon)| = \det (Dg_\varepsilon)$. Since $g_\varepsilon^* \Omega_0 = (\det (Dg_\varepsilon)) \cdot \Omega_0$ (compare with [HK95, proposition 5.1.3.]) we finally conclude:

$$\mu (g_\varepsilon (A)) = \int_A (\det (Dg_\varepsilon)) \cdot \Omega_0 = \int_A g_\varepsilon^* \Omega_0 = \int_A \Omega_0 = \mu (A).$$

Consequently g_ε is a measure-preserving diffeomorphism on $\mathbb{R}^2$ satisfying the desired properties. □

Let $\tilde{g}_b : \mathbb{S}^1 \times [0,1]^{m-1} \to \mathbb{S}^1 \times [0,1]^{m-1}$ be the smooth measure-preserving diffeomorphism $\tilde{g}_b (\theta, r_1, ..., r_{m-1}) = (\theta + b \cdot r_1, r_1, ..., r_{m-1})$ and denote $\left[0, \frac{1}{a}\right] \times \left[0, \frac{\varepsilon}{b \cdot a}\right] \times [\delta, 1-\delta]^{m-2}$ by $\Delta_{a,b,\varepsilon,\delta}$. Using the map $D_{a,b,\varepsilon} : \mathbb{R}^m \to \mathbb{R}^m, (\theta, r_1, ..., r_{m-1}) \mapsto \left(a \cdot \theta, \frac{b \cdot a}{\varepsilon} \cdot r_1, r_2, ..., r_{m-1}\right)$ and g_ε from Lemma 3.6 we define the measure-preserving diffeomorphism $g_{a,b,\varepsilon,\delta} : \Delta_{a,b,\varepsilon,\delta} \to \tilde{g}_b (\Delta_{a,b,\varepsilon,\delta})$ by setting $g_{a,b,\varepsilon,\delta} = D^{-1}_{a,b,\varepsilon} \circ (g_\varepsilon, \mathrm{id}_{\mathbb{R}^{m-2}}) \circ D_{a,b,\varepsilon}$. Using the fact that $\frac{a \cdot b \cdot \delta}{\varepsilon} \in \mathbb{Z}$ we extend it to a smooth diffeomorphism $g_{a,b,\varepsilon,\delta} : \left[0, \frac{1}{a}\right] \times [\delta, 1-\delta]^{m-1} \to \tilde{g}_b \left(\left[0, \frac{1}{a}\right] \times [\delta, 1-\delta]^{m-1}\right)$ by the description:

$$g_{a,b,\varepsilon,\delta} \left(\theta, r_1 + l \cdot \frac{\varepsilon}{b \cdot a}, r_2, ..., r_{m-1}\right) = \left(l \cdot \frac{\varepsilon}{a}, l \cdot \frac{\varepsilon}{b \cdot a}, \vec{0}\right) + g_{a,b,\varepsilon,\delta} (\theta, r_1, ..., r_{m-1})$$

for $r_1 \in \left[0, \frac{\varepsilon}{b \cdot a}\right]$ and some $l \in \mathbb{Z}$ satisfying $\frac{b \cdot a \cdot \delta}{\varepsilon} \leq l \leq \frac{b \cdot a}{\varepsilon} - \frac{b \cdot a \cdot \delta}{\varepsilon} - 1$. Since this map coincides with the map $\tilde{g}_b$ in a neighbourhood of the boundary we can extend it to a map $g_{a,b,\varepsilon,\delta} : \left[0, \frac{1}{a}\right] \times [0,1]^{m-1} \to \tilde{g}_b \left(\left[0, \frac{1}{a}\right] \times [0,1]^{m-1}\right)$ by setting it equal to $\tilde{g}_b$ on $\left[0, \frac{1}{a}\right] \times \left([0,1]^{m-1} \setminus [\delta, 1-\delta]^{m-1}\right)$. We summarize the properties of this map as follows:

Lemma 3.7. *The constructed map* $g_{a,b,\varepsilon,\delta} : \left[0, \frac{1}{a}\right] \times [0,1]^{m-1} \to \tilde{g}_b \left(\left[0, \frac{1}{a}\right] \times [0,1]^{m-1}\right)$ *satisfies*

1. *For any set* $V \subset \left[0, \frac{1}{a}\right] \times [\delta, 1-\delta]^{m-1}$ *with* $\left[\frac{\varepsilon}{a}, \frac{1-\varepsilon}{a}\right] \subset \pi_\theta(V)$ *and* $\left[\frac{l\varepsilon + \varepsilon^2}{b \cdot a}, \frac{(l+1)\varepsilon - \varepsilon^2}{b \cdot a}\right] \subset \pi_{r_1}(V) \subset \left[\frac{l\varepsilon}{b \cdot a}, \frac{(l+1)\varepsilon}{b \cdot a}\right]$ *we have* $g_{a,b,\varepsilon,\delta}(V) = \tilde{g}_b(V)$. π_θ *and* π_{r_1} *denote the projection to the particular coordinate.*
2. *On any set* $V \subset \left[0, \frac{1}{a}\right] \times [\delta, 1-\delta]^{m-1}$ *with* $\pi_\theta(V) \subset \left[\frac{4\varepsilon}{a}, \frac{1-4\varepsilon}{a}\right]$ *and* $\pi_{r_1}(V) \subset \left[\frac{l\varepsilon + 4\varepsilon^2}{b \cdot a}, \frac{(l+1)\varepsilon - 4\varepsilon^2}{b \cdot a}\right]$ *the map* $g_{a,b,\varepsilon,\delta}$ *acts as the translation by* $\left(l \cdot \frac{\varepsilon}{a}, l \cdot \frac{\varepsilon}{b \cdot a}, \vec{0}\right)$.
3. $g_{a,b,\varepsilon,\delta}$ *coincides with* $\tilde{g}_b$ *on* $\left[0, \frac{1}{a}\right] \times \left([0,1]^{m-1} \setminus [\delta, 1-\delta]^{m-1}\right)$.

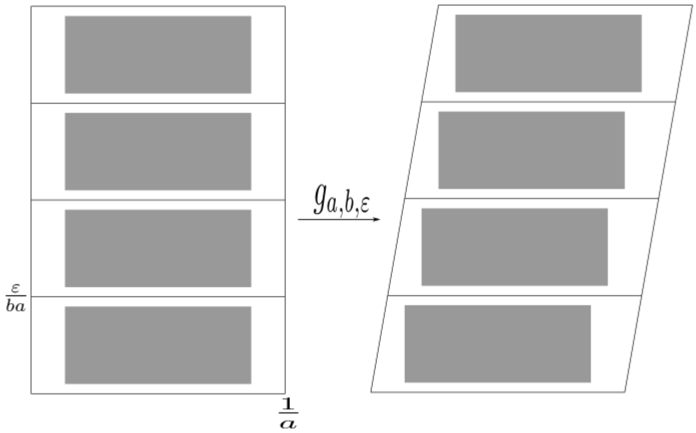

Figure 2: The action of the map $g_{a,b,\varepsilon}$.

We initially construct the smooth measure-preserving diffeomorphism g_n on the fundamental sector:

$$g_n = g_{2q_n^m,[n\cdot q_n^\sigma],\frac{1}{8n^4},\frac{1}{32n^4}}.$$

Since g_n coincides with the map $\tilde{g}_{[n\cdot q_n^\sigma]}$ in a neighbourhood of the boundary, we can extend it to a smooth measure-preserving diffeomorphism on $\mathbb{S}^1 \times [0,1]^{m-1}$ using the description $g_n \circ R_{\frac{l}{q_n}} = R_{\frac{l}{q_n}} \circ g_n$ for $l \in \mathbb{Z}$. Furthermore, we note that the subsequent constructions are done in such a way that $260n^4$ divides q_n (see Lemma 5.9) and so the assumption $\frac{a\cdot b\cdot \delta}{\varepsilon} = \frac{a\cdot b}{4} \in \mathbb{Z}$ is satisfied. Indeed, this map g_n satisfies the following useful properties:

Proposition 3.8. *The constructed map g_n satisfies:*

1. *For any set $V \subset \mathbb{S}^1 \times \left[\frac{1}{32n^4}, 1-\frac{1}{32n^4}\right]^{m-1}$ with*

$$\left[\frac{l_1+\frac{1}{8n^4}}{2q_n^m}, \frac{l_1+1-\frac{1}{8n^4}}{2q_n^m}\right] \subset \pi_\theta(V) \subset \left[\frac{l_1}{2q_n^m}, \frac{l_1+1}{2q_n^m}\right]$$
$$\text{and } \left[\frac{l_2+\frac{1}{8n^4}}{16n^4\cdot q_n^m\cdot[nq_n^\sigma]}, \frac{l_2+1-\frac{1}{8n^4}}{16n^4\cdot q_n^m\cdot[nq_n^\sigma]}\right] \subset \pi_{r_1}(V) \subset \left[\frac{l_2}{16n^4\cdot q_n^m\cdot[nq_n^\sigma]}, \frac{l_2+1}{16n^4\cdot q_n^m\cdot[nq_n^\sigma]}\right],$$

   where $l_1, l_2 \in \mathbb{Z}$, we have $g_n(V) = \tilde{g}_{[nq_n^\sigma]}(V)$.

2. *For every element $\hat{I}_n \in \eta_n$ we have $g_n\left(\hat{I}_n\right) = \tilde{g}_{[nq_n^\sigma]}\left(\hat{I}_n\right)$.*

3. *On any set $V \subset \mathbb{S}^1 \times \left[\frac{1}{32n^4}, 1-\frac{1}{32n^4}\right]^{m-1}$ with*

$$\pi_\theta(V) \subset \left[\frac{l_1+\frac{1}{2n^4}}{2q_n^m}, \frac{l_1+1-\frac{1}{2n^4}}{2q_n^m}\right] \quad \text{and} \quad \pi_{r_1}(V) \subset \left[\frac{l_2+\frac{1}{2n^4}}{16n^4\cdot q_n^m\cdot[nq_n^\sigma]}, \frac{l_2+1-\frac{1}{2n^4}}{16n^4\cdot q_n^m\cdot[nq_n^\sigma]}\right],$$

   where $l_1, l_2 \in \mathbb{Z}$, the map g_n acts as an isometry.

The first property will be used in Lemma 4.6 to show that $g_n\left(\Phi_n\left(\hat{I}_n\right)\right)=\tilde{g}_{[nq_n^\sigma]}\left(\Phi_n\left(\hat{I}_n\right)\right)$ for every $\hat{I}_n\in\eta_n$. The third one will guarantee in Lemma 7.1 that g_n acts as an isometry on $\phi_n\left(\check{I}_n\right)$ for every $\check{I}_n\in\zeta_n$.

Proof. The first property is an immediate consequence of Lemma 3.7, 1., since we have $g_n = g_{2q_n^m,[n\cdot q_n^\sigma],\frac{1}{8n^4},\frac{1}{32n^4}}$ on the domain under consideration. This definition of g_n gives us also the third property due to Lemma 3.7, 2.
In order to prove the second part we initially consider a partition element $\hat{I}_n\in\eta_n$ on $\left[0,\frac{1}{2q_n}\right]\times[0,1]^{m-1}$ and want to examine the effect of $g_n=g_{2q_n^m,[n\cdot q_n^\sigma],\frac{1}{8n^4},\frac{1}{32n^4}}$ on it. In the r_1-coordinate we use the fact that there is $u_1\in\mathbb{Z}$ such that

$$\frac{1}{26n^4q_n}=u_1\cdot\frac{\varepsilon}{b\cdot a}=u_1\cdot\frac{1}{8n^4\cdot[nq_n^\sigma]\cdot 2q_n^m},$$

where we use the fact that $260n^4$ divides q_n (Lemma 5.9). Also, with respect to the θ-coordinate there is $u_2\in\mathbb{Z}$ such that

$$\frac{1}{52n^4q_n}=u_2\cdot\frac{1}{a}=u_2\cdot\frac{1}{2q_n^m}.$$

This implies the second property with the aid of Lemma 3.7, 1. Next, we want to prove the statement for partition elements $\hat{I}_n\in\eta_n$ on $\left[\frac{1}{2q_n},\frac{1}{q_n}\right]\times[0,1]^{m-1}$. With regard to the r_1-coordinate there is $u_2\in\mathbb{Z}$ such that

$$\frac{1}{10n^4q_n}=u_2\cdot\frac{\varepsilon}{b\cdot a}=u_2\cdot\frac{1}{8n^4\cdot[nq_n^\sigma]\cdot 2q_n^m}$$

since $260n^4$ divides q_n. Considering the θ-coordinate we exploit $\frac{1}{20n^4\cdot q_n^m}<\frac{\varepsilon}{a}=\frac{1}{16n^4\cdot q_n^m}$. Then the claim follows from Lemma 3.7, 1. □

3.3 The conjugation map ϕ_n

The conjugation map ϕ_n will be composed of maps $\tilde{\phi}_{\lambda,\varepsilon,i,j,\mu,\delta,\varepsilon_2}$, where $j\in\{2,...,m\}$, $\varepsilon,\varepsilon_2\in\left(0,\frac{1}{4}\right)$ and $\lambda,\mu\in\mathbb{N}$. Moreover, $\delta\in(0,1)$ such that $\frac{1}{\delta}\in\mathbb{N}$ and $\frac{1}{\delta}$ divides μ. In the construction of the map $\tilde{\phi}_{\lambda,\varepsilon,1,j,\mu,\delta,\varepsilon_2}$ we will use maps C_λ causing a stretch by λ in the first coordinate and so-called "quasi-rotations" $\varphi_{\varepsilon,1,j}$ constructed in Lemma 3.9 with the aid of "Moser's trick" similar to [FS05, Lemma 5.3.]. With these maps we will also define a family of "inner rotations" $\psi_{\mu,\delta,1,j,\varepsilon_2}$ in order to get that $\tilde{\phi}_{\lambda,\varepsilon,1,j,\mu,\delta,\varepsilon_2}$ acts as an isometry on specific cuboids (see Proposition 3.11, 3.): A map of the form $C_\lambda^{-1}\circ\varphi_{\varepsilon,1,j}\circ C_\lambda$ as in [FS05] and [FSW07] would cause an expansion by λ in one coordinate and by λ^{-1} in another, so far away from being an isometry. The "inner rotations" will cause that C_λ and C_λ^{-1} act on the same coordinate on the elements $\check{I}_n\in\zeta_n$.

Lemma 3.9. *For every $\varepsilon\in\left(0,\frac{1}{4}\right)$ and every $i,j\in\{1,...,m\}$ there exists a smooth measure-preserving diffeomorphism $\varphi_{\varepsilon,i,j}$ on $\mathbb{R}^m$ which is the rotation in the x_i-x_j-plane by $\pi/2$ about the point $\left(\frac{1}{2},...,\frac{1}{2}\right)\in\mathbb{R}^m$ on $[2\varepsilon,1-2\varepsilon]^m$ and coincides with the identity outside of $[\varepsilon,1-\varepsilon]^m$.*

Proof. The proof is similar to the proof of Lemma 3.6. (See also [GK00, section 4.6] for a geometrical argument of the proof.) □

Furthermore, for $\lambda\in\mathbb{N}$ we define the maps $C_\lambda(x_1,x_2,...,x_m)=(\lambda\cdot x_1,x_2,...,x_m)$ and $D_\lambda(x_1,...,x_m)=(\lambda\cdot x_1,\lambda\cdot x_2,...,\lambda\cdot x_m)$. Let $\mu\in\mathbb{N}$, $\frac{1}{\delta}\in\mathbb{N}$ and assume $\frac{1}{\delta}$ divides μ. We construct a diffeomorphism $\psi_{\mu,\delta,i,j,\varepsilon_2}$ in the following way:

- Consider $[0,1-2\cdot\delta]^m$: Since $\frac{1}{\delta}$ divides μ, we can divide $[0,1-2\cdot\delta]^m$ into cubes of side length $\frac{1}{\mu}$.
- Under the map D_μ any of these cubes of the form $\prod_{i=1}^m\left[\frac{l_i}{\mu},\frac{l_i+1}{\mu}\right]$ with $l_i\in\mathbb{N}$ is mapped onto $\prod_{i=1}^m[l_i,l_i+1]$.
- On $[0,1]^m$ we will use the diffeomorphism $\varphi^{-1}_{\varepsilon_2,i,j}$ constructed in Lemma 3.9 . Since this is the identity outside of $\Delta(\varepsilon_2)$, we can extend it to a diffeomorphism $\bar{\varphi}^{-1}_{\varepsilon_2,i,j}$ on $\mathbb{R}^m$ using the instruction $\bar{\varphi}^{-1}_{\varepsilon_2,i,j}(x_1+k_1,x_2+k_2,...,x_m+k_m)=(k_1,...,k_m)+\varphi^{-1}_{\varepsilon_2,i,j}(x_1,x_2,...,x_m)$, where $k_i\in\mathbb{Z}$ and $x_i\in[0,1]$.
- Now we define the smooth measure-preserving diffeomorphism
$$\tilde{\psi}_{\mu,\delta,i,j,\varepsilon_2}=D_\mu^{-1}\circ\bar{\varphi}^{-1}_{\varepsilon_2,i,j}\circ D_\mu \quad : \quad [0,1-2\delta]^m\to[0,1-2\delta]^m$$
- With this we define
$$\psi_{\mu,\delta,i,j,\varepsilon_2}(x_1,...,x_m)=$$
$$\begin{cases}\left(\left[\tilde{\psi}_{\mu,\delta,i,j,\varepsilon_2}(x_1-\delta,...,x_m-\delta)\right]_1+\delta,...,\left[\tilde{\psi}_{\mu,\delta,i,j,\varepsilon_2}(x_1-\delta,...,x_m-\delta)\right]_m+\delta\right) & \text{on } [\delta,1-\delta]^m\\ (x_1,...,x_m) & \text{otherwise}\end{cases}$$
This is a smooth map because $\tilde{\psi}_{\mu,\delta,i,j,\varepsilon_2}$ is the identity in a neighbourhood of the boundary by the construction.

Remark 3.10. For every set $W=\prod_{i=1}^m\left[\frac{l_i}{\mu}+r_i,\frac{l_i+1}{\mu}-r_i\right]$ where $l_i\in\mathbb{Z}$ and $r_i\in\mathbb{R}$ satisfies $|r_i\cdot\mu|\leq\varepsilon_2$ we have $\psi_{\mu,\delta,i,j,\varepsilon_2}(W)=W$.

Using these maps we build the following smooth measure-preserving diffeomorphism:
$$\tilde{\phi}_{\lambda,\varepsilon,i,j,\mu,\delta,\varepsilon_2}:\left[0,\frac{1}{\lambda}\right]\times[0,1]^{m-1}\to\left[0,\frac{1}{\lambda}\right]\times[0,1]^{m-1},\quad \tilde{\phi}_{\lambda,\varepsilon,i,j,\mu,\delta,\varepsilon_2}=C_\lambda^{-1}\circ\psi_{\mu,\delta,i,j,\varepsilon_2}\circ\varphi_{\varepsilon,i,j}\circ C_\lambda$$
Afterwards, $\tilde{\phi}_{\lambda,\varepsilon,i,j,\mu,\delta,\varepsilon_2}$ is extended to a diffeomorphism on $\mathbb{S}^1\times[0,1]^{m-1}$ by the description $\tilde{\phi}_{\lambda,\varepsilon,i,j,\mu,\delta,\varepsilon_2}\left(x_1+\frac{1}{\lambda},x_2,...,x_m\right)=\left(\frac{1}{\lambda},0,...,0\right)+\tilde{\phi}_{\lambda,\varepsilon,i,j,\mu,\delta,\varepsilon_2}(x_1,x_2,...,x_m)$. This map satisfies the following properties:

Proposition 3.11. *Let $j\in\{2,...,m\}$, $\varepsilon,\varepsilon_2\in\left(0,\frac{1}{4}\right)$, $2\varepsilon<\varepsilon_2$, and $\lambda,\mu\in\mathbb{N}$. Moreover, let $\delta\in(0,1)$ such that $2\varepsilon_2<\delta$, $\frac{1}{\delta}\in\mathbb{N}$ and $\frac{1}{\delta}$ divides μ. Then there is a smooth measure-preserving $\frac{1}{\lambda}$-equivariant diffeomorphism $\tilde{\phi}_{\lambda,\varepsilon,1,j,\mu,\delta,\varepsilon_2}:\mathbb{S}^1\times[0,1]^{m-1}\to\mathbb{S}^1\times[0,1]^{m-1}$ such that*

1. *Let $t_s\in\mathbb{Z}$, $\lceil 2\varepsilon\mu\rceil\leq t_s\leq\mu-\lceil 2\varepsilon\mu\rceil-1$, for $s=1,...,m$ and $|u_s|\leq\varepsilon_2$ for $s=0,...,m$. Then we have*
$$\tilde{\phi}^{-1}_{\lambda,\varepsilon,1,j,\mu,\delta,\varepsilon_2}\left(\left[\frac{t_1+u_0}{\lambda\mu},\frac{1}{\lambda}-\frac{t_1+u_1}{\lambda\mu}\right]\times\prod_{s=2}^m\left[\frac{t_s+u_s}{\mu},\frac{t_s+1-u_s}{\mu}\right]\right)$$
$$=\left[\frac{t_j+u_j}{\lambda\mu},\frac{t_j+1-u_j}{\lambda\mu}\right]\times\prod_{s=2}^{j-1}\left[\frac{t_s+u_s}{\mu},\frac{t_s+1-u_s}{\mu}\right]$$
$$\times\left[\frac{t_1+u_1}{\mu},1-\frac{t_1+u_0}{\mu}\right]\times\prod_{s=j+1}^{m}\left[\frac{t_s+u_s}{\mu},\frac{t_s+1-u_s}{\mu}\right]$$

2. Let $t_s \in \mathbb{Z}$, $\lceil \delta\mu \rceil \leq t_s \leq \mu - \lceil \delta\mu \rceil - 1$ for $s = 1, ..., m$ and V be contained in

$$\left[\frac{t_1 + 2\varepsilon_2}{\lambda\mu}, \frac{t_1 + 1 - 2\varepsilon_2}{\lambda\mu}\right] \times \prod_{i=2}^{m} \left[\frac{t_i + 2\varepsilon_2}{\mu}, \frac{t_i + 1 - 2\varepsilon_2}{\mu}\right].$$

When applying $\tilde{\phi}_{\lambda,\varepsilon,1,j,\mu,\delta,\varepsilon_2}$ on V the occurring maps $\varphi_{\varepsilon,1,j}$ and $\varphi^{-1}_{\varepsilon_2,1,j}$ act as the respective rotations.

3. $\tilde{\phi}_{\lambda,\varepsilon,1,j,\mu,\delta,\varepsilon_2}$ acts as an isometry on each cuboid

$$\left[\frac{t_1 + 2\varepsilon_2}{\lambda\mu}, \frac{t_1 + 1 - 2\varepsilon_2}{\lambda\mu}\right] \times \prod_{s=2}^{m} \left[\frac{t_s + 2\varepsilon_2}{\mu}, \frac{t_s + 1 - 2\varepsilon_2}{\mu}\right],$$

where $t_s \in \mathbb{Z}$, $\lceil 2\varepsilon\mu \rceil \leq t_s \leq \mu - \lceil 2\varepsilon\mu \rceil - 1$, for $s = 1, ..., m$.

For convenience we will use the notation $\tilde{\phi}^{(j)}_{\lambda,\mu} = \tilde{\phi}_{\lambda,\frac{1}{60n^4},1,j,\mu,\frac{1}{10n^4},\frac{1}{22n^4}}$. With this we define the diffeomorphism ϕ_n on the fundamental sector:

- On $\left[0, \frac{1}{2 \cdot q_n}\right] \times [0,1]^{m-1}$ we put

$$\phi_n = \tilde{\phi}^{(m)}_{2q_n,q_n} \circ \tilde{\phi}^{(m-1)}_{2q_n^2,q_n} \circ ... \circ \tilde{\phi}^{(2)}_{2q_n^{m-1},q_n}.$$

- On $\left[\frac{1}{2 \cdot q_n}, \frac{1}{q_n}\right] \times [0,1]^{m-1}$ we put

$$\phi_n = \mathrm{id}$$

This is a smooth map because ϕ_n coincides with the identity in a neighbourhood of the different sections.
Now we extend ϕ_n to a diffeomorphism on $\mathbb{S}^1 \times [0,1]^{m-1}$ using the description $\phi_n \circ R_{\frac{1}{q_n}} = R_{\frac{1}{q_n}} \circ \phi_n$.

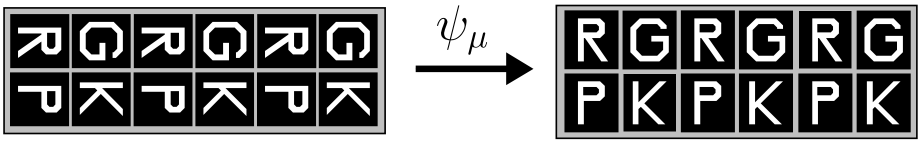

Figure 3: The map ψ_μ has the useful property of rotating several small cuboids individually while being the identity outside of a neighborhood of them.

We summarize the useful properties of the constructed map ϕ_n in the subsequent Proposition.

Proposition 3.12. *The smooth measure-preserving diffeomorphism ϕ_n satisfies:*

1. By applying ϕ_n^{-1} on any partition element $\hat{I}_n \in \eta_n$ of the form

$$\left[\frac{1}{52n^4 \cdot q_n}, \frac{1}{2q_n} - \frac{1}{52n^4 \cdot q_n}\right] \times \prod_{i=2}^{m} \left[\frac{j_i}{q_n} + \frac{1}{26n^4 \cdot q_n}, \frac{j_i + 1}{q_n} - \frac{1}{26n^4 \cdot q_n}\right],$$

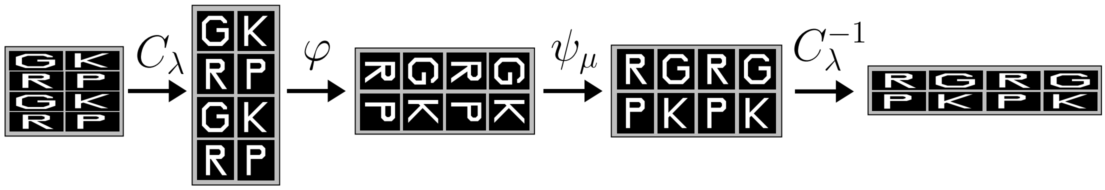

Figure 4: The map ϕ_n is constructed as concatenation of a stretch map C_λ, a rotation φ, the map ψ_μ mentioned before, and C_λ^{-1} (the inverse of the stretch map). The map thus constructed has the very useful property of stretching a cuboid (illustrated here by the underlying grey rectangle) in one direction (similar to what a hyperbolic map would do), yet it is almost an isometry on all of the smaller cuboids (illustrated here by black squares with letters). In particular, a partition element $\hat{I} \in \eta_n$ (the leftmost grey rectangle) is mapped to a set that has size almost 1 in one of its coordinates.

we get

$$\left[\frac{j_m}{2q_n^2} + \frac{j_{m-1}}{2q_n^3} + \cdots + \frac{j_2}{2q_n^m} + \frac{1}{52n^4 \cdot q_n^m}, \frac{j_m}{2q_n^2} + \frac{j_{m-1}}{2q_n^3} + \cdots + \frac{j_2+1}{2q_n^m} - \frac{1}{52n^4 \cdot q_n^m}\right]$$
$$\times \left[\frac{1}{26n^4}, 1 - \frac{1}{26n^4}\right]^{m-1}.$$

2. *Let* $j_i \in \mathbb{Z}$, $\lceil \frac{q_n}{10n^4} \rceil \leq j_i \leq q_n - \lceil \frac{q_n}{10n^4} \rceil - 1$, *for* $i = 2, ..., m$ *and* $j_1^{(l)} \in \mathbb{Z}$, $\lceil \frac{q_n}{10n^4} \rceil \leq j_1^{(l)} \leq q_n - \lceil \frac{q_n}{10n^4} \rceil - 1$, *for* $l = 1, ..., m-1$ *and* $u_0, u_1 \geq \frac{1}{11n^4}$. *Then* ϕ_n *maps*

$$\left[\frac{j_1^{(1)}}{2q_n^2} + \cdots + \frac{j_1^{(m-1)}}{2q_n^m} + \frac{u_0}{2q_n^m}, \frac{j_1^{(1)}}{2q_n^2} + \cdots + \frac{j_1^{(m-1)}+1}{2q_n^m} - \frac{u_1}{2q_n^m}\right]$$
$$\times \prod_{i=2}^{m} \left[\frac{j_i}{q_n} + \frac{1}{10n^4 \cdot q_n}, \frac{j_i+1}{q_n} - \frac{1}{10n^4 \cdot q_n}\right],$$

to

$$\left[\frac{1}{2q_n} - \frac{j_m}{2q_n^2} - \cdots - \frac{j_3}{2q_n^{m-1}} - \frac{j_2+1}{2q_n^m} + \frac{1}{20n^4 \cdot q_n^m} + a_n,\right.$$
$$\left.\frac{1}{2q_n} - \frac{j_m}{2q_n^2} - \cdots - \frac{j_3}{2q_n^{m-1}} - \frac{j_2}{2q_n^m} - \frac{1}{20n^4 \cdot q_n^m} + a_n\right]$$
$$\times \prod_{i=2}^{m} \left[\frac{j_1^{(m+1-i)}}{q_n} + \frac{1}{10n^4 \cdot q_n}, \frac{j_1^{(m+1-i)}+1}{q_n} - \frac{1}{10n^4 \cdot q_n}\right].$$

3. ϕ_n *acts as an isometry on every* $\check{I}_n \in \zeta_n$.

Proof. All these properties are immediate consequences of the corresponding statements in Proposition 3.11, the choice of parameters in the definition of ϕ_n and the positions of the partition elements. $\square$

The first two properties will enable us to prove in Lemma 4.5 that $\Phi_n = \phi_n \circ R^{m_n}_{\alpha_{n+1}} \circ \phi_n^{-1}$ maps partition elements $\hat{I}_n \in \eta_n$ almost uniformly in the $r_1,\dots,r_{m-1}$-coordinates. With the aid of the third statement we will show in Lemma 7.1 that h_n acts as an isometry on any element of ζ_n.

4 (γ,ϵ)-distribution

We introduce the central notion for the proof of the weak mixing-property in section 6:

Definition 4.1. Let $\Phi : M \to M$ be a diffeomorphism and $J \subset [0,1]^{m-1}$. We say that an element $\hat{I}$ of a partial partition is (γ,ϵ)-distributed on J under Φ if the following properties are satisfied:

- $\Phi\left(\hat{I}\right)$ is contained in a set of the form $[c, c+\gamma] \times [0,1]^{m-1}$ for some $c \in \mathbb{S}^1$.
- $\pi_{\vec{r}}\left(\Phi\left(\hat{I}\right)\right) \supset J$. Here $\pi_{\vec{r}}$ denotes the projection on the $(r_1,\dots,r_{m-1})$-coordinates (i.e., the last $m-1$ coordinates; the first one is the θ-coordinate).
- For every $(m-1)$-dimensional interval $\tilde{J} \subseteq J$ it holds:
$$\left|\frac{\mu\left(\hat{I}\cap\Phi^{-1}\left(\mathbb{S}^1\times\tilde{J}\right)\right)}{\mu\left(\hat{I}\right)} - \frac{\mu^{(m-1)}\left(\tilde{J}\right)}{\mu^{(m-1)}(J)}\right| \leq \epsilon\cdot\frac{\mu^{(m-1)}\left(\tilde{J}\right)}{\mu^{(m-1)}(J)},$$
where $\mu^{(m-1)}$ is the Lebesgue measure on $[0,1]^{m-1}$.

Remark 4.2. Analogous to [FS05] we will call the third property "almost uniform distribution" of $\hat{I}$ in the $r_1,..,r_{m-1}$-coordinates. In the following we will often write it in the form
$$\left|\mu\left(\hat{I}\cap\Phi^{-1}\left(\mathbb{S}^1\times\tilde{J}\right)\right)\cdot\mu^{(m-1)}(J) - \mu\left(\hat{I}\right)\cdot\mu^{(m-1)}\left(\tilde{J}\right)\right| \leq \epsilon\cdot\mu\left(\hat{I}\right)\cdot\mu^{(m-1)}\left(\tilde{J}\right).$$

In the next step we define the sequence of natural numbers $(m_n)_{n\in\mathbb{N}}$:
$$\begin{aligned} m_n &= \min\left\{m \leq q_{n+1} \ : \ m\in\mathbb{N},\ \inf_{k\in\mathbb{Z}}\left|m\cdot\frac{p_{n+1}}{q_{n+1}} - \frac{1}{2\cdot q_n} + \frac{k}{q_n}\right| \leq \frac{260\cdot(n+1)^4}{q_{n+1}}\right\} \\ &= \min\left\{m \leq q_{n+1} \ : \ m\in\mathbb{N},\ \inf_{k\in\mathbb{Z}}\left|m\cdot\frac{q_n\cdot p_{n+1}}{q_{n+1}} - \frac{1}{2} + k\right| \leq \frac{260\cdot(n+1)^4\cdot q_n}{q_{n+1}}\right\}\end{aligned}$$

Lemma 4.3. *The set* $\left\{m \leq q_{n+1} \ : \ m\in\mathbb{N},\ \inf_{k\in\mathbb{Z}}\left|m\cdot\frac{q_n\cdot p_{n+1}}{q_{n+1}} - \frac{1}{2} + k\right| \leq \frac{260(n+1)^4\cdot q_n}{q_{n+1}}\right\}$ *is nonempty for every* $n\in\mathbb{N}$*, i.e.,* m_n *exists.*

Proof. In Lemma 5.9 we will construct the sequence $\alpha_n = \frac{p_n}{q_n}$ in such a way that $q_n = 260n^4\cdot\tilde{q}_n$ and $p_n = 260n^4\cdot\tilde{p}_n$ with $\tilde{p}_n, \tilde{q}_n$ relatively prime. Therefore, the set $\left\{j\cdot\frac{q_n\cdot p_{n+1}}{q_{n+1}} \ : \ j = 1,\dots,q_{n+1}\right\}$ contains $\frac{q_{n+1}}{260(n+1)^4\cdot\gcd(q_n,\tilde{q}_{n+1})}$ different equally distributed points on $\mathbb{S}^1$. Hence there are at least $\frac{q_{n+1}}{260(n+1)^4\cdot q_n}$ different such points and so for every $x\in\mathbb{S}^1$ there is a $j\in\{1,\dots,q_{n+1}\}$ such that $\inf_{k\in\mathbb{Z}}\left|x - j\cdot\frac{q_n\cdot p_{n+1}}{q_{n+1}} + k\right| \leq \frac{260(n+1)^4\cdot q_n}{q_{n+1}}$. In particular, this is true for $x = \frac{1}{2}$. □

Remark 4.4. We define

$$a_n = \left(m_n \cdot \frac{p_{n+1}}{q_{n+1}} - \frac{1}{2 \cdot q_n} \right) \mod \frac{1}{q_n}$$

By the above construction of m_n it holds that $|a_n| \leq \frac{260 \cdot (n+1)^4}{q_{n+1}}$. In Lemma 5.9 we will see that it is possible to choose $q_{n+1} \geq 80 \cdot 260 \cdot (n+1)^4 \cdot n^4 \cdot q_n^m$. Thus, we get:

$$|a_n| \leq \frac{1}{80 \cdot n^4 \cdot q_n^m}.$$

By this choice of the number m_n, $R^{m_n}_{\alpha_{n+1}}$ causes a translation to the different domain of definition of the map ϕ_n. In order to deal with partition elements of the second kind we introduce the so-called "good set" $J_n \subset [0,1]^{m-1}$ in the $\vec{r}$-coordinates:

$$J_n = \bigcup \prod_{i=1}^{m-1} \left[\frac{t_i}{q_n} + \frac{1}{10n^4 \cdot q_n}, \frac{t_i+1}{q_n} - \frac{1}{10n^4 \cdot q_n} \right], \tag{1}$$

where the union is taken over all $t_i \in \mathbb{Z}$, $\left\lceil \frac{q_n}{10n^4} \right\rceil \leq t_i \leq q_n - \left\lceil \frac{q_n}{10n^4} \right\rceil - 1$, for $i = 1, ..., m-1$. Altogether, the following property is satisfied by our constructions:

Lemma 4.5. *We consider the map $\Phi_n := \phi_n \circ R^{m_n}_{\alpha_{n+1}} \circ \phi_n^{-1}$ with the conjugating maps ϕ_n defined in section 3.3.*

1. *Under Φ_n the elements of the partition η_n of the first kind are $\left(\frac{1}{2q_n^m}, \frac{1}{n}\right)$-distributed on $J = \prod_{i=2}^{m} \left[\frac{1}{26n^4}, 1 - \frac{1}{26n^4}\right]$.*
2. *The elements of the partition η_n of the second kind are $\left(\frac{1}{2q_n^m}, \frac{1}{n}\right)$-distributed on J_n under Φ_n.*

Proof. We consider a partition element $\hat{I}_{n,1}$ on $\left[0, \frac{1}{2q_n}\right] \times [0,1]^{m-1}$. Then we compute $\phi_n^{-1}\left(\hat{I}_{n,1}\right)$ with the aid of Proposition 3.12, 1.:

$$\left[\frac{j_m}{2q_n^2} + \frac{j_{m-1}}{2q_n^3} + \dots + \frac{j_2}{2q_n^m} + \frac{1}{52n^4 \cdot q_n^m}, \frac{j_m}{2q_n^2} + \frac{j_{m-1}}{2q_n^3} + \dots + \frac{j_2+1}{2q_n^m} - \frac{1}{52n^4 \cdot q_n^m} \right] \times \prod_{i=2}^{m} \left[\frac{1}{26n^4}, 1 - \frac{1}{26n^4} \right].$$

By our choice of the number m_n the subsequent application of $R^{m_n}_{\alpha_{n+1}}$ yields a translation by $\frac{1}{2q_n}$ modulo $\frac{1}{q_n}$ except for the "error term" a_n introduced in Remark 4.4. In particular, $R^{m_n}_{\alpha_{n+1}} \circ \phi_n^{-1}\left(\hat{I}_{n,1}\right)$ is positioned in another domain of definition of the map ϕ_n, namely we have $\phi_n = \mathrm{id}$. Hence, $\Phi_n\left(\hat{I}_{n,1}\right)$ is equal to

$$\left[\frac{1}{2q_n} + \frac{j_m}{2q_n^2} + \frac{j_{m-1}}{2q_n^3} + \dots + \frac{j_2}{2q_n^m} + \frac{1}{52n^4 \cdot q_n^m} + a_n, \frac{1}{2q_n} + \frac{j_m}{2q_n^2} + \frac{j_{m-1}}{2q_n^3} + \dots + \frac{j_2+1}{2q_n^m} - \frac{1}{52n^4 \cdot q_n^m} + a_n \right]$$
$$\times \prod_{i=2}^{m} \left[\frac{1}{26n^4}, 1 - \frac{1}{26n^4} \right].$$

Thus, such a set $\Phi_n\left(\hat{I}_{n,1}\right)$ has a θ-width of at most $\frac{1}{2q_n^m}$. Moreover, we see $\pi_{\vec{r}}\left(\Phi_n\left(\hat{I}_{n,1}\right)\right) = \prod_{i=2}^{m} \left[\frac{1}{26n^4}, 1 - \frac{1}{26n^4}\right] = J$. With the notation $A_\theta := \pi_\theta\left(\Phi_n\left(\hat{I}_{n,1}\right)\right)$ we have $\Phi_n\left(\hat{I}_{n,1}\right) = A_\theta \times J$

and so for every $(m-1)$-dimensional interval $\tilde{J} \subseteq J$:

$$\frac{\mu\left(\hat{I}_{n,1} \cap \Phi_n^{-1}\left(\mathbb{S}^1 \times \tilde{J}\right)\right)}{\mu\left(\hat{I}_{n,1}\right)} = \frac{\mu\left(\Phi_n\left(\hat{I}_{n,1}\right) \cap \mathbb{S}^1 \times \tilde{J}\right)}{\mu\left(\Phi_n\left(\hat{I}_{n,1}\right)\right)} = \frac{\tilde{\lambda}\left(A_\theta\right) \cdot \mu^{(m-1)}\left(\tilde{J}\right)}{\tilde{\lambda}\left(A_\theta\right) \cdot \mu^{(m-1)}\left(J\right)} = \frac{\mu^{(m-1)}\left(\tilde{J}\right)}{\mu^{(m-1)}\left(J\right)}$$

because Φ_n is measure-preserving. Hence, we can choose $\epsilon = 0$ in the definition of a (γ, ϵ)-distribution.

In order to prove the second statement, we consider a partition element $\hat{I}_{n,2}$ on $\left[\frac{1}{2q_n}, \frac{1}{q_n}\right] \times [0,1]^{m-1}$. Since ϕ_n^{-1} acts as the identity on it and $R^{m_n}_{\alpha_{n+1}}$ yields a translation by $\frac{1}{2q_n}$ modulo $\frac{1}{q_n}$ except for the "error term" a_n, we can compute the image of $\hat{I}_{n,2}$ under $R^{m_n}_{\alpha_{n+1}} \circ \phi_n^{-1}$:

$$\bigcup \left[\frac{j_1^{(1)}}{2q_n^2} + \dots + \frac{j_1^{(m-1)}}{2q_n^m} + \frac{1}{20n^4 \cdot q_n^m} + a_n, \frac{j_1^{(1)}}{2q_n^2} + \dots + \frac{j_1^{(m-1)}+1}{2q_n^m} - \frac{1}{20n^4 \cdot q_n^m} + a_n\right]$$
$$\times \prod_{i=2}^{m} \left[\frac{j_i}{q_n} + \frac{1}{10n^4 \cdot q_n}, \frac{j_i+1}{q_n} - \frac{1}{10n^4 \cdot q_n}\right].$$

Applying $\phi_n = \tilde{\phi}^{(m)}_{2q_n,q_n} \circ \tilde{\phi}^{(m-1)}_{2q_n^2,q_n} \circ \dots \circ \tilde{\phi}^{(2)}_{2q_n^{m-1},q_n}$ yields due to Proposition 3.12, 2., and the bounds on a_n in Remark 4.4:

$$\bigcup \left[\frac{1}{2q_n} - \frac{j_m}{2q_n^2} - \dots - \frac{j_3}{2q_n^{m-1}} - \frac{j_2+1}{2q_n^m} + \frac{1}{20n^4 \cdot q_n^m} + a_n, \frac{1}{2q_n} - \frac{j_m}{2q_n^2} - \dots - \frac{j_2}{2q_n^m} - \frac{1}{20n^4 \cdot q_n^m} + a_n\right]$$
$$\times \prod_{i=2}^{m} \left[\frac{j_1^{(m+1-i)}}{q_n} + \frac{1}{10n^4 \cdot q_n}, \frac{j_1^{(m+1-i)}+1}{q_n} - \frac{1}{10n^4 \cdot q_n}\right].$$

Obviously, $\pi_{\vec{r}}\left(\Phi_n\left(\hat{I}_{n,1}\right)\right) = J_n$. By the same calculations as above we can choose $\epsilon = 0$ in the definition of a (γ, ϵ)-distribution . □

Furthermore, we show the next property concerning the conjugating map g_n constructed in section 3.2:

Lemma 4.6. *For every $\hat{I}_n \in \eta_n$ we have: $g_n\left(\Phi_n\left(\hat{I}_n\right)\right) = \tilde{g}_{[nq_n^\sigma]}\left(\Phi_n\left(\hat{I}_n\right)\right)$.*

Proof. In the proof of the precedent Lemma 4.5 we computed $\Phi_n\left(\hat{I}_{n,k}\right)$ for partition elements $\hat{I}_{n,k}$ of both kinds. Now we have to examine the effect of $g_n = g_{2q_n^m,[n\cdot q_n^\sigma],\frac{1}{8n^4},\frac{1}{32n^4}}$ on it.
Since $260n^4$ divides q_n by Lemma 5.9, there is $u_1 \in \mathbb{Z}$ such that

$$\frac{1}{26n^4} = u_1 \cdot \frac{\varepsilon}{b \cdot a} = u_1 \cdot \frac{1}{8n^4 \cdot [nq_n^\sigma] \cdot 2q_n^m}.$$

By the bound on a_n we have

$$\frac{1}{52n^4 q_n^m} + a_n < \frac{\varepsilon}{a} = \frac{1}{16n^4 \cdot q_n^m}$$

and so the boundary of $\Phi_n\left(\hat{I}_{n,1}\right)$ lies in the domain where $g_{2q_n^m,[n\cdot q_n^\sigma],\frac{1}{8n^4},\frac{1}{32n^4}} = \tilde{g}_{[nq_n^\sigma]}$ according to Proposition 3.8, 1.

Similarly, we examine the action of g_n on $\Phi_n\left(\hat{I}_{n,2}\right)$. Since $260n^4$ divides q_n by Lemma 5.9, there is $u_2\in\mathbb{Z}$ such that

$$\frac{1}{10n^4\cdot q_n}=u_2\cdot\frac{\varepsilon}{b\cdot a}=u_2\cdot\frac{1}{8n^4\cdot[nq_n^\sigma]\cdot 2q_n^m}.$$

By the bound on a_n the boundary of $\Phi_n\left(\hat{I}_{n,2}\right)$ lies in the domain where $g_{2q_n^m,[n\cdot q_n^\sigma],\frac{1}{8n^4},\frac{1}{32n^4}}=\tilde{g}_{[nq_n^\sigma]}$ according to Proposition 3.8, 1, once again.
Additionally we observe

$$\mu\left(\Phi_n\left(\hat{I}_{n,k}\right)\right)\geq\frac{1}{a}\cdot\left(1-\frac{1}{5n^4}\right)^{2m-1}. \tag{2}$$

□

5 Convergence of $(f_n)_{n\in\mathbb{N}}$ in Diff$^\infty(M)$

In the following we show that the sequence of constructed measure-preserving smooth diffeomorphisms $f_n=H_n\circ R_{\alpha_{n+1}}\circ H_n^{-1}$ converges. For this purpose, we need a couple of results concerning the conjugation maps.

5.1 Properties of the conjugation maps ϕ_n and H_n

In order to find estimates on the norms $|||H_n|||_k$ we will need the next technical result which is an application of the chain rule:

Lemma 5.1. *Let $\phi:=\tilde{\phi}^{(m)}_{\lambda_m,\mu_m}\circ...\circ\tilde{\phi}^{(2)}_{\lambda_2,\mu_2}$, $j\in\{1,...,m\}$ and $k\in\mathbb{N}$. For any multi-index $\vec{a}$ with $|\vec{a}|=k$ the partial derivative $D_{\vec{a}}[\phi]_j$ consists of a sum of products of at most $(m-1)\cdot k$ terms of the form*

$$D_{\vec{b}}\left(\left[\tilde{\phi}^{(i)}_{\lambda_i,\mu_i}\right]_l\right)\circ\tilde{\phi}^{(i-1)}_{\lambda_{i-1},\mu_{i-1}}\circ...\circ\tilde{\phi}^{(2)}_{\lambda_2,\mu_2},$$

where $l\in\{1,...,m\}$, $i\in\{2,...,m\}$ and $\vec{b}$ is a multi-index with $\left|\vec{b}\right|\leq k$.

In the same way we obtain a similar statement holding for the inverses:

Lemma 5.2. *Let $\psi:=\left(\tilde{\phi}^{(2)}_{\lambda_2,\mu_2}\right)^{-1}\circ...\circ\left(\tilde{\phi}^{(m)}_{\lambda_m,\mu_m}\right)^{-1}$, $j\in\{1,...,m\}$ and $k\in\mathbb{N}$. For any multi-index $\vec{a}$ with $|\vec{a}|=k$ the partial derivative $D_{\vec{a}}[\psi]_j$ consists of a sum of products of at most $(m-1)\cdot k$ terms of the following form*

$$D_{\vec{b}}\left(\left[\left(\tilde{\phi}^{(i)}_{\lambda_i,\mu_i}\right)^{-1}\right]_l\right)\circ\left(\tilde{\phi}^{(i+1)}_{\lambda_{i+1},\mu_{i+1}}\right)^{-1}\circ...\circ\left(\tilde{\phi}^{(m)}_{\lambda_m,\mu_m}\right)^{-1},$$

where $l\in\{1,...,m\}$, $i\in\{2,...,m\}$ and $\vec{b}$ is a multi-index with $\left|\vec{b}\right|\leq k$.

Remark 5.3. In the proof of the following lemmas we will use the formula of Faà di Bruno in several variables. It can be found in the paper *"A multivariate Faà di Bruno formula with applications"* ([CS96]) for example.
For this we introduce an ordering on $\mathbb{N}_0^d$: For multiindices $\vec{\mu}=(\mu_1,...,\mu_d)$ and $\vec{\nu}=(\nu_1,...,\nu_d)$ in $\mathbb{N}_0^d$ we will write $\vec{\mu}\prec\vec{\nu}$, if one of the following properties is satisfied:

1. $|\vec{\mu}|<|\vec{\nu}|$, where $|\vec{\mu}|=\sum_{i=1}^d\mu_i$.

2. $|\vec{\mu}| = |\vec{\nu}|$ and $\mu_1 < \nu_1$.

3. $|\vec{\mu}| = |\vec{\nu}|$, $\mu_i = \nu_i$ for $1 \leq i \leq k$ and $\mu_{k+1} < \nu_{k+1}$ for some $1 \leq k < d$.

In other words, we compare by order and then lexicographically. Additionally we will use these notations:

- For $\vec{\nu} = (\nu_1, ..., \nu_d) \in \mathbb{N}_0^d$:

$$\vec{\nu}! = \prod_{i=1}^{d} \nu_i!$$

- For $\vec{\nu} = (\nu_1, ..., \nu_d) \in \mathbb{N}_0^d$ and $\vec{z} = (z_1, ..., z_d) \in \mathbb{R}^d$:

$$\vec{z}^{\,\vec{\nu}} = \prod_{i=1}^{d} z_i^{\nu_i}$$

Then we get for the composition $h(x_1, ..., x_d) := f\left(g^{(1)}(x_1, ..., x_d), ..., g^{(m)}(x_1, ..., x_d)\right)$ with sufficiently differentiable functions $f : \mathbb{R}^m \to \mathbb{R}$, $g^{(i)} : \mathbb{R}^d \to \mathbb{R}$ and a multi-index $\vec{\nu} \in \mathbb{N}_0^d$ with $|\vec{\nu}| = n$:

$$D_{\vec{\nu}} h = \sum_{\vec{\lambda} \in \mathbb{N}_0^m \text{ with } 1 \leq |\vec{\lambda}| \leq n} D_{\vec{\lambda}} f \cdot \sum_{s=1}^{n} \sum_{p_s(\vec{\nu}, \vec{\lambda})} \vec{\nu}! \cdot \prod_{j=1}^{s} \frac{\left[D_{\vec{l}_j} \vec{g}\right]^{\vec{k}_j}}{\vec{k}_j! \cdot \left(\vec{l}_j!\right)^{|\vec{k}_j|}}$$

Here $\left[D_{\vec{l}_j} \vec{g}\right]$ denotes $\left(D_{\vec{l}_j} g^{(1)}, ..., D_{\vec{l}_j} g^{(m)}\right)$ and

$$p_s\left(\vec{\nu}, \vec{\lambda}\right) :=$$

$$\left\{\left(\vec{k}_1, ..., \vec{k}_s, \vec{l}_1, ..., \vec{l}_s\right) : \vec{k}_i \in \mathbb{N}_0^m, \left|\vec{k}_i\right| > 0, \vec{l}_i \in \mathbb{N}_0^d, 0 \prec \vec{l}_1 \prec ... \prec \vec{l}_s, \sum_{i=1}^{s} \vec{k}_i = \vec{\lambda} \text{ and } \sum_{i=1}^{s} \left|\vec{k}_i\right| \cdot \vec{l}_i = \vec{\nu}\right\}$$

With the aid of these technical results we can prove an estimate on the norms of the map ϕ_n:

Lemma 5.4. *For every $k \in \mathbb{N}$ it holds that*

$$|||\phi_n|||_k \leq C \cdot q_n^{m^2 \cdot k},$$

where C is a constant depending on m, k and n, but is independent of q_n.

Proof. First of all we consider the map $\tilde{\phi}_{\lambda,\mu} := \tilde{\phi}_{\lambda,\varepsilon,i,j,\mu,\delta,\varepsilon_2} = C_{\lambda}^{-1} \circ \psi_{\mu,\delta,i,j,\varepsilon_2} \circ \varphi_{\varepsilon,i,j} \circ C_{\lambda}$ introduced in subsection 3.3:

$$\tilde{\phi}_{\lambda,\mu}(x_1, ..., x_m) =$$

$$\left(\frac{1}{\lambda}\left[\psi_\mu \circ \varphi_\varepsilon\right]_1 (\lambda x_1, x_2, ..., x_m), \left[\psi_\mu \circ \varphi_\varepsilon\right]_2 (\lambda x_1, x_2, ..., x_m), ..., \left[\psi_\mu \circ \varphi_\varepsilon\right]_m (\lambda x_1, x_2, ..., x_m)\right).$$

Let $k \in \mathbb{N}$. We compute for a multi-index $\vec{a}$ with $0 \leq |\vec{a}| \leq k$: $\left\|D_{\vec{a}}\left[\tilde{\phi}_{\lambda,\mu}\right]_1\right\|_0 \leq \lambda^{k-1} \cdot |||\psi_\mu \circ \varphi_\varepsilon|||_k$ and for $r \in \{2, ..., m\}$: $\left\|D_{\vec{a}}\left[\tilde{\phi}_{\lambda,\mu}\right]_r\right\|_0 \leq \lambda^k \cdot |||\psi_\mu \circ \varphi_\varepsilon|||_k$.
Therefore, we examine the map ψ_μ. For any multi-index $\vec{a}$ with $0 \leq |\vec{a}| \leq k$ and $r \in \{1, ..., m\}$ we

obtain: $\left\|D_{\vec{a}}\left[\psi_\mu\right]_r\right\|_0 \leq \mu^{k-1}\cdot|||\varphi_{\varepsilon_2}|||_k = C_{k,\varepsilon_2}\cdot\mu^{k-1}$ and analogously $\left\|D_{\vec{a}}\left[\psi_\mu^{-1}\right]_r\right\|_0 \leq C_{k,\varepsilon_2}\cdot\mu^{k-1}$.
Hence: $|||\psi_\mu|||_k \leq C\cdot\mu^{k-1}$.
In the next step we use the formula of Faà di Bruno mentioned in remark 5.3. With it we compute for any multi-index $\vec{\nu}$ with $|\vec{\nu}| = k$:

$$
\begin{aligned}
\left\|D_{\vec{\nu}}\left[\left(\psi_\mu\circ\varphi_\varepsilon\right)^{-1}\right]_r\right\|_0 &= \left\|D_{\vec{\nu}}\left[\varphi_\varepsilon^{-1}\circ\psi_\mu^{-1}\right]_r\right\|_0 \\
&= \left\|\sum_{\vec{\lambda}\in\mathbb{N}_0^m, 1\leq|\vec{\lambda}|\leq k} D_{\vec{\lambda}}\left[\varphi_\varepsilon^{-1}\right]_r \sum_{s=1}^{k} \sum_{(\vec{k}_1,\dots,\vec{k}_s,\vec{l}_1,\dots,\vec{l}_s)\in p_s(\vec{\nu},\vec{\lambda})} \vec{\nu}! \prod_{j=1}^{s} \frac{\left[D_{\vec{l}_j}\psi_\mu^{-1}\right]^{\vec{k}_j}}{\vec{k}_j!\cdot\left(\vec{l}_j!\right)^{|\vec{k}_j|}}\right\|_0 \\
&= \left\|\sum_{\vec{\lambda}\in\mathbb{N}_0^m, 1\leq|\vec{\lambda}|\leq k} D_{\vec{\lambda}}\left[\varphi_\varepsilon^{-1}\right]_r \cdot \sum_{s=1}^{k} \sum_{p_s(\vec{\nu},\vec{\lambda})} \vec{\nu}! \cdot \prod_{j=1}^{s} \frac{\prod_{t=1}^{m}\left(D_{\vec{l}_j}\left[\psi_\mu^{-1}\right]_t\right)^{\vec{k}_{j_t}}}{\vec{k}_j!\cdot\left(\vec{l}_j!\right)^{|\vec{k}_j|}}\right\|_0 \\
&\leq \sum_{\vec{\lambda}\in\mathbb{N}_0^m, 1\leq|\vec{\lambda}|\leq k} \left\|D_{\vec{\lambda}}\left[\varphi_\varepsilon^{-1}\right]_r\right\|_0 \cdot \sum_{s=1}^{k} \sum_{p_s(\vec{\nu},\vec{\lambda})} \vec{\nu}! \cdot \prod_{j=1}^{s} \frac{\prod_{t=1}^{m}\left\|D_{\vec{l}_j}\left[\psi_\mu^{-1}\right]_t\right\|_0^{\vec{k}_{j_t}}}{\vec{k}_j!\cdot\left(\vec{l}_j!\right)^{|\vec{k}_j|}} \\
&\leq \sum_{\vec{\lambda}\in\mathbb{N}_0^m \text{ with } 1\leq|\vec{\lambda}|\leq k} \left\|D_{\vec{\lambda}}\left[\varphi_\varepsilon^{-1}\right]_r\right\|_0 \cdot \sum_{s=1}^{k} \sum_{p_s(\vec{\nu},\vec{\lambda})} \vec{\nu}! \cdot \prod_{j=1}^{s} \frac{|||\psi_\mu^{-1}|||_{|\vec{l}_j|}^{\sum_{t=1}^{m}\vec{k}_{j_t}}}{\vec{k}_j!\cdot\left(\vec{l}_j!\right)^{|\vec{k}_j|}} \\
&= \sum_{\vec{\lambda}\in\mathbb{N}_0^m \text{ with } 1\leq|\vec{\lambda}|\leq k} \left\|D_{\vec{\lambda}}\left[\varphi_\varepsilon^{-1}\right]_r\right\|_0 \cdot \sum_{s=1}^{k} \sum_{p_s(\vec{\nu},\vec{\lambda})} \vec{\nu}! \cdot \prod_{j=1}^{s} \frac{|||\psi_\mu^{-1}|||_{|\vec{l}_j|}^{|\vec{k}_j|}}{\vec{k}_j!\cdot\left(\vec{l}_j!\right)^{|\vec{k}_j|}}
\end{aligned}
$$

As seen above: $|||\psi_\mu^{-1}|||_{|\vec{l}_j|}^{|\vec{k}_j|} \leq C\cdot\mu^{|\vec{k}_j|\cdot|\vec{l}_j|}$. Hereby: $\prod_{j=1}^{s}|||\psi_\mu^{-1}|||_{|\vec{l}_j|}^{|\vec{k}_j|} \leq \hat{C}\cdot\mu^{\sum_{j=1}^{s}|\vec{l}_j|\cdot|\vec{k}_j|}$, where $\hat{C}$ is independent of μ. By definition of the set $p_s\left(\vec{\nu},\vec{\lambda}\right)$ we have $\sum_{i=1}^{s}\left|\vec{k}_i\right|\cdot\vec{l}_i = \vec{\nu}$. Hence:

$$
k = |\vec{\nu}| = \left|\sum_{i=1}^{s}\left|\vec{k}_i\right|\cdot\vec{l}_i\right| = \sum_{t=1}^{m}\left(\sum_{i=1}^{s}\left|\vec{k}_i\right|\cdot\vec{l}_i\right)_t = \sum_{t=1}^{m}\sum_{i=1}^{s}\left|\vec{k}_i\right|\cdot\vec{l}_{i_t} = \sum_{i=1}^{s}\left|\vec{k}_i\right|\cdot\left(\sum_{t=1}^{m}\vec{l}_{i_t}\right) = \sum_{i=1}^{s}\left|\vec{k}_i\right|\cdot\left|\vec{l}_i\right|
$$

This shows $\prod_{j=1}^{s}|||\psi_\mu^{-1}|||_{|\vec{l}_j|}^{|\vec{k}_j|} \leq \hat{C}\cdot\mu^k$ and finally $\left\|D_{\vec{\nu}}\left[\left(\psi_\mu\circ\varphi_\varepsilon\right)^{-1}\right]_r\right\|_0 \leq C\cdot\mu^k$. Analogously we compute $\left\|D_{\vec{\nu}}\left[\psi_\mu\circ\varphi_\varepsilon\right]_r\right\|_0 \leq C\cdot|||\psi_\mu|||_k \leq C\cdot\mu^{k-1}$. Altogether, we obtain $|||\psi_\mu\circ\varphi_\varepsilon|||_k \leq C\cdot\mu^k$. Hereby, we estimate $\left\|D_{\vec{a}}\left[\tilde{\phi}_{\lambda,\mu}\right]_r\right\|_0 \leq C\cdot\lambda^k\cdot\mu^k$ and analogously $\left\|D_{\vec{a}}\left[\tilde{\phi}_{\lambda,\mu}^{-1}\right]_r\right\|_0 \leq C\cdot\lambda^k\cdot\mu^k$. In conclusion this yields $|||\tilde{\phi}_{\lambda,\mu}|||_k \leq C\cdot\mu^k\cdot\lambda^k$.
In the next step we consider $\phi := \tilde{\phi}_{\lambda_m,\mu_m}^{(m)}\circ\dots\circ\tilde{\phi}_{\lambda_2,\mu_2}^{(2)}$. Let $\lambda_{\max} := \max\{\lambda_2,\dots,\lambda_m\}$ as well as $\mu_{\max} := \max\{\mu_2,\dots,\mu_m\}$. Inductively we will show $|||\phi|||_k \leq \tilde{C}\cdot\lambda_{\max}^{(m-1)\cdot k}\cdot\mu_{\max}^{(m-1)\cdot k}$ for every $k\in\mathbb{N}$, where $\tilde{C}$ is a constant independent of λ_i and μ_i.
Start: $k = 1$
Let $l\in\{1,\dots,m\}$ be arbitrary. By Lemma 5.1 a partial derivative of $[\phi]_l$ of first order consists of a sum of products of at most $m-1$ first order partial derivatives of functions $\tilde{\phi}_{\lambda_j,\mu_j}^{(j)}$. Therewith,

we obtain using $|||\tilde{\phi}^{(j)}_{\lambda_j,\mu_j}|||_1 \leq C\cdot\lambda_{\max}\cdot\mu_{\max}$ the estimate $\|D_i[\phi]_l\|_0 \leq C_1\cdot\lambda^{m-1}_{\max}\cdot\mu^{m-1}_{\max}$ for every $i\in\{1,...,m\}$, where C_1 is a constant independent of λ and μ.
With the aid of Lemma 5.2 we obtain the same statement for $\phi^{-1} = \left(\tilde{\phi}^{(2)}_{\lambda_2,\mu_2}\right)^{-1}\circ...\circ\left(\tilde{\phi}^{(m)}_{\lambda_m,\mu_m}\right)^{-1}$.
Hence, we conclude: $|||\phi|||_1 \leq \tilde{C}_1\cdot\lambda^{m-1}_{\max}\cdot\mu^{m-1}_{\max}$.
Assumption: The claim is true for $k\in\mathbb{N}$.
Induction step $k\to k+1$:
In the proof of Lemma 5.1 one observes that at the transition $k\to k+1$ in the product of at most $(m-1)\cdot k$ terms of the form $D_{\vec{b}}\left(\left[\tilde{\phi}^{(i)}_{\lambda_i,\mu_i}\right]_l\right)\circ\tilde{\phi}^{(i-1)}_{\lambda_{i-1},\mu_{i-1}}\circ...\circ\tilde{\phi}^{(2)}_{\lambda_2,\mu_2}$ one is replaced by a product of a term $\left(D_jD_{\vec{b}}\left[\tilde{\phi}^{(i)}_{\lambda_i,\mu_i}\right]_l\right)\circ\tilde{\phi}^{(i-1)}_{\lambda_{i-1},\mu_{i-1}}\circ...\circ\tilde{\phi}^{(2)}_{\lambda_2,\mu_2}$ with $j\in\{1,...,m\}$ and at most $m-2$ partial derivatives of first order. Because of $|||\tilde{\phi}^{(i)}_{\lambda_i,\mu_i}|||_{k+1}\leq C\cdot\lambda^{k+1}_{\max}\cdot\mu^{k+1}_{\max}$ and $|||\tilde{\phi}^{(j)}_{\lambda_j,\mu_j}|||_1\leq C\cdot\lambda_{\max}\cdot\mu_{\max}$ the $\lambda_{\max}$-exponent as well as the $\mu_{\max}$-exponent increase by at most $1+(m-2)\cdot 1 = m-1$.
In the same spirit one uses the proof of Lemma 5.2 to show that also in case of ϕ^{-1} the $\lambda_{\max}$-exponent as well as the $\mu_{\max}$-exponent increase by at most $m-1$.
Using the assumption we conclude

$$|||\phi|||_{k+1}\leq\hat{C}\cdot\lambda^{k\cdot(m-1)+m-1}_{\max}\cdot\mu^{k\cdot(m-1)+m-1}_{\max}=\hat{C}\cdot\lambda^{(k+1)\cdot(m-1)}_{\max}\cdot\mu^{(k+1)\cdot(m-1)}_{\max}.$$

So the proof by induction is completed.

In the setting of our explicit construction of the map ϕ_n in section 3.3 we have $\varepsilon_1=\frac{1}{60\cdot n^4}$, $\varepsilon_2=\frac{1}{22\cdot n^4}$, $\lambda_{\max}=2q_n^{m-1}$ and $\mu_{\max}=q_n$. Thus:

$$\begin{aligned}|||\phi_n|||_k &\leq \tilde{C}(m,k,n)\cdot\left(2q_n^{m-1}\right)^{(m-1)\cdot k}\cdot q_n^{(m-1)\cdot k}\\ &\leq C(m,k,n)\cdot q_n^{m^2\cdot k},\end{aligned}$$

where $C(m,k,n)$ is a constant independent of q_n. □

In the next step we consider the map $h_n=g_n\circ\phi_n$, where g_n is constructed in section 3.2:

Lemma 5.5. *For every $k\in\mathbb{N}$ it holds:*

$$|||h_n|||_k\leq\bar{C}\cdot q_n^{2\cdot m^2\cdot k},$$

where $\bar{C}$ is a constant depending on m, k and n, but is independent of q_n.

Proof. Outside of $\mathbb{S}^1\times[\delta,1-\delta]^{m-1}$, i.e. $g_n=\tilde{g}_{[nq_n^{\sigma}]}$, we have:

$$\begin{aligned}&h_n(x_1,...,x_m)=g_n\circ\phi_n(x_1,...,x_m)\\ &=\left([\phi_n(x_1,...,x_m)]_1+[n\cdot q_n^{\sigma}]\cdot[\phi_n(x_1,...,x_m)]_2,[\phi_n(x_1,...,x_m)]_2,...,[\phi_n(x_1,...,x_m)]_m\right)\end{aligned}$$

and

$$\begin{aligned}&h_n^{-1}(x_1,...,x_m)=\phi_n^{-1}\circ g_n^{-1}(x_1,...,x_m)\\ &=\left(\left[\phi_n^{-1}(x_1-[n\cdot q_n^{\sigma}]\cdot x_2,x_2,...,x_m)\right]_1,...,[\phi_n(x_1-[n\cdot q_n^{\sigma}]\cdot x_2,x_2,...,x_m)]_m\right).\end{aligned}$$

Since $\sigma<1$ we can estimate:

$$|||h_n|||_k\leq 2\cdot[n\cdot q_n^{\sigma}]^k\cdot|||\phi_n|||_k\leq\bar{C}(m,k,n)\cdot q_n^{\sigma\cdot k}\cdot q_n^{m^2\cdot k}\leq\bar{C}(m,k,n)\cdot q_n^{2\cdot m^2\cdot k}$$

with a constant $\bar{C}(m,k,n)$ independent of q_n.
In the other case we have

$$g_n \circ \phi_n(x_1,...,x_m) = \left([g_{a,b,\varepsilon}([\phi_n]_1,[\phi_n]_2)]_1,[g_{a,b,\varepsilon}([\phi_n]_1,[\phi_n]_2)]_2,[\phi_n]_3,...,[\phi_n]_m\right).$$

We will use the formula of Faà di Bruno as above for any multi-index $\vec{\nu}$ with $|\vec{\nu}| = k$ and $r \in \{1,...,m\}$:

$$\left\|D_{\vec{\nu}}[h_n]_r\right\|_0 = \left\|D_{\vec{\nu}}[g_{a,b,\varepsilon}\circ\phi_n]_r\right\|_0$$
$$\leq \sum_{\vec{\lambda}\in\mathbb{N}_0^m \text{ with } 1\leq|\vec{\lambda}|\leq k} \left\|D_{\vec{\lambda}}[g_{a,b,\varepsilon}]_r\right\|_0 \cdot \sum_{s=1}^{k}\sum_{p_s(\vec{\nu},\vec{\lambda})}\vec{\nu}!\cdot\prod_{j=1}^{s}\frac{|||\phi_n|||_{|\vec{l}_j|}^{|\vec{k}_j|}}{\vec{k}_j!\cdot\left(\vec{l}_j!\right)^{|\vec{k}_j|}}$$

By Lemma 5.4 we have $|||\phi_n|||_k \leq C\cdot q_n^{m^2\cdot k}$, where C is a constant independent of q_n. As above we show $\prod_{j=1}^{s}|||\phi_n|||_{|\vec{l}_j|}^{|\vec{k}_j|} \leq \hat{C}\cdot q_n^{\left(\sum_{j=1}^{s}|\vec{l}_j|\cdot|\vec{k}_j|\right)\cdot m^2} = \hat{C}\cdot q_n^{m^2\cdot k}$, where $\hat{C}$ is a constant independent of q_n.
Furthermore, we examine the map $g_{a,b,\varepsilon,\delta} = D_{a,b,\varepsilon}^{-1}\circ g_\varepsilon\circ D_{a,b,\varepsilon}$ for $a,b\in\mathbb{Z}$ and obtain

$$|||g_{a,b,\varepsilon,\delta}|||_k \leq \left(\frac{b\cdot a}{\varepsilon}\right)^k\cdot|||g_\varepsilon|||_k = C_{\varepsilon,k}\cdot b^k\cdot a^k.$$

By our constructions in section 3.2 we have $b = [n\cdot q_n^\sigma] \leq n\cdot q_n^\sigma$, $a = 2q_n^m$ and $\varepsilon = \frac{1}{8n^4}$. Hence: $|||g_n|||_k \leq C_{n,k}\cdot q_n^{\sigma\cdot k}\cdot q_n^{k\cdot m} \leq C_{n,k}\cdot q_n^{k\cdot(m+1)}$. Finally, we conclude: $\left\|D_{\vec{\nu}}[h_n]_r\right\|_0 \leq C\cdot q_n^{k\cdot(m+1)}\cdot q_n^{k\cdot m^2} \leq C\cdot q_n^{2\cdot k\cdot m^2}$.
In the next step we consider $h_n^{-1} = \phi_n^{-1}\circ g_{a,b,\varepsilon}^{-1}$. For $r\in\{1,...,m\}$ and any multi-index $\vec{\nu}$ with $|\vec{\nu}| = k$ we obtain using the formula of Faà di Bruno again:

$$\left\|D_{\vec{\nu}}\left[h_n^{-1}\right]_r\right\|_0 = \left\|D_{\vec{\nu}}\left[\phi_n^{-1}\circ g_n^{-1}\right]_r\right\|_0$$
$$\leq \sum_{\vec{\lambda}\in\mathbb{N}_0^m \text{ with } 1\leq|\vec{\lambda}|\leq k} \left\|D_{\vec{\lambda}}\left[\phi_n^{-1}\right]_r\right\|_0 \cdot \sum_{s=1}^{k}\sum_{p_s(\vec{\nu},\vec{\lambda})}\vec{\nu}!\cdot\prod_{j=1}^{s}\frac{|||g_n|||_{|\vec{l}_j|}^{|\vec{k}_j|}}{\vec{k}_j!\cdot\left(\vec{l}_j!\right)^{|\vec{k}_j|}}$$

As above we show $\prod_{j=1}^{s}|||g_n|||_{|\vec{l}_j|}^{|\vec{k}_j|} \leq \hat{C}\cdot q_n^{k\cdot(m+1)}$, where $\hat{C}$ is a constant independent of q_n. Since $|||\phi_n|||_k \leq C\cdot q_n^{k\cdot m^2}$ we get

$$\left\|D_{\vec{\nu}}\left[h_n^{-1}\right]_r\right\|_0 \leq \check{C}\cdot q_n^{k\cdot(m+1)}\cdot q_n^{k\cdot m^2} \leq \check{C}\cdot q_n^{2\cdot k\cdot m^2},$$

where $\check{C}$ is a constant independent of q_n.
Thus, we finally obtain $|||h_n|||_k \leq C(n,k,m)\cdot q_n^{2\cdot m^2\cdot k}$. □

Finally, we are able to prove an estimate on the norms of the map H_n:

Lemma 5.6. *For every $k\in\mathbb{N}$ we get:*

$$|||H_n|||_k \leq \check{C}\cdot q_n^{2\cdot m^2\cdot k},$$

where $\check{C}$ is a constant depending solely on m, k, n and H_{n-1}. Since H_{n-1} is independent of q_n in particular, the same is true for $\check{C}$.

Proof. Let $k \in \mathbb{N}$, $r \in \{1, ..., m\}$ and $\vec{\nu} \in \mathbb{N}_0^m$ be a multi-index with $|\vec{\nu}| = k$. As above we estimate:

$$\begin{aligned}\|D_{\vec{\nu}}[H_n]_r\|_0 &= \|D_{\vec{\nu}}[H_{n-1}\circ h_n]_r\|_0 \\ &\leq \sum_{\vec{\lambda}\in\mathbb{N}_0^m \text{ with } 1\leq|\vec{\lambda}|\leq k} \left\|D_{\vec{\lambda}}[H_{n-1}]_r\right\|_0 \cdot \sum_{s=1}^{k} \sum_{p_s(\vec{\nu},\vec{\lambda})} \vec{\nu}! \cdot \prod_{j=1}^{s} \frac{|||h_n|||_{|\vec{l}_j|}^{|\vec{k}_j|}}{\vec{k}_j! \cdot \left(\vec{l}_j!\right)^{|\vec{k}_j|}}\end{aligned}$$

and compute using Lemma 5.5: $\prod_{j=1}^{s} |||h_n|||_{|\vec{l}_j|}^{|\vec{k}_j|} \leq \hat{C} \cdot q_n^{2\cdot m^2\cdot k}$, where $\hat{C}$ is a constant independent of q_n. Since H_{n-1} is independent of q_n we conclude:

$$\|D_{\vec{\nu}}[H_n]_r\|_0 \leq \check{C} \cdot q_n^{2\cdot m^2\cdot k},$$

where $\check{C}$ is a constant independent of q_n.
In the same way we prove an analogous estimate of $\left\|D_{\vec{\nu}}\left[H_n^{-1}\right]_r\right\|_0$ and verify the claim. □

In particular, we see that this norm can be estimated by a power of q_n.

5.2 Proof of convergence

For the proof of the convergence of the sequence $(f_n)_{n\in\mathbb{N}}$ in the Diff$^\infty(M)$-topology the next result, that can be found in [FSW07, Lemma 4], is very useful.

Lemma 5.7. *Let $k \in \mathbb{N}_0$ and h be a C^∞-diffeomorphism on M. Then we get for every $\alpha, \beta \in \mathbb{R}$:*

$$d_k\left(h\circ R_\alpha\circ h^{-1}, h\circ R_\beta\circ h^{-1}\right) \leq C_k \cdot |||h|||_{k+1}^{k+1} \cdot |\alpha-\beta|,$$

where the constant C_k depends solely on k and m. In particular $C_0 = 1$.

In the following Lemma we show that under some assumptions on the sequence $(\alpha_n)_{n\in\mathbb{N}}$ the sequence $(f_n)_{n\in\mathbb{N}}$ converges to $f \in \mathcal{A}_\alpha(M)$ in the Diff$^\infty(M)$-topology. Afterwards, we will show that we can fulfil these conditions (see Lemma 5.9).

Lemma 5.8 (Criterion for Diff$^\infty$-Convergence)**.** *Let $\varepsilon > 0$ be arbitrary and $(k_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers satisfying $\sum_{n=1}^{\infty} \frac{1}{k_n} < \varepsilon$. Furthermore, we assume that in our constructions the following conditions are fulfilled:*

$$|\alpha-\alpha_1| < \varepsilon \quad \textit{and} \quad |\alpha-\alpha_n| \leq \frac{1}{2\cdot k_n \cdot C_{k_n}\cdot |||H_n|||_{k_n+1}^{k_n+1}} \textit{ for every } n\in\mathbb{N},$$

where C_{k_n} are the constants from Lemma 5.7.

1. *Then the sequence of diffeomorphisms $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ converges in the Diff$^\infty(M)$-topology to a measure-preserving smooth diffeomorphism f, for which $d_\infty(f, R_\alpha) < 3\cdot\varepsilon$ holds.*

2. *Also the sequence of diffeomorphisms $\hat{f}_n = H_n \circ R_\alpha \circ H_n^{-1} \in \mathcal{A}_\alpha(M)$ converges to f in the Diff$^\infty(M)$-topology. Hence $f \in \mathcal{A}_\alpha(M)$.*

Proof. 1. According to our construction it holds $h_n \circ R_{\alpha_n} = R_{\alpha_n} \circ h_n$ and hence

$$\begin{aligned} f_{n-1} &= H_{n-1} \circ R_{\alpha_n} \circ H_{n-1}^{-1} = H_{n-1} \circ R_{\alpha_n} \circ h_n \circ h_n^{-1} \circ H_{n-1}^{-1} \\ &= H_{n-1} \circ h_n \circ R_{\alpha_n} \circ h_n^{-1} \circ H_{n-1}^{-1} = H_n \circ R_{\alpha_n} \circ H_n^{-1}. \end{aligned}$$

Applying Lemma 5.7 we obtain for every $k, n \in \mathbb{N}$:
(3)

$$d_k(f_n, f_{n-1}) = d_k\left(H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}, H_n \circ R_{\alpha_n} \circ H_n^{-1}\right) \leq C_k \cdot |||H_n|||_{k+1}^{k+1} \cdot |\alpha_{n+1} - \alpha_n|$$

In section 2.2 we assumed $|\alpha - \alpha_n| \overset{n\to\infty}{\longrightarrow} 0$ monotonically. Using the triangle inequality we obtain $|\alpha_{n+1} - \alpha_n| \leq |\alpha_{n+1} - \alpha| + |\alpha - \alpha_n| \leq 2 \cdot |\alpha - \alpha_n|$ and therefore equation (3) becomes:

$$d_k(f_n, f_{n-1}) \leq C_k \cdot |||H_n|||_{k+1}^{k+1} \cdot 2 \cdot |\alpha_n - \alpha|.$$

By the assumptions of this Lemma it follows for every $k \leq k_n$:

$$(4)\quad d_k(f_n, f_{n-1}) \leq d_{k_n}(f_n, f_{n-1}) \leq C_{k_n} \cdot |||H_n|||_{k_n+1}^{k_n+1} \cdot 2 \cdot \frac{1}{2 \cdot k_n \cdot C_{k_n} \cdot |||H_n|||_{k_n+1}^{k_n+1}} \leq \frac{1}{k_n}$$

In the next step we show that for arbitrary $k \in \mathbb{N}$ $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{Diff}^k(M)$, i.e. $\lim_{n,m\to\infty} d_k(f_n, f_m) = 0$. For this purpose, we calculate:

$$(5)\qquad \lim_{n\to\infty} d_k(f_n, f_m) \leq \lim_{n\to\infty} \sum_{i=m+1}^{n} d_k(f_i, f_{i-1}) = \sum_{i=m+1}^{\infty} d_k(f_i, f_{i-1}).$$

We consider the limit process $m \to \infty$, i.e. we can assume $k \leq k_m$ and obtain from equations (4) and (5):

$$\lim_{n,m\to\infty} d_k(f_n, f_m) \leq \lim_{m\to\infty} \sum_{i=m+1}^{\infty} \frac{1}{k_i} = 0.$$

Since $\mathrm{Diff}^k(M)$ is complete, the sequence $(f_n)_{n\in\mathbb{N}}$ converges consequently in $\mathrm{Diff}^k(M)$ for every $k \in \mathbb{N}$. Thus, the sequence converges in $\mathrm{Diff}^{\infty}(M)$ by definition.

Furthermore, we estimate:

$$(6)\qquad d_\infty(R_\alpha, f) = d_\infty\left(R_\alpha, \lim_{n\to\infty} f_n\right) \leq d_\infty(R_\alpha, R_{\alpha_1}) + \sum_{n=1}^{\infty} d_\infty(f_n, f_{n-1}),$$

where we used the notation $f_0 = R_{\alpha_1}$.
By explicit calculations we obtain $d_k(R_\alpha, R_{\alpha_1}) = d_0(R_\alpha, R_{\alpha_1}) = |\alpha - \alpha_1|$ for every $k \in \mathbb{N}$, hence

$$d_\infty(R_\alpha, R_{\alpha_1}) = \sum_{k=1}^{\infty} \frac{|\alpha - \alpha_1|}{2^k \cdot (1 + d_k(R_\alpha, R_{\alpha_1}))} \leq |\alpha - \alpha_1| \cdot \sum_{k=1}^{\infty} \frac{1}{2^k} = |\alpha - \alpha_1|.$$

Additionally it holds:

$$\begin{aligned} \sum_{n=1}^{\infty} d_\infty(f_n, f_{n-1}) &= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{d_k(f_n, f_{n-1})}{2^k \cdot (1 + d_k(f_n, f_{n-1}))} \\ &= \sum_{n=1}^{\infty} \left(\sum_{k=1}^{k_n} \frac{d_k(f_n, f_{n-1})}{2^k \cdot (1 + d_k(f_n, f_{n-1}))} + \sum_{k=k_n+1}^{\infty} \frac{d_k(f_n, f_{n-1})}{2^k \cdot (1 + d_k(f_n, f_{n-1}))} \right) \end{aligned}$$

As seen above $d_k(f_n, f_{n-1}) \leq \frac{1}{k_n}$ for every $k \leq k_n$. Hereby, it follows further:

$$\sum_{n=1}^{\infty} d_{\infty}(f_n, f_{n-1}) \leq \sum_{n=1}^{\infty}\left(\frac{1}{k_n}\cdot\sum_{k=1}^{k_n}\frac{1}{2^k} + \sum_{k=k_n+1}^{\infty}\frac{d_k(f_n, f_{n-1})}{2^k\cdot(1+d_k(f_n, f_{n-1}))}\right)$$
$$\leq \sum_{n=1}^{\infty}\frac{1}{k_n} + \sum_{n=1}^{\infty}\sum_{k=k_n+1}^{\infty}\frac{1}{2^k}.$$

Because of $\sum_{k=k_n+1}^{\infty}\frac{1}{2^k} = 2 - \sum_{k=0}^{k_n}\frac{1}{2^k} = \left(\frac{1}{2}\right)^{k_n} \leq \frac{1}{k_n}$ we conclude:

$$\sum_{n=1}^{\infty} d_{\infty}(f_n, f_{n-1}) \leq \sum_{n=1}^{\infty}\frac{1}{k_n} + \sum_{n=1}^{\infty}\frac{1}{k_n} < 2\cdot\varepsilon.$$

Hence, using equation (6) we obtain the desired estimate $d_{\infty}(f, R_{\alpha}) < 3\cdot\varepsilon$.

2. We have to show: $\hat{f}_n \to f$ in $\mathrm{Diff}^{\infty}(M)$.
For it we compute with the aid of Lemma 5.7 for every $n \in \mathbb{N}$ and $k \leq k_n$:

$$d_k\left(f_n, \hat{f}_n\right) \leq d_{k_n}\left(H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}, H_n \circ R_{\alpha} \circ H_n^{-1}\right)$$
$$\leq C_{k_n}\cdot |||H_n|||_{k_n+1}^{k_n+1}\cdot|\alpha_{n+1}-\alpha| \leq C_{k_n}\cdot|||H_n|||_{k_n+1}^{k_n+1}\cdot|\alpha_n-\alpha|$$
$$\leq C_{k_n}\cdot|||H_n|||_{k_n+1}^{k_n+1}\cdot\frac{1}{2\cdot k_n\cdot C_{k_n}\cdot|||H_n|||_{k_n+1}^{k_n+1}} = \frac{1}{2\cdot k_n} \leq \frac{1}{k_n}.$$

Fix some $k \in \mathbb{N}$.
Claim: $\forall\delta > 0\ \ \exists N\ \ \forall n \geq N:\ \ d_k\left(f, \hat{f}_n\right) < \delta$, i.e. $\hat{f}_n \to f$ in $\mathrm{Diff}^k(M)$.
Proof: Let $\delta > 0$ be given. Since $f_n \to f$ in $\mathrm{Diff}^{\infty}(M)$ we have $f_n \to f$ in $\mathrm{Diff}^k(M)$ in particular. Hence, there is $n_1 \in \mathbb{N}$, such that $d_k(f, f_n) < \frac{\delta}{2}$ for every $n \geq n_1$. Because of $k_n \to \infty$ we conclude the existence of $n_2 \in \mathbb{N}$, such that $\frac{1}{k_n} < \frac{\delta}{2}$ for every $n \geq n_2$, as well as the existence of $n_3 \in \mathbb{N}$, such that $k_n \geq k$ for every $n \geq n_3$. Then we obtain for every $n \geq \max\{n_1, n_2, n_3\}$:

$$d_k\left(f, \hat{f}_n\right) \leq d_k(f, f_n) + d_k\left(f_n, \hat{f}_n\right) < \frac{\delta}{2} + d_{k_n}\left(f_n, \hat{f}_n\right) \leq \frac{\delta}{2} + \frac{1}{k_n} < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Hence, the claim is proven.

In the next step we show: $\lim_{n\to\infty} d_{\infty}\left(\hat{f}_n, f\right) = 0$. For this purpose, we examine:

$$d_{\infty}\left(f_n, \hat{f}_n\right) = \sum_{k=1}^{k_n}\frac{d_k\left(f_n, \hat{f}_n\right)}{2^k\cdot\left(1+d_k\left(f_n, \hat{f}_n\right)\right)} + \sum_{k=k_n+1}^{\infty}\frac{d_k\left(f_n, \hat{f}_n\right)}{2^k\cdot\left(1+d_k\left(f_n, \hat{f}_n\right)\right)}$$
$$\leq \frac{1}{k_n}\cdot\sum_{k=1}^{k_n}\frac{1}{2^k} + \sum_{k=k_n+1}^{\infty}\frac{1}{2^k} \leq \frac{1}{k_n} + \left(\frac{1}{2}\right)^{k_n}.$$

Consequently $\lim_{n\to\infty} d_\infty\left(f_n, \hat{f}_n\right) = 0$. With it we compute:

$$\begin{aligned}\lim_{n\to\infty} d_\infty\left(f, \hat{f}_n\right) &= \lim_{n\to\infty} d_\infty\left(\lim_{m\to\infty} f_m, \hat{f}_n\right) = \lim_{n\to\infty}\lim_{m\to\infty} d_\infty\left(f_m, \hat{f}_n\right)\\ &\leq \lim_{n\to\infty}\lim_{m\to\infty}\left(\sum_{i=n+1}^{m} d_\infty\left(f_i, f_{i-1}\right) + d_\infty\left(f_n, \hat{f}_n\right)\right)\\ &= \lim_{n\to\infty}\sum_{i=n+1}^{\infty} d_\infty\left(f_i, f_{i-1}\right) + \lim_{n\to\infty} d_\infty\left(f_n, \hat{f}_n\right) = 0.\end{aligned}$$

As asserted we obtain: $\lim_{n\to\infty} d_\infty\left(\hat{f}_n, f\right) = 0$.

□

As announced we show that we can satisfy the conditions from Lemma 5.8 in our constructions:

Lemma 5.9. *Let $(k_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers with $\sum_{n=1}^{\infty}\frac{1}{k_n} < \infty$ and C_{k_n} be the constants from Lemma 5.7. For any Liouvillean number α there exists a sequence $\alpha_n = \frac{p_n}{q_n}$ of rational numbers with the property that $260n^4$ divides q_n, such that our conjugation maps H_n constructed in section 3.2 and 3.3 fulfil the following conditions:*

1. *For every $n\in\mathbb{N}$:*
$$|\alpha - \alpha_n| < \frac{1}{2\cdot k_n\cdot C_{k_n}\cdot |||H_n|||_{k_n+1}^{k_n+1}}.$$
2. *For every $n\in\mathbb{N}$:*
$$|\alpha - \alpha_n| < \frac{1}{2^{n+1}\cdot q_n\cdot |||H_n|||_1}.$$
3. *For every $n\in\mathbb{N}$:*
$$\|DH_{n-1}\|_0 < \frac{\ln(q_n)}{n}.$$

Proof. In Lemma 5.6 we saw $|||H_n|||_{k_n+1} \leq \check{C}_n\cdot q_n^{2\cdot m^2\cdot(k_n+1)}$, where the constant $\check{C}_n$ was independent of q_n. Thus, we can choose $q_n \geq \check{C}_n$ for every $n\in\mathbb{N}$. Hence, we obtain: $|||H_n|||_{k_n+1} \leq q_n^{3\cdot m^2\cdot(k_n+1)}$.
Besides $q_n \geq \check{C}_n$ we keep the mentioned condition $q_n \geq 80\cdot 260\cdot n^4\cdot(n-1)^4\cdot q_{n-1}^m$ in mind. Furthermore, we can demand $\|DH_{n-1}\|_0 < \frac{\ln(q_n)}{n}$ from q_n because H_{n-1} is independent of q_n. Since α is a Liouvillean number, we find a sequence of rational numbers $\tilde{\alpha}_n = \frac{\tilde{p}_n}{\tilde{q}_n}$, $\tilde{p}_n, \tilde{q}_n$ relatively prime, under the above restrictions (formulated for $\tilde{q}_n$) satisfying:

$$|\alpha - \tilde{\alpha}_n| = \left|\alpha - \frac{\tilde{p}_n}{\tilde{q}_n}\right| < \frac{|\alpha - \alpha_{n-1}|}{2^{n+1}\cdot k_n\cdot C_{k_n}\cdot(260n^4)^{1+3\cdot m^2\cdot(k_n+1)^2}\cdot\tilde{q}_n^{1+3\cdot m^2\cdot(k_n+1)^2}}$$

Put $q_n := 260n^4\cdot\tilde{q}_n$ and $p_n := 260n^4\cdot\tilde{p}_n$. Then we obtain:

$$|\alpha - \alpha_n| < \frac{|\alpha - \alpha_{n-1}|}{2^{n+1}\cdot k_n\cdot C_{k_n}\cdot q_n^{1+3\cdot m^2\cdot(k_n+1)^2}}.$$

So we have $|\alpha - \alpha_n| \overset{n\to\infty}{\longrightarrow} 0$ monotonically. Because of $|||H_n|||_{k_n+1}^{k_n+1} \leq q^{3\cdot m^2\cdot(k_n+1)^2}$ this yields: $|\alpha-\alpha_n| < \frac{1}{2^{n+1}\cdot q_n\cdot k_n\cdot C_{k_n}\cdot|||H_n|||_{k_n+1}^{k_n+1}}$. Thus, the first property of this Lemma is fulfilled.
Furthermore, we note $k_n \geq 1$ and $C_{k_n} \geq 1$ by Lemma 5.7. Thus, $q_n \cdot k_n \cdot C_{k_n} \geq q_n$. Moreover, $|||H_n|||_1 \geq \|H_n\|_0 = 1$, because $H_n : \mathbb{S}^1 \times [0,1]^{m-1} \to \mathbb{S}^1 \times [0,1]^{m-1}$ is a diffeomorphism. Hence, $|||H_n|||_{k_n+1}^{k_n+1} \geq |||H_n|||_1$. Altogether, we conclude $2^{n+1}\cdot q_n\cdot k_n\cdot C_{k_n}\cdot|||H_n|||_{k_n+1}^{k_n+1} \geq 2^{n+1}\cdot q_n\cdot|||H_n|||_1$ and so:

$$|\alpha-\alpha_n| < \frac{1}{2^{n+1}\cdot q_n\cdot k_n\cdot C_{k_n}\cdot|||H_n|||_{k_n+1}^{k_n+1}} \leq \frac{1}{2^{n+1}\cdot q_n\cdot|||H_n|||_1}, \tag{7}$$

i.e. we verified the second property. □

Remark 5.10. Lemma 5.9 shows that the conditions of Lemma 5.8 are satisfied. Therefore, our sequence of constructed diffeomorphisms f_n converges in the $\mathrm{Diff}^\infty(M)$-topology to a diffeomorphism $f \in \mathcal{A}_\alpha(M)$. In addition, for every $\varepsilon > 0$ we can choose the parameters by Lemma 5.8 in such a way, that $d_\infty(f, R_\alpha) < \varepsilon$ holds.

To apply Proposition 6.6 we need another result:

Lemma 5.11. *Let $(\alpha_n)_{n\in\mathbb{N}}$ be constructed as in Lemma 5.9. Then it holds for every $n \in \mathbb{N}$ and for every $\tilde{m} \leq q_{n+1}$:*

$$d_0\left(f^{\tilde{m}}, f_n^{\tilde{m}}\right) \leq \frac{1}{2^n}.$$

Proof. In the proof of Lemma 5.8 we observed $f_{i-1} = H_i \circ R_{\alpha_i} \circ H_i^{-1}$ for every $i \in \mathbb{N}$. Hereby and with the help of Lemma 5.7 we compute:

$$d_0\left(f_i^{\tilde{m}}, f_{i-1}^{\tilde{m}}\right) = d_0\left(H_i \circ R_{\tilde{m}\cdot\alpha_{i+1}} \circ H_i^{-1}, H_i \circ R_{\tilde{m}\cdot\alpha_i} \circ H_i^{-1}\right) \leq |||H_i|||_1 \cdot \tilde{m} \cdot 2 \cdot |\alpha - \alpha_i|.$$

Since $\tilde{m} \leq q_{n+1} \leq q_i$ we conclude for every $i > n$ using equation (7) :

$$d_0\left(f_i^{\tilde{m}}, f_{i-1}^{\tilde{m}}\right) \leq |||H_i|||_1 \cdot \tilde{m} \cdot 2 \cdot |\alpha-\alpha_i| \leq |||H_i|||_1 \cdot \tilde{m} \cdot 2 \cdot \frac{1}{2^{i+1}\cdot q_i \cdot |||H_i|||_1} \leq \frac{\tilde{m}}{q_i}\cdot\frac{1}{2^i} \leq \frac{1}{2^i}.$$

Thus, for every $\tilde{m} \leq q_{n+1}$ we get the claimed result:

$$d_0\left(f^{\tilde{m}}, f_n^{\tilde{m}}\right) = \lim_{k\to\infty} d_0\left(f_k^{\tilde{m}}, f_n^{\tilde{m}}\right) \leq \lim_{k\to\infty} \sum_{i=n+1}^{k} d_0\left(f_i^{\tilde{m}}, f_{i-1}^{\tilde{m}}\right) \leq \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \left(\frac{1}{2}\right)^n.$$

□

Remark 5.12. Note that the sequence $(m_n)_{n\in\mathbb{N}}$ defined in section 4 meets the mentioned condition $m_n \leq q_{n+1}$ and hence Lemma 5.11 can be applied to it.

6 Proof of weak mixing

In this section we will prove that our constructed diffeomorphisms on $M = \mathbb{S}^1 \times [0,1]^{m-1}$ are weakly mixing. For the derivation we need a couple of lemmas. The first one expresses the weak mixing property on the elements of a partial partition ν_n generally:

Lemma 6.1. *Let $f \in \mathrm{Diff}^{\infty}(M,\mu)$, $(m_n)_{n\in\mathbb{N}}$ be a sequence of natural numbers and $(\nu_n)_{n\in\mathbb{N}}$ be a sequence of partial partitions satisfying $\nu_n \to \varepsilon$ the following property: For every m-dimensional cube $A \subseteq \mathbb{S}^1 \times (0,1)^{m-1}$ and for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $n \geq N$ and for every $\Gamma_n \in \nu_n$ we have*

$$\left|\mu\left(\Gamma_n \cap f^{-m_n}(A)\right) - \mu(\Gamma_n)\cdot\mu(A)\right| \leq 3\cdot\epsilon\cdot\mu(\Gamma_n)\cdot\mu(A). \tag{8}$$

Then f is weakly mixing.

In our case the partial partition ν_n will be the image of η_n from section 3.1.1 under the measure-preserving map $H_{n-1}\circ g_n$.

Proof. A diffeomorphism f is weakly mixing if for all measurable sets $A, B \subseteq M$ it holds:

$$\lim_{n\to\infty}\left|\mu\left(B\cap f^{-m_n}(A)\right) - \mu(B)\cdot\mu(A)\right| = 0.$$

Since every measurable set in $M = \mathbb{S}^1\times[0,1]^{m-1}$ can be approximated by a countable disjoint union of m-dimensional cubes in $\mathbb{S}^1\times(0,1)^{m-1}$ in arbitrary precision, we only have to prove the statement in case that A is a m-dimensional cube in $\mathbb{S}^1\times(0,1)^{m-1}$.
Hence, we consider an arbitrary m-dimensional cube $A\subset\mathbb{S}^1\times(0,1)^{m-1}$. Moreover, let $B\subseteq M$ be a measurable set. Since $\nu_n\to\varepsilon$ for every $\epsilon\in(0,1]$ there are $n\in\mathbb{N}$ and a set $\hat{B}=\bigcup_{i\in\Lambda}\Gamma_n^i$, where $\Gamma_n^i\in\nu_n$ and Λ is a countable set of indices, such that $\mu\left(B\triangle\hat{B}\right)<\epsilon\cdot\mu(B)\cdot\mu(A)$. We obtain for sufficiently large n:

$$\begin{aligned}
&\left|\mu\left(B\cap f^{-m_n}(A)\right)-\mu(B)\cdot\mu(A)\right|\\
&\leq\left|\mu\left(B\cap f^{-m_n}(A)\right)-\mu\left(\hat{B}\cap f^{-m_n}(A)\right)\right|+\left|\mu\left(\hat{B}\cap f^{-m_n}(A)\right)-\mu\left(\hat{B}\right)\cdot\mu(A)\right|\\
&\quad+\left|\mu\left(\hat{B}\right)\cdot\mu(A)-\mu(B)\cdot\mu(A)\right|\\
&=\left|\mu\left(B\cap f^{-m_n}(A)\right)-\mu\left(\hat{B}\cap f^{-m_n}(A)\right)\right|\\
&\quad+\left|\mu\left(\bigcup_{i\in\Lambda}\left(\Gamma_n^i\cap f^{-m_n}(A)\right)\right)-\mu\left(\bigcup_{i\in\Lambda}\Gamma_n^i\right)\cdot\mu(A)\right|+\mu(A)\cdot\left|\mu\left(\hat{B}\right)-\mu(B)\right|\\
&\leq\mu\left(\hat{B}\triangle B\right)+\left|\sum_{i\in\Lambda}\mu\left(\Gamma_n^i\cap f^{-m_n}(A)\right)-\mu\left(\Gamma_n^i\right)\cdot\mu(A)\right|+\mu(A)\cdot\mu\left(\hat{B}\triangle B\right)\\
&\leq\epsilon\cdot\mu(B)\cdot\mu(A)+\sum_{i\in\Lambda}\left(\left|\mu\left(\Gamma_n^i\cap f^{-m_n}(A)\right)-\mu\left(\Gamma_n^i\right)\cdot\mu(A)\right|\right)+\epsilon\cdot\mu(A)^2\cdot\mu(B)\\
&\leq\sum_{i\in\Lambda}\left(3\cdot\epsilon\cdot\mu\left(\Gamma_n^i\right)\cdot\mu(A)\right)+2\cdot\epsilon\cdot\mu(A)\cdot\mu(B)=3\cdot\epsilon\cdot\mu(A)\cdot\mu\left(\bigcup_{i\in\Lambda}\hat{I}_n^i\right)+2\cdot\epsilon\cdot\mu(A)\cdot\mu(B)\\
&=3\cdot\epsilon\cdot\mu(A)\cdot\mu\left(\hat{B}\right)+2\cdot\epsilon\cdot\mu(A)\cdot\mu(B)\leq3\cdot\epsilon\cdot\mu(A)\cdot\left(\mu(B)+\mu\left(\hat{B}\triangle B\right)\right)+2\cdot\epsilon\cdot\mu(A)\cdot\mu(B)\\
&\leq5\cdot\epsilon\cdot\mu(A)\cdot\mu(B)+3\cdot\epsilon^2\cdot\mu(A)^2\cdot\mu(B).
\end{aligned}$$

This estimate shows $\lim_{n\to\infty}\left|\mu\left(B\cap f^{-m_n}(A)\right)-\mu(B)\cdot\mu(A)\right|=0$, because ϵ can be chosen arbitrarily small. □

In property (8) we want to replace f by f_n:

Lemma 6.2. *Let $f = \lim_{n\to\infty} f_n$ be a diffeomorphism obtained by the constructions in the preceding sections and $(m_n)_{n\in\mathbb{N}}$ be a sequence of natural numbers fulfilling $d_0\left(f^{m_n}, f_n^{m_n}\right) < \frac{1}{2^n}$. Furthermore, let $(\nu_n)_{n\in\mathbb{N}}$ be a sequence of partial partitions satisfying $\nu_n \to \varepsilon$ and the following property: For every m-dimensional cube $A \subseteq \mathbb{S}^1 \times (0,1)^{m-1}$ and for every $\epsilon \in (0,1]$ there exists $N \in \mathbb{N}$ such that for every $n \geq N$ and for every $\Gamma_n \in \nu_n$ we have*

$$\left|\mu\left(\Gamma_n \cap f_n^{-m_n}(A)\right) - \mu(\Gamma_n)\cdot\mu(A)\right| \leq \epsilon\cdot\mu(\Gamma_n)\cdot\mu(A). \tag{9}$$

Then f is weakly mixing.

Proof. We want to show that the requirements of Lemma 6.1 are fulfilled. This implies that f is weakly mixing.
For it let $A \subseteq \mathbb{S}^1 \times (0,1)^{m-1}$ be an arbitrary m-dimensional cube and $\epsilon \in (0,1]$.
We consider two m-dimensional cubes $A_1, A_2 \subset \mathbb{S}^1 \times (0,1)^{m-1}$ with $A_1 \subset A \subset A_2$ as well as $\mu(A \triangle A_i) < \epsilon \cdot \mu(A)$ and for sufficiently large n: $\operatorname{dist}(\partial A, \partial A_i) > \frac{1}{2^n}$ for $i = 1, 2$.

If n is sufficiently large, we obtain for $\Gamma_n \in \nu_n$ and for $i = 1, 2$ by the assumptions of this Lemma:

$$\left|\mu\left(\Gamma_n \cap f_n^{-m_n}(A_i)\right) - \mu(\Gamma_n)\cdot\mu(A_i)\right| \leq \epsilon\cdot\mu(\Gamma_n)\cdot\mu(A_i).$$

Herefrom we conclude $(1-\epsilon)\cdot\mu(\Gamma_n)\cdot\mu(A_1) \leq \mu\left(\Gamma_n \cap f_n^{-m_n}(A_1)\right)$ on the one hand and $\mu\left(\Gamma_n \cap f_n^{-m_n}(A_2)\right) \leq (1+\epsilon)\cdot\mu(\Gamma_n)\cdot\mu(A_2)$ on the other hand. Because of $d_0\left(f^{m_n}, f_n^{m_n}\right) < \frac{1}{2^n}$ the following relations are true:

$$\begin{aligned} f_n^{m_n}(x) \in A_1 &\Longrightarrow f^{m_n}(x) \in A, \\ f^{m_n}(x) \in A &\Longrightarrow f_n^{m_n}(x) \in A_2. \end{aligned}$$

Thus: $\mu\left(\Gamma_n \cap f_n^{-m_n}(A_1)\right) \leq \mu\left(\Gamma_n \cap f^{-m_n}(A)\right) \leq \mu\left(\Gamma_n \cap f_n^{-m_n}(A_2)\right)$.
Altogether, it holds: $(1-\epsilon)\cdot\mu(\Gamma_n)\cdot\mu(A_1) \leq \mu\left(\Gamma_n \cap f^{-m_n}(A)\right) \leq (1+\epsilon)\cdot\mu(\Gamma_n)\cdot\mu(A_2)$.
Therewith, we obtain the following estimate from above:

$$\begin{aligned} &\mu\left(\Gamma_n \cap f^{-m_n}(A)\right) - \mu(\Gamma_n)\cdot\mu(A) \\ &\leq (1+\epsilon)\cdot\mu(\Gamma_n)\cdot\mu(A_2) - \mu(\Gamma_n)\cdot\mu(A_2) + \mu(\Gamma_n)\cdot(\mu(A_2) - \mu(A)) \\ &\leq \epsilon\cdot\mu(\Gamma_n)\cdot\mu(A_2) + \mu(\Gamma_n)\cdot\mu(A_2\triangle A) \leq \epsilon\cdot\mu(\Gamma_n)\cdot(\mu(A) + \mu(A_2\triangle A)) + \epsilon\cdot\mu(\Gamma_n)\cdot\mu(A) \\ &\leq 2\cdot\epsilon\cdot\mu(\Gamma_n)\cdot\mu(A) + \epsilon^2\cdot\mu(\Gamma_n)\cdot\mu(A) \leq 3\cdot\epsilon\cdot\mu(\Gamma_n)\cdot\mu(A) \end{aligned}$$

Furthermore, we deduce the following estimate from below in an analogous way:

$$\mu\left(\Gamma_n \cap f^{-m_n}(A)\right) - \mu(\Gamma_n)\cdot\mu(A) \geq -3\cdot\epsilon\cdot\mu(\Gamma_n)\cdot\mu(A)$$

Hence, we get: $\left|\mu\left(\Gamma_n \cap f^{-m_n}(A)\right) - \mu(\Gamma_n)\cdot\mu(A)\right| \leq 3\cdot\epsilon\cdot\mu(\Gamma_n)\cdot\mu(A)$, i.e. the requirements of Lemma 6.1 are met. □

Now we concentrate on the setting of our explicit constructions:

Lemma 6.3. *Consider the sequence of partial partitions $(\eta_n)_{n\in\mathbb{N}}$ constructed in section 3.1.1 and the diffeomorphisms g_n from chapter 3.2. Furthermore, let $(H_n)_{n\in\mathbb{N}}$ be a sequence of measure-preserving smooth diffeomorphisms satisfying $\|DH_{n-1}\| \leq \frac{\ln(q_n)}{n}$ for every $n \in \mathbb{N}$ and define the partial partitions $\nu_n = \left\{\Gamma_n = H_{n-1}\circ g_n\left(\hat{I}_n\right) \ : \ \hat{I}_n \in \eta_n\right\}$.*
Then we get $\nu_n \to \varepsilon$.

Proof. By construction $\eta_n = \left\{ \hat{I}_n^i : i \in \Lambda_n \right\}$, where Λ_n is a countable set of indices. Because of $\eta_n \to \varepsilon$ it holds $\lim_{n\to\infty} \mu \left(\bigcup_{i\in\Lambda_n} \hat{I}_n^i \right) = 1$. Since $H_{n-1} \circ g_n$ is measure-preserving, we conclude:

$$\lim_{n\to\infty} \mu \left(\bigcup_{i\in\Lambda_n} \Gamma_n^i \right) = \lim_{n\to\infty} \mu \left(\bigcup_{i\in\Lambda_n} H_{n-1} \circ g_n \left(\hat{I}_n^i \right) \right) = \lim_{n\to\infty} \mu \left(H_{n-1} \circ g_n \left(\bigcup_{i\in\Lambda_n} \hat{I}_n^i \right) \right) = 1.$$

For any m-dimensional cube with side length l_n it holds: $\operatorname{diam}(W_n) = \sqrt{m} \cdot l_n$. Because every element of the partition η_n is contained in a cube of side length $\frac{1}{q_n}$ it follows for every $i \in \Lambda_n$: $\operatorname{diam}\left(\hat{I}_n^i \right) \leq \sqrt{m} \cdot \frac{1}{q_n}$. Furthermore, we saw in Proposition 3.8: $g_n \left(\hat{I}_n^i \right) = \tilde{g}_{[nq_n^\sigma]} \left(\hat{I}_n^i \right)$ for every $i \in \Lambda_n$. Hence, for every $\Gamma_n^i = H_{n-1} \circ \tilde{g}_{[nq_n^\sigma]} \left(I_n^i \right)$:

$$\operatorname{diam}\left(\Gamma_n^i \right) \leq \|DH_{n-1}\|_0 \cdot \left\| D\tilde{g}_{[nq_n^\sigma]} \right\|_0 \cdot \operatorname{diam}\left(\hat{I}_n^i \right) \leq \frac{\ln(q_n)}{n} \cdot [n \cdot q_n^\sigma] \cdot \frac{\sqrt{m}}{q_n} \leq \sqrt{m} \cdot q_n^{\sigma-1} \cdot \ln(q_n).$$

Because of $\sigma < 1$ we conclude $\lim_{n\to\infty} \operatorname{diam}\left(\Gamma_n^i \right) = 0$ and consequently $\nu_n \to \varepsilon$. □

In the following the Lebesgue measures on $\mathbb{S}^1$, $[0,1]^{m-2}$, $[0,1]^{m-1}$ are denoted by $\tilde{\lambda}$, $\mu^{(m-2)}$ and $\tilde{\mu}$ respectively. The next technical result is needed in the proof of Lemma 6.5.

Lemma 6.4. *Given an interval on the r_1-axis of the form $K = \left[\frac{k_1 \cdot \varepsilon}{b \cdot a}, \frac{k_2 \cdot \varepsilon}{b \cdot a} \right]$, where $k_1, k_2 \in \mathbb{Z}$ with $\frac{b\cdot a}{\varepsilon} \cdot \delta \leq k_1 < k_2 \leq \frac{b\cdot a}{\varepsilon} - \frac{b\cdot a}{\varepsilon} \cdot \delta$, and a $(m-2)$-dimensional interval Z in $(r_2, ..., r_{m-1})$, let $K_{c,\gamma}$ denote the cuboid $[c, c+\gamma] \times K \times Z$ for some $\gamma > 0$. We consider the diffeomorphism $g_{a,b,\varepsilon,\delta}$ constructed in subsection 3.2 and an interval $L = [l_1, l_2]$ of $\mathbb{S}^1$ satisfying $\tilde{\lambda}(L) \geq 4 \cdot \frac{1-2\varepsilon}{a} - \gamma$. If $b \cdot \lambda(K) > 2$, then for the set $Q := \pi_{\vec{r}} \left(K_{c,\gamma} \cap g_{a,b,\varepsilon,\delta}^{-1} (L \times K \times Z) \right)$ we have:*

$$\begin{aligned}
&\left| \tilde{\mu}(Q) - \lambda(K) \cdot \tilde{\lambda}(L) \cdot \mu^{(m-2)}(Z) \right| \\
&\leq \left(\frac{2}{b} \cdot \tilde{\lambda}(L) + \frac{2 \cdot \gamma}{b} + \gamma \cdot \lambda(K) + 4 \cdot \frac{1-2\varepsilon}{a} \cdot \lambda(K) + 8 \cdot \frac{1-2\varepsilon}{b \cdot a} \right) \cdot \mu^{(m-2)}(Z).
\end{aligned}$$

Proof. We consider the diffeomorphism $\tilde{g}_b : M \to M$, $(\theta, r_1, ..., r_{m-1}) \mapsto (\theta + b \cdot r_1, r_1, ..., r_{m-1})$ and the set:

$$\begin{aligned}
Q_b &:= \pi_{\vec{r}} \left(K_{c,\gamma} \cap \tilde{g}_b^{-1} (L \times K \times Z) \right) \\
&= \{ (r_1, r_2, ..., r_{m-1}) \in K \times Z \ : \ (\theta + b \cdot r_1, \vec{r}) \in L \times K \times Z, \theta \in [c, c+\gamma] \} \\
&= \{ (r_1, r_2, ..., r_{m-1}) \in K \times Z \ : \ b \cdot r_1 \in [l_1 - c - \gamma, l_2 - c] \mod 1 \}.
\end{aligned}$$

The interval $b \cdot K$ seen as an interval in $\mathbb{R}$ does not intersect more than $b \cdot \lambda(K) + 2$ and not less than $b \cdot \lambda(K) - 2$ intervals of the form $[i, i+1]$ with $i \in \mathbb{Z}$. By construction of the map $g_{a,b,\varepsilon,\delta}$ it holds for $\Delta_l := \left[\frac{l \cdot \varepsilon}{b \cdot a}, \frac{(l+1) \cdot \varepsilon}{b \cdot a} \right]$ in consideration: $\pi_{\vec{r}} (g_{a,b,\varepsilon,\delta}([c, c+\gamma] \times \Delta_l \times Z)) = \Delta_l \times Z$.

Claim: A resulting interval on the r_1-axis of $K_{c,\gamma} \cap \tilde{g}_b^{-1} (L \times K \times Z)$ and the corresponding r_1-projection of $K_{c,\gamma} \cap g_{a,b,\varepsilon}^{-1} (L \times K \times Z)$ can differ by a length of at most $4 \cdot \frac{1-2\varepsilon}{b \cdot a}$.

Proof: If $\{c\} \times \Delta_l \times Z$ (resp. $\{c+\gamma\} \times \Delta_l \times Z$) are contained in the domain, where $g_{a,b,\varepsilon} = \tilde{g}_b$, the left (resp. the right) boundaries of $\pi_\theta (g_{a,b,\varepsilon,\delta}([c, c+\gamma] \times \Delta_l \times Z))$ and $\pi_\theta (\tilde{g}_b([c, c+\gamma] \times \Delta_l \times Z))$ coincide. Otherwise, i.e. $c \in \left(\frac{k}{a} + \varepsilon, \frac{k+1}{a} - \varepsilon \right)$ (resp. $c + \gamma \in \left(\frac{k}{a} + \varepsilon, \frac{k+1}{a} - \varepsilon \right)$) the sets $\pi_\theta (g_{a,b,\varepsilon,\delta}(\{c\} \times \Delta_l \times Z))$ and $\pi_\theta (\tilde{g}_b(\{c\} \times \Delta_l \times Z))$ (resp. $\pi_\theta (g_{a,b,\varepsilon,\delta}(\{c+\gamma\} \times \Delta_l \times Z))$ and $\pi_\theta (\tilde{g}_b(\{c+\gamma\} \times \Delta_l \times Z))$) differ by a length of at most $\frac{1-2\varepsilon}{a}$. Since $\pi_\theta (\tilde{g}_b(\{u\} \times \Delta_l \times Z))$ for arbitrary $u \in \mathbb{S}^1$ has a length of $\frac{\varepsilon}{a}$ on the θ-axis, this discrepancy will be equalised after at most $\frac{1-2\varepsilon}{a} : \frac{\varepsilon}{a} = \frac{1-2\varepsilon}{\varepsilon}$ blocks Δ_l on the r_1-axis. Thus, the resulting interval on the r_1-axis of $K_{c,\gamma} \cap \tilde{g}_b^{-1}(L \times K \times Z)$ and the corresponding r_1-projection of $K_{c,\gamma} \cap g_{a,b,\varepsilon}^{-1}(L \times K \times Z)$ can differ by a length of at most $4 \cdot \frac{1-2\varepsilon}{\varepsilon} \cdot \frac{\varepsilon}{b \cdot a} = 4 \cdot (1-2\varepsilon) \frac{1}{b \cdot a}$. □

Therefore, we compute on the one side:

$$\begin{aligned}
&\tilde{\mu}(Q) \leq (b \cdot \lambda(K) + 2) \cdot \left(\frac{l_2 - (l_1 - \gamma)}{b} + 4 \cdot \frac{1-2\varepsilon}{b \cdot a}\right) \cdot \mu^{(m-2)}(Z) \\
&= \left(\lambda(K) \cdot \tilde{\lambda}(L) + 2 \cdot \frac{\tilde{\lambda}(L)}{b} + \lambda(K) \cdot \gamma + \frac{2 \cdot \gamma}{b} + 4 \cdot \lambda(K) \cdot \frac{1-2\varepsilon}{a} + 8 \cdot \frac{1-2\varepsilon}{b \cdot a}\right) \cdot \mu^{(m-2)}(Z)
\end{aligned}$$

and on the other side

$$\begin{aligned}
&\tilde{\mu}(Q) \geq (b \cdot \lambda(K) - 2) \cdot \left(\frac{l_2 - (l_1 - \gamma)}{b} - 4 \cdot \frac{1-2\varepsilon}{b \cdot a}\right) \cdot \mu^{(m-2)}(Z) \\
&= \left(\lambda(K) \cdot \tilde{\lambda}(L) - 2 \cdot \frac{\tilde{\lambda}(L)}{b} + \lambda(K) \cdot \gamma - \frac{2 \cdot \gamma}{b} - 4 \cdot \lambda(K) \cdot \frac{1-2\varepsilon}{a} + 8 \cdot \frac{1-2\varepsilon}{b \cdot a}\right) \cdot \mu^{(m-2)}(Z).
\end{aligned}$$

Both equations together yield:

$$\begin{aligned}
&\left|\tilde{\mu}(Q) - \lambda(K) \cdot \tilde{\lambda}(L) \cdot \mu^{(m-2)}(Z) - \gamma \cdot \lambda(K) \cdot \mu^{(m-2)}(Z) - 8 \cdot \frac{1-2\varepsilon}{b \cdot a} \cdot \mu^{(m-2)}(Z)\right| \\
&\leq \left(\frac{2}{b} \cdot \tilde{\lambda}(L) + \frac{2 \cdot \gamma}{b} + 4 \cdot \lambda(K) \cdot \frac{1-2\varepsilon}{a}\right) \cdot \mu^{(m-2)}(Z).
\end{aligned}$$

The claim follows because

$$\begin{aligned}
&\left|\tilde{\mu}(Q) - \lambda(K) \cdot \tilde{\lambda}(L) \cdot \mu^{(m-2)}(Z)\right| - \gamma \cdot \lambda(K) \cdot \mu^{(m-2)}(Z) - 8 \cdot \frac{1-2\varepsilon}{b \cdot a} \cdot \mu^{(m-2)}(Z) \\
&\leq \left|\tilde{\mu}(Q) - \lambda(K) \cdot \tilde{\lambda}(L) \cdot \mu^{(m-2)}(Z) - \gamma \cdot \lambda(K) \cdot \mu^{(m-2)}(Z) - 8 \cdot \frac{1-2\varepsilon}{b \cdot a} \cdot \mu^{(m-2)}(Z)\right|.
\end{aligned}$$

□

Lemma 6.5. *Let n be sufficiently large, g_n as in section 3.2 and $\hat{I}_n \in \eta_n$, where η_n is the partial partition constructed in section 3.1.1. For the diffeomorphism ϕ_n constructed in section 3.3 and m_n as in chapter 4 we consider $\Phi_n = \phi_n \circ R^{m_n}_{\alpha_{n+1}} \circ \phi_n^{-1}$.*
Then for every m-dimensional cube S of side length $q_n^{-\sigma}$ lying in $\mathbb{S}^1 \times \left[\frac{1}{10n^4}, 1 - \frac{1}{10n^4}\right]^{m-1}$ we get

$$\left|\mu\left(\hat{I} \cap \Phi_n^{-1} \circ g_n^{-1}(S)\right) - \mu\left(\hat{I}\right) \cdot \mu(S)\right| \leq \frac{42 + 2m}{n} \cdot \mu\left(\hat{I}\right) \cdot \mu(S). \tag{10}$$

In other words this Lemma tells us that a partition element is “almost uniformly distributed” under $g_n \circ \Phi_n$ on the whole manifold $M = \mathbb{S}^1 \times [0,1]^{m-1}$.

Proof. Let S be a m-dimensional cube with side length $q_n^{-\sigma}$ lying in $\mathbb{S}^1 \times \left[\frac{1}{10n^4}, 1 - \frac{1}{10n^4}\right]^{m-1}$. Furthermore, we denote:

$$S_\theta = \pi_\theta(S) \qquad S_{r_1} = \pi_{r_1}(S) \qquad S_{\tilde{\vec{r}}} = \pi_{(r_2,\dots,r_{m-1})}(S) \qquad S_r = S_{r_1} \times S_{\tilde{\vec{r}}} = \pi_{\vec{r}}(S)$$

Obviously: $\tilde{\lambda}(S_\theta) = \lambda(S_{r_1}) = q_n^{-\sigma}$ and $\tilde{\lambda}(S_\theta) \cdot \lambda(S_{r_1}) \cdot \mu^{(m-2)}\left(S_{\tilde{\vec{r}}}\right) = \mu(S) = q_n^{-m\sigma}$.
Recalling the parameters in the definition of $g_n = g_{a,b,\varepsilon,\delta}$ we introduce the set $\Delta_l = \left[\frac{l\varepsilon}{ba}, \frac{(l+1)\varepsilon}{ba}\right]$ for $l \in \mathbb{Z}$, $0 \leq l \leq \frac{b \cdot a}{\varepsilon} - 1$.

- In case of a partition element $\hat{I}_n \in \eta_n$ of the first kind we define

$$\tilde{S}_{r_1} := \bigcup_{\Delta_l \subseteq S_{r_1}} \Delta_l; \quad \tilde{S}_r := \bigcup_{\Delta_l \subseteq S_{r_1}} \Delta_l \times S_{\tilde{\vec{r}}} \quad \text{as well as} \quad \tilde{S} := S_\theta \times \tilde{S}_r \subseteq S.$$

Here $\left|\mu\left(\tilde{S}\right) - \mu(S)\right| = \mu\left(S \setminus \tilde{S}\right) \leq 2 \cdot \frac{\varepsilon}{b \cdot a} \cdot \tilde{\lambda}(S_\theta) \cdot \mu^{(m-2)}\left(S_{\tilde{\vec{r}}}\right) \leq 2 \cdot \frac{\varepsilon}{a} \cdot \mu(S)$, where we used $b = [n \cdot q_n^\sigma] \geq q_n^\sigma$ in case of $n > 4$. Since Φ_n and g_n are measure-preserving, we additionally obtain: $\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(g_n^{-1}(S)\right)\right) - \mu\left(\hat{I} \cap \Phi_n^{-1}\left(g_n^{-1}\left(\tilde{S}\right)\right)\right)\right| \leq \mu\left(S \setminus \tilde{S}\right) \leq 2 \cdot \frac{\varepsilon}{a} \cdot \mu(S)$. In equation 2 in the proof of Lemma 4.6 we observed $\mu\left(\Phi_n\left(\hat{I}\right)\right) \geq \frac{1}{a} \cdot \left(1 - \frac{1}{5n^4}\right)^{2m-1}$. Hence:

$$\begin{aligned}&\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(g_n^{-1}(S)\right)\right) - \mu\left(\hat{I} \cap \Phi_n^{-1}\left(g_n^{-1}\left(\tilde{S}\right)\right)\right)\right| \leq 2 \cdot \frac{\varepsilon}{a} \cdot \mu(S) \\ &\leq 2 \cdot \frac{\varepsilon}{\left(1 - \frac{1}{5n^4}\right)^{2m-1}} \cdot \mu(S) \cdot \mu\left(\Phi_n\left(\hat{I}\right)\right) \leq 4 \cdot \varepsilon \cdot \mu(S) \cdot \mu\left(\hat{I}\right).\end{aligned}$$

- In case of a partition element $\hat{I}_n \in \eta_n$ of the second kind we define

$$\tilde{S}_{r_1} := \bigcup_{\Delta_l \subseteq S_{r_1}} \Delta_l; \quad \tilde{S}_r := \left(\bigcup_{\Delta_l \subseteq S_{r_1}} \Delta_l \times S_{\tilde{\vec{r}}}\right) \cap J_n \quad \text{as well as} \quad \tilde{S} := S_\theta \times \tilde{S}_r \subseteq S$$

Once again, we want to estimate $\mu\left(S \setminus \tilde{S}\right)$. As seen above we have $\lambda\left(S_{r_1} \setminus \tilde{S}_{r_1}\right) \leq 2 \cdot \frac{\varepsilon}{b \cdot a}$. Since the cube of side length $q_n^{-\sigma}$ in the $\vec{r}$-coordinates contains at least $\frac{q_n}{q_n^\sigma} - 2$ and at most $\frac{q_n}{q_n^\sigma} + 2$ intervals of the form $\left[\frac{l}{q_n}, \frac{l+1}{q_n}\right]$ for some $l \in \mathbb{Z}$ in each of those coordinates, we estimate $\mu\left(S \setminus \tilde{S}\right) \leq \frac{1}{n} \cdot \mu(S)$ for n sufficiently large. Moreover, we observed $\pi_{\vec{r}}\left(\Phi_n\left(\hat{I}_n\right)\right) \subset J_n$ in the proof of Lemma 4.5. Hereby, we get

$$\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(g_n^{-1}(S)\right)\right) - \mu\left(\hat{I} \cap \Phi_n^{-1}\left(g_n^{-1}\left(\tilde{S}\right)\right)\right)\right| \leq \tilde{\lambda}(S_\theta) \cdot 2 \cdot \frac{\varepsilon}{ba} \cdot \mu^{(m-2)}\left(S_{\tilde{\vec{r}}}\right) \leq 2 \cdot \frac{\varepsilon}{a} \cdot \mu(S).$$

Then we continue as in the previous case.

For partition elements of both kinds Φ_n $\left(\frac{1}{2 \cdot q_n^m}, \frac{1}{n}\right)$-distributes the partition element $\hat{I}_n \in \eta_n$ on a set J according to Lemma 4.5, in particular $\Phi_n\left(\hat{I}_n\right) \subseteq [c, c+\gamma] \times [0,1]^{m-1}$ for some $c \in \mathbb{S}^1$ and some $\gamma \leq \frac{1}{2 \cdot q_n^m}$. On the other hand, we saw $\gamma \geq \frac{1-2\varepsilon}{a}$ in the proof of Lemma 4.6.

Using the triangle inequality we obtain

$$\begin{aligned}&\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(g_n^{-1}(S)\right)\right) - \mu\left(\hat{I}\right) \cdot \mu(S)\right| \\ \leq &\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(g_n^{-1}(S)\right)\right) - \mu\left(\hat{I} \cap \Phi_n^{-1}\left(g_n^{-1}\left(\tilde{S}\right)\right)\right)\right| + \frac{1}{\tilde{\mu}(J)}\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(g_n^{-1}\left(\tilde{S}\right)\right)\right)\tilde{\mu}(J) - \mu\left(\hat{I}\right)\mu\left(\tilde{S}\right)\right| \\ &+ \frac{\mu\left(\hat{I}\right)}{\tilde{\mu}(J)} \cdot \left|\mu\left(\tilde{S}\right) - \mu(S)\right| + \frac{1 - \tilde{\mu}(J)}{\tilde{\mu}(J)} \cdot \mu\left(\hat{I}\right) \cdot \mu(S).\end{aligned}$$

In both cases, Bernoulli's inequality yields: $\tilde{\mu}(J) \geq \left(1-\frac{1}{n}\right)^{m-1} \geq 1+(m-1)\cdot\left(-\frac{1}{n}\right) = 1-\frac{m-1}{n}$. Hence we obtain for $n > 2\cdot(m-1)$: $\tilde{\mu}(J) \geq \frac{1}{2}$ and so: $\frac{1-\tilde{\mu}(J)}{\tilde{\mu}(J)} \leq 2\cdot(1-\tilde{\mu}(J)) \leq \frac{2\cdot(m-1)}{n}$. Thus, we obtain:

$$\begin{aligned}&\left|\mu\left(\hat{I}\cap\Phi_n^{-1}\left(g_n^{-1}(S)\right)\right)-\mu\left(\hat{I}\right)\cdot\mu(S)\right|\\ &\leq 2\cdot\left|\mu\left(\hat{I}\cap\Phi_n^{-1}\left(g_n^{-1}\left(\tilde{S}\right)\right)\right)\tilde{\mu}(J)-\mu\left(\hat{I}\right)\mu\left(\tilde{S}\right)\right|+4\varepsilon\cdot\mu(S)\,\mu\left(\hat{I}\right)+\frac{2m}{n}\mu(S)\,\mu\left(\hat{I}\right)\end{aligned} \tag{11}$$

Next, we want to estimate the first summand. By construction of the map $g_n = g_{a,b,\varepsilon,\delta}$ and the definition of $\tilde{S}$ it holds: $\Phi_n\left(\hat{I}\right)\cap g_n^{-1}\left(\tilde{S}\right) \subseteq [c,c+\gamma]\times\tilde{S}_r =: K_{c,\gamma}$. Considering the proof of Lemma 4.6 again, we obtain $g_n(K_{c,\gamma}) = \tilde{g}_{[nq_n^\sigma]}(K_{c,\gamma})$ (since c and $c+\gamma$ are in the domain where $g_n = \tilde{g}_{[nq_n^\sigma]}$ holds).
Because of Lemma 4.5 $2\gamma \leq \frac{2}{2\cdot q_n^m} < q_n^{-\sigma}$ for $n > 2$. So we can define a cuboid $S_1 \subseteq \tilde{S}$, where $S_1 := [s_1+\gamma, s_2-\gamma]\times\tilde{S}_r$ using the notation $S_\theta = [s_1,s_2]$. We examine the two sets

$$Q := \pi_{\vec{r}}\left(K_{c,\gamma}\cap g_n^{-1}\left(S_\theta\times\tilde{S}_r\right)\right) \qquad Q_1 := \pi_{\vec{r}}\left(K_{c,\gamma}\cap g_n^{-1}\left([s_1+\gamma,s_2-\gamma]\times\tilde{S}_r\right)\right)$$

As seen above $\Phi_n\left(\hat{I}\right)\cap g_n^{-1}\left(\tilde{S}\right)\subseteq K_{c,\gamma}$. Hence $\Phi_n\left(\hat{I}\right)\cap g_n^{-1}\left(\tilde{S}\right)\subseteq \Phi_n\left(\hat{I}\right)\cap g_n^{-1}\left(\tilde{S}\right)\cap K_{c,\gamma}$, which implies $\Phi_n\left(\hat{I}\right)\cap g_n^{-1}\left(\tilde{S}\right)\subseteq\Phi_n\left(\hat{I}\right)\cap\left(\mathbb{S}^1\times Q\right)$.

Claim: On the other hand: $\Phi_n\left(\hat{I}\right)\cap\left(\mathbb{S}^1\times Q_1\right)\subseteq\Phi_n\left(\hat{I}\right)\cap g_n^{-1}\left(\tilde{S}\right)$.

Proof of the claim: For $(\theta,\vec{r})\in\Phi_n\left(\hat{I}\right)\cap\left(\mathbb{S}^1\times Q_1\right)$ arbitrary it holds $(\theta,\vec{r})\in\Phi_n\left(\hat{I}\right)$, i.e. $\theta\in[c,c+\gamma]$, and $\vec{r}\in\pi_{\vec{r}}\left(K_{c,\gamma}\cap g_n^{-1}\left([s_1+\gamma,s_2-\gamma]\times\tilde{S}_r\right)\right)$, i.e. in particular $\vec{r}\in\tilde{S}_r$. This implies the existence of $\bar{\theta}\in[c,c+\gamma]$ satisfying $\left(\bar{\theta},\vec{r}\right)\in K_{c,\gamma}\cap g_n^{-1}(S_1)$. Hence, there are $\beta\in[s_1+\gamma,s_2-\gamma]$ and $\vec{r}_1\in\tilde{S}_r$, such that $g_n\left(\bar{\theta},\vec{r}\right)=(\beta,\vec{r}_1)$. Because of $\bar{\theta}\in[c,c+\gamma]$ and $\vec{r}\in\tilde{S}_r$ the point $\left(\bar{\theta},\vec{r}\right)$ is contained in one cuboid of the form $\Delta_{a,b,\varepsilon}$. Since $\theta\in[c,c+\gamma]$, $(\theta,\vec{r})$ is contained in the same $\Delta_{a,b,\varepsilon}$. Thus, $\pi_{\vec{r}}(g_n(\theta,\vec{r}))\in\tilde{S}_r$. Furthermore, $g_n(\theta,\vec{r})$ and $g_n\left(\bar{\theta},\vec{r}\right)$ are in a distance of at most γ on the θ-axis, because $\theta,\bar{\theta}\in[c,c+\gamma]$, i.e. $\left|\theta-\bar{\theta}\right|\leq\gamma$, $g_n(K_{c,\gamma})=\tilde{g}_{[nq_n^\sigma]}(K_{c,\gamma})$ and the map $\tilde{g}_{[nq_n^\sigma]}$ preserves the distances on the θ-axis. Thus, there are $\bar{\beta}\in[s_1,s_2]$ and $\vec{r}_2\in\tilde{S}_r$ such that $g_n(\theta,\vec{r})=\left(\bar{\beta},\vec{r}_2\right)$. So $(\theta,\vec{r})\in\Phi_n\left(\hat{I}\right)\cap g_n^{-1}\left(\tilde{S}\right)$. □
Altogether, the following inclusions are true:

$$\Phi_n\left(\hat{I}\right)\cap\left(\mathbb{S}^1\times Q_1\right)\subseteq\Phi_n\left(\hat{I}\right)\cap g_n^{-1}\left(\tilde{S}\right)\subseteq\Phi_n\left(\hat{I}\right)\cap\left(\mathbb{S}^1\times Q\right).$$

Thus, we obtain:

$$\begin{aligned}&\left|\mu\left(\hat{I}\cap\Phi_n^{-1}\left(g_n^{-1}(\tilde{S})\right)\right)\cdot\tilde{\mu}(J)-\mu\left(\hat{I}\right)\cdot\mu\left(\tilde{S}\right)\right|\\ &\leq\max\left(\left|\mu\left(\hat{I}\cap\Phi_n^{-1}\left(\mathbb{S}^1\times Q\right)\right)\cdot\tilde{\mu}(J)-\mu\left(\hat{I}\right)\cdot\mu\left(\tilde{S}\right)\right|,\right.\\ &\qquad\left.\left|\mu\left(\hat{I}\cap\Phi_n^{-1}\left(\mathbb{S}^1\times Q_1\right)\right)\cdot\tilde{\mu}(J)-\mu\left(\hat{I}\right)\cdot\mu\left(\tilde{S}\right)\right|\right)\end{aligned} \tag{12}$$

We want to apply Lemma 6.4 for $K=\tilde{S}_{r_1}$, $L=S_\theta$, $Z=S_{\tilde{\vec{r}}}$ and $b=[n\cdot q_n^\sigma]$ (note that $4\cdot\frac{1-2\varepsilon}{a}-\gamma\leq3\cdot\frac{1-2\varepsilon}{a}\leq\frac{3}{2\cdot q_n^m}<\frac{1}{q_n^\sigma}=\tilde{\lambda}(L)$ because of the mentioned relation $\gamma\geq\frac{1-2\varepsilon}{a}$ and for

$n > 4$: $b \cdot \lambda(K) = [nq_n^\sigma] \cdot q_n^{-\sigma} \geq \frac{1}{2} n q_n^\sigma \cdot q_n^{-\sigma} > 2$):

$$\begin{aligned}
&\left|\tilde{\mu}(Q) - \mu\left(\tilde{S}\right)\right| \\
&\leq \left(\frac{2}{[n \cdot q_n^\sigma]} \cdot \tilde{\lambda}(S_\theta) + \frac{2\gamma}{[n \cdot q_n^\sigma]} + \gamma \cdot \lambda\left(\tilde{S}_{r_1}\right) + 4 \cdot \frac{1-2\varepsilon}{a} \lambda\left(\tilde{S}_{r_1}\right) + 8 \cdot \frac{1-2\varepsilon}{[nq_n^\sigma] \cdot a}\right) \cdot \mu^{(m-2)}\left(S_{\tilde{\vec{r}}}\right) \\
&\leq \left(\frac{4}{n \cdot q_n^\sigma} \tilde{\lambda}(S_\theta) + \frac{4}{n \cdot q_n^\sigma \cdot q_n^\sigma} + \frac{1}{n \cdot q_n^\sigma} \lambda(S_{r_1}) + 4 \cdot \frac{1-2\varepsilon}{2 \cdot q_n^m} \lambda(S_{r_1}) + \frac{16 \cdot (1-2\varepsilon)}{n \cdot q_n^\sigma \cdot 2 \cdot q_n^m}\right) \cdot \mu^{(m-2)}\left(S_{\tilde{\vec{r}}}\right) \\
&\leq \frac{14}{n} \cdot \mu(S).
\end{aligned}$$

In particular, we receive from this estimate: $\frac{14}{n} \cdot \mu(S) \geq \tilde{\mu}(Q) - \mu\left(\tilde{S}\right) \geq \tilde{\mu}(Q) - \mu(S)$, hence: $\tilde{\mu}(Q) \leq \left(1 + \frac{14}{n}\right) \cdot \mu(S) \leq 4 \cdot \mu(S)$.
Analogously, we obtain: $\tilde{\mu}(Q_1) \leq 4 \cdot \mu(S)$ as well as $\left|\tilde{\mu}(Q_1) - \mu\left(\tilde{S}_1\right)\right| \leq \frac{14}{n} \cdot \mu(S)$.
Since Q as well as Q_1 are a finite union of disjoint $(m-1)$-dimensional intervals contained in J and Φ_n $\left(\frac{1}{2q_n^m}, \frac{1}{n}\right)$-distributes the interval $\hat{I}$ on J, we get:

$$\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(\mathbb{S}^1 \times Q\right)\right) \cdot \tilde{\mu}(J) - \mu\left(\hat{I}\right) \cdot \tilde{\mu}(Q)\right| \leq \frac{1}{n} \cdot \mu\left(\hat{I}\right) \cdot \tilde{\mu}(Q) \leq \frac{4}{n} \cdot \mu\left(\hat{I}\right) \cdot \mu(S)$$

as well as

$$\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(\mathbb{S}^1 \times Q_1\right)\right) \cdot \tilde{\mu}(J) - \mu\left(\hat{I}\right) \cdot \tilde{\mu}(Q_1)\right| \leq \frac{1}{n} \cdot \mu\left(\hat{I}\right) \cdot \tilde{\mu}(Q_1) \leq \frac{4}{n} \cdot \mu\left(\hat{I}\right) \cdot \mu(S).$$

Now we can proceed

$$\begin{aligned}
&\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(\mathbb{S}^1 \times Q\right)\right) \cdot \tilde{\mu}(J) - \mu\left(\hat{I}\right) \cdot \mu\left(\tilde{S}\right)\right| \\
&\leq \left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(\mathbb{S}^1 \times Q\right)\right) \cdot \tilde{\mu}(J) - \mu\left(\hat{I}\right) \cdot \tilde{\mu}(Q)\right| + \mu\left(\hat{I}\right) \cdot \left|\tilde{\mu}(Q) - \mu\left(\tilde{S}\right)\right| \\
&\leq \frac{4}{n} \cdot \mu\left(\hat{I}\right) \cdot \mu(S) + \mu\left(\hat{I}\right) \cdot \frac{14}{n} \cdot \mu(S) = \frac{18}{n} \cdot \mu\left(\hat{I}\right) \cdot \mu(S).
\end{aligned}$$

Noting that $\mu(S_1) = \mu\left(\tilde{S}\right) - 2\gamma \cdot \tilde{\mu}\left(\tilde{S}_r\right)$ and so $\mu\left(\tilde{S}\right) - \mu(S_1) \leq 2 \cdot \frac{1}{n \cdot q_n^\sigma} \cdot \tilde{\mu}\left(\tilde{S}_r\right) \leq \frac{2}{n} \cdot \mu(S)$ we obtain in the same way as above:

$$\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(\mathbb{S}^1 \times Q_1\right)\right) \cdot \tilde{\mu}(J) - \mu\left(\hat{I}\right) \cdot \mu\left(\tilde{S}\right)\right| \leq \frac{20}{n} \cdot \mu\left(\hat{I}\right) \cdot \mu(S).$$

Using equation (12) this yields:

$$\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(g_n^{-1}\left(\tilde{S}\right)\right)\right) \cdot \tilde{\mu}(J) - \mu\left(\hat{I}\right) \cdot \mu\left(\tilde{S}\right)\right| \leq \frac{20}{n} \cdot \mu\left(\hat{I}\right) \cdot \mu(S).$$

Finally, we conclude with the aid of equation (11) because of $\varepsilon = \frac{1}{8n^4}$:

$$\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(g_n^{-1}(S)\right)\right) - \mu\left(\hat{I}\right) \cdot \mu(S)\right| \leq \frac{42 + 2m}{n} \cdot \mu\left(\hat{I}\right) \cdot \mu(S).$$

□

Now we are able to prove the desired weak mixing-property.

Proposition 6.6 (Proof of weak mixing). *Let $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ and the sequence $(m_n)_{n\in\mathbb{N}}$ be constructed as in the previous sections. Then $f = \lim_{n\to\infty} f_n$ is weakly mixing.*

Proof. By Lemma 5.11 we have $d_0\left(f^{m_n}, f_n^{m_n}\right) < \frac{1}{2^n}$ for every $n \in \mathbb{N}$. To apply Lemma 6.2 we consider the partial partitions $\nu_n := H_{n-1} \circ g_n\left(\eta_n\right)$. As proven in Lemma 6.3 these partial partitions satisfy $\nu_n \to \varepsilon$. We have to establish equation (9). To do so, let $\varepsilon > 0$ and a m-dimensional cube $A \subseteq \mathbb{S}^1 \times (0,1)^{m-1}$ be given. There exists $N \in \mathbb{N}$ such that $A \subseteq \mathbb{S}^1 \times \left[\frac{1}{n^4}, 1-\frac{1}{n^4}\right]^{m-1}$ for every $n \geq N$. Furthermore, we note that $f_n^{m_n} = H_n \circ R_{\alpha_{n+1}}^{m_n} \circ H_n^{-1} = H_{n-1} \circ g_n \circ \Phi_n \circ g_n^{-1} \circ H_{n-1}^{-1}$.
Let S_n be a m-dimensional cube of side length $q_n^{-\sigma}$ contained in $\mathbb{S}^1 \times \left[\frac{1}{n^4}, 1-\frac{1}{n^4}\right]^{m-1}$. We look at $C_n := H_{n-1}\left(S_n\right)$, $\Gamma_n \in \nu_n$, and compute (since g_n and H_{n-1} are measure-preserving):

$$\left|\mu\left(\Gamma_n \cap f_n^{-m_n}\left(C_n\right)\right) - \mu\left(\Gamma_n\right)\cdot\mu\left(C_n\right)\right| = \left|\mu\left(\hat{I}_n \cap \Phi_n^{-1}\circ g_n^{-1}\left(S_n\right)\right) - \mu\left(\hat{I}_n\right)\cdot\mu\left(S_n\right)\right|.$$

We continue by applying Lemma 6.5:

$$\left|\mu\left(\Gamma_n \cap f_n^{-m_n}\left(C_n\right)\right) - \mu\left(\Gamma_n\right)\cdot\mu\left(C_n\right)\right| \leq \frac{42+2\cdot m}{n}\cdot\mu\left(\hat{I}_n\right)\cdot\mu\left(S_n\right)$$

Moreover, we recall that $\left\|DH_{n-1}\right\|_0 \leq \frac{\ln(q_n)}{n}$ by Lemma 5.9, 3. Then we get that $\operatorname{diam}(C_n) \leq \left\|DH_{n-1}\right\|_0 \cdot \operatorname{diam}\left(S_n\right) \leq \sqrt{m}\cdot\frac{\ln(q_n)}{q_n^{\sigma}}$, i.e. $\operatorname{diam}(C_n) \to 0$ as $n \to \infty$. Thus, we can approximate A by a countable disjoint union of sets $C_n = H_{n-1}\left(S_n\right)$ with $S_n \subseteq \mathbb{S}^1 \times \left[\frac{1}{n^4}, 1-\frac{1}{n^4}\right]^{m-1}$ a m-dimensional cube of side length $q_n^{-\sigma}$ with given precision, assuming that n is chosen to be large enough. Consequently for sufficiently large n there are sets $A_1 = \dot{\bigcup}_{i\in\Sigma_n^1} C_n^i$ and $A_2 = \dot{\bigcup}_{i\in\Sigma_n^2} C_n^i$ with countable sets Σ_n^1 and Σ_n^2 of indices satisfying $A_1 \subseteq A \subseteq A_2$ as well as $|\mu(A) - \mu(A_i)| \leq \frac{\epsilon}{3}\cdot\mu(A)$ for $i = 1, 2$.
Additionally we choose n such that $\frac{42+2\cdot m}{n} < \frac{\epsilon}{3}$ holds. It follows that

$$\begin{aligned}
&\mu\left(\Gamma_n \cap f_n^{-m_n}(A)\right) - \mu\left(\Gamma_n\right)\cdot\mu(A)\\
&\leq \mu\left(\Gamma_n \cap f_n^{-m_n}\left(A_2\right)\right) - \mu\left(\Gamma_n\right)\cdot\mu\left(A_2\right) + \mu\left(\Gamma_n\right)\cdot\left(\mu\left(A_2\right) - \mu(A)\right)\\
&\leq \sum_{i\in\Sigma_n^2}\left(\mu\left(\Gamma_n \cap f_n^{-m_n}\left(C_n^i\right)\right) - \mu\left(\Gamma_n\right)\cdot\mu\left(C_n^i\right)\right) + \frac{\epsilon}{3}\cdot\mu\left(\Gamma_n\right)\cdot\mu(A)\\
&\leq \sum_{i\in\Sigma_n^2}\left(\frac{42+2\cdot m}{n}\cdot\mu\left(\hat{I}_n\right)\cdot\mu\left(S_n^i\right)\right) + \frac{\epsilon}{3}\cdot\mu\left(\Gamma_n\right)\cdot\mu(A)\\
&= \frac{42+2\cdot m}{n}\cdot\mu\left(\Gamma_n\right)\cdot\mu\left(\bigcup_{i\in\Sigma_n^2} C_n^i\right) + \frac{\epsilon}{3}\cdot\mu\left(\Gamma_n\right)\cdot\mu(A) \leq \frac{\epsilon}{3}\cdot\mu\left(\Gamma_n\right)\cdot\mu\left(A_2\right) + \frac{\epsilon}{3}\cdot\mu\left(\Gamma_n\right)\cdot\mu(A)\\
&= \frac{\epsilon}{3}\cdot\mu\left(\Gamma_n\right)\cdot\mu(A) + \frac{\epsilon}{3}\cdot\mu\left(\Gamma_n\right)\cdot\left(\mu\left(A_2\right) - \mu(A)\right) + \frac{\epsilon}{3}\cdot\mu\left(\Gamma_n\right)\cdot\mu(A) \leq \epsilon\cdot\mu\left(\Gamma_n\right)\cdot\mu(A).
\end{aligned}$$

Analogously, we estimate that $\mu\left(\Gamma_n \cap f_n^{-m_n}(A)\right) - \mu\left(\Gamma_n\right)\cdot\mu(A) \geq -\epsilon\cdot\mu\left(\Gamma_n\right)\cdot\mu(A)$. Both estimates enable us to conclude that $\left|\mu\left(\Gamma_n \cap f_n^{-m_n}(A)\right) - \mu\left(\Gamma_n\right)\cdot\mu(A)\right| \leq \epsilon\cdot\mu\left(\Gamma_n\right)\cdot\mu(A)$. □

7 Construction of the measurable f-invariant Riemannian metric

Let ω_0 denote the standard Riemannian metric on $M = \mathbb{S}^1 \times [0,1]^{m-1}$. The following Lemma shows that the conjugation map $h_n = g_n \circ \phi_n$ constructed in section 3 is an isometry with respect to ω_0 on the elements of the partial partition ζ_n.

Lemma 7.1. *Let $\check{I}_n \in \zeta_n$. Then $h_n|_{\check{I}_n}$ is an isometry with respect to ω_0.*

Proof. Let $\check{I}_{n,k} \in \zeta_n$ be a partition element on $\left[\frac{k}{2q_n}, \frac{k+1}{2q_n}\right] \times [0,1]^{m-1}$. In case of k even, ϕ_n acts as an isometry on $\check{I}_{n,k}$ by Proposition 3.12, 3., and $\phi_n\left(\check{I}_{n,k}\right)$ is equal to

$$\left[\frac{k}{2q_n} - \frac{j_m}{2q_n^2} - \dots - \frac{j_3}{2q_n^{m-1}} - \frac{j_2^{(1)}+1}{2q_n^m} + \frac{1}{n^4 \cdot 2q_n^m}, \frac{k}{2q_n} - \frac{j_m}{2q_n^2} - \dots - \frac{j_3}{2q_n^{m-1}} - \frac{j_2^{(1)}}{2q_n^m} - \frac{1}{n^4 \cdot 2q_n^m}\right]$$
$$\times \left[\frac{j_1^{(m-1)}}{q_n} + \frac{j_2^{(2)}}{q_n^2} + \dots + \frac{j_2^{(m)}}{q_n^m} + \frac{j_2^{(m+1)}}{16n^4 \cdot q_n^m \cdot [nq_n^\sigma]} + \frac{1}{16n^8 \cdot q_n^m \cdot [nq_n^\sigma]},\right.$$
$$\left.\frac{j_1^{(m-1)}}{q_n} + \frac{j_2^{(2)}}{q_n^2} + \dots + \frac{j_2^{(m)}}{q_n^m} + \frac{j_2^{(m+1)}+1}{16n^4 \cdot q_n^m \cdot [nq_n^\sigma]} - \frac{1}{16n^8 \cdot q_n^m \cdot [nq_n^\sigma]}\right]$$
$$\times \prod_{i=3}^{m}\left[\frac{j_1^{(m+1-i)}}{q_n} + \frac{1}{n^4 \cdot q_n}, \frac{j_1^{(m+1-i)}+1}{q_n} - \frac{1}{n^4 \cdot q_n}\right].$$

By Proposition 3.8, 3., $g_n = g_{2q_n^m, [n \cdot q_n^\sigma], \frac{1}{8n^4}, \frac{1}{32n^4}}$ acts as an isometry on this set.
In case of k odd ϕ_n acts as the identity on the element $\check{I}_{n,k}$. By the shape of this set and Proposition 3.8, 3., g_n acts as an isometry on it. □

This Lemma implies that $h_n^{-1}|_{h_n\left(\check{I}_n\right)}$ is an isometry as well.
In the following we construct the f-invariant measurable Riemannian metric. This construction parallels the approach in [GK00, section 4.8]. For it we put $\omega_n := \left(H_n^{-1}\right)^* \omega_0$. Each ω_n is a smooth Riemannian metric because it is the pullback of a smooth metric via a $C^\infty(M)$-diffeomorphism. Since $R^*_{\alpha_{n+1}}\omega_0 = \omega_0$ the metric ω_n is f_n-invariant:

$$\begin{aligned} f_n^*\omega_n &= \left(H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}\right)^* \left(H_n^{-1}\right)^* \omega_0 = \left(H_n^{-1}\right)^* R^*_{\alpha_{n+1}} H_n^* \left(H_n^{-1}\right)^* \omega_0 = \left(H_n^{-1}\right)^* R^*_{\alpha_{n+1}}\omega_0 \\ &= \left(H_n^{-1}\right)^* \omega_0 = \omega_n. \end{aligned}$$

With the succeeding Lemmas we show that the limit $\omega_\infty := \lim_{n\to\infty}\omega_n$ exists μ-almost everywhere and is the desired f-invariant Riemannian metric.

Lemma 7.2. *The sequence $(\omega_n)_{n\in\mathbb{N}}$ converges μ-a.e. to a limit ω_∞*

Proof. For every $N \in \mathbb{N}$ we have for every $k > 0$:

$$\omega_{N+k} = \left(H_{N+k}^{-1}\right)^* \omega_0 = \left(h_{N+k}^{-1} \circ \dots \circ h_{N+1}^{-1} \circ H_N^{-1}\right)^* \omega_0 = \left(H_N^{-1}\right)^* \left(h_{N+k}^{-1} \circ \dots \circ h_{N+1}^{-1}\right)^* \omega_0.$$

Since the elements of the partition ζ_n cover M except a set of measure at most $\frac{4m}{n^2}$ by Remark 3.4, Lemma 7.1 shows that ω_{N+k} coincides with $\omega_N = \left(H_N^{-1}\right)^* \omega_0$ on a set of measure at least $1 - \sum_{n=N+1}^{\infty} \frac{4m}{n^2}$. As this measure approaches 1 for $N \to \infty$, the sequence $(\omega_n)_{n\in\mathbb{N}}$ converges on a set of full measure. □

Lemma 7.3. *The limit ω_∞ is a measurable Riemannian metric.*

Proof. The limit ω_∞ is a measurable map because it is the pointwise limit of the smooth metrics ω_n, which in particular are measurable. By the same reasoning $\omega_\infty|_p$ is symmetric for μ-almost every $p \in M$. Furthermore, ω_∞ is positive definite because ω_n is positive definite for every $n \in \mathbb{N}$ and ω_∞ coincides with ω_N on $T_1M \otimes T_1M$ minus a set of measure at most $\sum_{n=N+1}^{\infty} \frac{4m}{n^2}$. Since this is true for every $N \in \mathbb{N}$, ω_∞ is positive definite on a set of full measure. □

Remark 7.4. In the proof of the subsequent Lemma we will need Egoroff's theorem (for example [Ha65, §21, Theorem A]): Let (N, d) denote a separable metric space. Given a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of N-valued measurable functions on a measure space (X, Σ, μ) and a measurable subset $A \subseteq X$, $\mu(A) < \infty$, such that $(\varphi_n)_{n\in\mathbb{N}}$ converges μ-a.e. on A to a limit function φ. Then for every $\varepsilon > 0$ there exists a measurable subset $B \subset A$ such that $\mu(B) < \varepsilon$ and $(\varphi_n)_{n\in\mathbb{N}}$ converges to φ uniformly on $A \setminus B$.

Lemma 7.5. *ω_∞ is f-invariant, i.e. $f^*\omega_\infty = \omega_\infty$ μ-a.e..*

Proof. By Lemma 7.2 the sequence $(\omega_n)_{n\in\mathbb{N}}$ converges in the C^∞-topology pointwise almost everywhere. Hence, we obtain using Egoroff's theorem: For every $\delta > 0$ there is a set $C_\delta \subseteq M$ such that $\mu(M \setminus C_\delta) < \delta$ and the convergence $\omega_n \to \omega_\infty$ is uniform on C_δ.
The function f was constructed as the limit of the sequence $(f_n)_{n\in\mathbb{N}}$ in the C^∞-topology. Thus, $\tilde{f}_n := f_n^{-1} \circ f \to \mathrm{id}$ in the C^∞-topology. Since M is compact, this convergence is uniform too. Furthermore, the smoothness of f implies $f^*\omega_\infty = f^* \lim_{n\to\infty} \omega_n = \lim_{n\to\infty} f^*\omega_n$. Therewith, we compute on C_δ: $f^*\omega_\infty = \lim_{n\to\infty} \left(\left(f_n \tilde{f}_n \right)^* \omega_n \right) = \lim_{n\to\infty} \left(\tilde{f}_n^* f_n^* \omega_n \right) = \lim_{n\to\infty} \tilde{f}_n^* \omega_n = \omega_\infty$, where we used the uniform convergence on C_δ in the last step. As this holds on every set C_δ with $\delta > 0$, it also holds on the set $\bigcup_{\delta>0} C_\delta$. This is a set of full measure and therefore the claim follows. □

Hence, the desired f-invariant measurable Riemannian metric ω_∞ is constructed and thus Proposition 2.8 is proven.

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