Weakly mixing diffeomorphisms with ergodic derivative extension in $\mathcal{A}_\alpha(M)$ for arbitrary Liouvillean number α

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Abstract

On any smooth compact connected manifold of dimension $m \geq 2$ admitting a smooth non-trivial circle action $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}$, $S_{t+1} = S_t$, preserving a smooth volume μ we construct weakly mixing C^∞-diffeomorphisms in $\mathcal{A}_\alpha(M) = \overline{\{h \circ S_\alpha \circ h^{-1} : h \in \mathrm{Diff}^\infty(M,\mu)\}}^{C^\infty}$ for every Liouvillean number $\alpha \in \mathbb{S}^1$ whose differential is ergodic with respect to a smooth measure in the projectivization of the tangent bundle. The proof is based on a quantitative version of the "approximation by conjugation"-method with explicitly defined conjugation maps, partial partitions and tower elements.

Key words: Smooth Ergodic Theory; Conjugation-approximation-method; almost isometries; weak mixing diffeomorphisms; projectivization of tangent bundle.

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Introduction

Let M be a smooth compact and connected manifold of dimension $m \geq 2$ with a non-trivial circle action $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}$, $S_{t+1} = S_t$ preserving a smooth volume μ. In their influential paper [AK70] D. V. Anosov and A. Katok introduced the so-called "approximation by conjugation"-method which enables the construction of smooth diffeomorphisms with specific ergodic properties (e. g. weakly mixing ones in [AK70], section 5, and weakly mixing diffeomorphisms that are uniformly rigid with respect to a prescribed sequence satisfying a growth condition ([Ku15])) or non-standard smooth realizations of measure-preserving systems (e. g. [AK70], section 6, [Be13] and [FSW07]). These diffeomorphisms are constructed as limits of conjugates $f_n = H_n \circ S_{\alpha_{n+1}} \circ H_n^{-1}$, where $\alpha_{n+1} = \alpha_n + \frac{1}{k_n \cdot l_n \cdot q_n^2} \in \mathbb{Q}$, $H_n = H_{n-1} \circ h_n$ and h_n is a measure-preserving diffeomorphism satisfying $S_{\frac{1}{q_n}} \circ h_n = h_n \circ S_{\frac{1}{q_n}}$. In each step the conjugation map h_n and the parameter k_n are chosen such that the diffeomorphism f_n imitates the desired property with a certain precision. Then the parameter l_n is chosen large enough to guarantee closeness of f_n to f_{n-1} in the C^∞-topology and so the convergence of the sequence $(f_n)_{n\in\mathbb{N}}$ to a limit diffeomorphism is provided. It is even possible to keep this limit diffeomorphism within any given C^∞-neighbourhood of the initial element S_{α_1} or, by applying a fixed diffeomorphism g first, of $g \circ S_{\alpha_1} \circ g^{-1}$. So the construction can be carried out in a neighbourhood of any diffeomorphism conjugate to an element of the action. Thus, $\mathcal{A}(M) = \overline{\{h \circ S_t \circ h^{-1} \ : \ t \in \mathbb{S}^1, h \in \mathrm{Diff}^\infty(M,\mu)\}}^{C^\infty}$ is a natural space for the produced diffeomorphisms. Moreover, we will consider the restricted space

$\mathcal{A}_\alpha(M) = \overline{\{h \circ S_\alpha \circ h^{-1} \ : h \in \text{Diff}^\infty(M,\mu)\}}^{C^\infty}$ for $\alpha \in \mathbb{S}^1$.
As mentioned above Anosov and Katok proved that the set of weakly mixing diffeomorphisms is generic (i. e. it is a dense G_δ-set) in $\mathcal{A}(M)$ in the $C^\infty(M)$-topology. In extension of it R. Gunesch and A. Katok constructed weakly mixing diffeomorphisms preserving a measurable Riemannian metric in [GKa00]. Actually, it follows from the respective proofs that both results are true in $\mathcal{A}_\alpha(M)$ for a G_δ-set of $\alpha \in \mathbb{S}^1$. However, both proofs do not give a full description of the set of $\alpha \in \mathbb{S}^1$ for which the particular result holds in $\mathcal{A}_\alpha(M)$. Such an investigation is started in [FS05]: B. Fayad and M. Saprykina showed that if $\alpha \in \mathbb{S}^1$ is Liouville, the set of weakly mixing diffeomorphisms is generic in the $C^\infty(M)$-topology in $\mathcal{A}_\alpha(M)$ in case of dimension 2. Generalising these results Gunesch and the author proved in [GKu15] that if $\alpha \in \mathbb{R}$ is Liouville, the set of volume-preserving diffeomorphisms, that are weakly mixing and preserve a measurable Riemannian metric, is dense in the C^∞-topology in $\mathcal{A}_\alpha(M)$. Recently, it has been proven that for every $\rho > 0$ and $m \geq 2$ there exists a weakly mixing real-analytic diffeomorphism $f \in \text{Diff}^\omega_\rho(\mathbb{T}^m,\mu)$ preserving a measurable Riemannian metric ([K1]).
Such diffeomorphisms preserving a measurable Riemannian metric are called IM-diffeomorphisms. In [GKa00], section 3, IM-diffeomorphisms and IM-group actions are discussed comprehensively. In particular, the existence of a measurable invariant metric for a diffeomorphism is equivalent to the existence of an invariant measure for the projectivized derivative extension which is absolutely continuous in the fibers. Hence, it is a natural question to study the ergodic properties of the projectivized derivative extension with respect to such a measure. Actually, the constructions in [GKa00] as well as [GKu15] are as non-ergodic as possible: Their projectivized derivative extensions are isomorphic to the direct product of the diffeomorphism in the base with the trivial action in the fibers so that each ergodic component intersects almost every fiber in a single point. In this paper we realise the other extreme possibility by constructing IM-diffeomorphisms whose differential is ergodic with respect to such a smooth measure in the projectivization of the tangent bundle:

Theorem 1. *Let M be a smooth compact and connected manifold of dimension $m \geq 2$ with a non-trivial circle action $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}$, $S_{t+1} = S_t$, preserving a smooth volume μ. If $\alpha \in \mathbb{R}$ is Liouville, there exists a volume-preserving weakly mixing diffeomorphism in $\mathcal{A}_\alpha(M)$, whose projectivized derivative extension is ergodic with respect to a measure in the projectivization of the tangent bundle which is absolutely continuous in the fibers.*
Moreover, for every Liouvillean number $\alpha \in \mathbb{R}$ the set of such diffeomorphisms is dense in the C^∞-topology in $\mathcal{A}_\alpha(M)$.

This construction provides the only known examples of measure-preserving diffeomorphisms whose differential is ergodic with respect to a smooth measure in the projectivization of the tangent bundle.

1 Preliminaries

1.1 Definitions and notations

We refer to [GKu15], section 2.1., for useful definitions and notations. In particular, we recall the notion of a partial partition which is a pairwise disjoint countable collection of measurable subsets of the manifold.
Additionally, we want to introduce the invariant measure for the projectivized derivative extension: Let $f : M \to M$ be a smooth diffeomorphism. On the tangent bundle TM we consider the derivative extension (f, df). Let $p \in M$. We can naturally identify the tangent space T_pM

with $\mathbb{R}^m$ which can be equipped with m-dimensional spherical coordinates $(r, \theta_1, ..., \theta_{m-1})$, where $r \in \mathbb{R}^+$, $\theta_1, ..., \theta_{m-2} \in [0, \pi]$ and $\theta_{m-1} \in [0, 2\pi)$. If x_i are the Cartesian coordinates, then

$$\begin{aligned} x_1 &= r \cdot \cos(\theta_1) \\ x_i &= r \cdot \prod_{j=1}^{i-1} \sin(\theta_j) \cdot \cos(\theta_i) \ \text{ for } i = 2, ..., m-1 \\ x_m &= r \cdot \prod_{j=1}^{m-1} \sin(\theta_j) \end{aligned}$$

Next, we consider its projective space $\mathbb{PR}^m$ and introduce the notation $[a_1, b_1] \times ... \times [a_{m-1}, b_{m-1}] \subset \mathbb{PR}^m$ which describes the allowed values for the spherical coordinates $\theta_1, ..., \theta_{m-1}$. This yields the projectivized tangent bundle which will be denoted by $\mathbb{P}TM$. In particular, we will use the notation $c \times \left[0, \frac{1}{k}\right]^{m-1} \subset \mathbb{P}TM$ with $c \subset M$ for the set in $\mathbb{P}TM$ with base points $x \in c$ and spherical coordinates $\theta_i \in \left[0, \frac{1}{k}\right]$. On the projectivized tangent bundle we consider the projectivized derivative extension of a diffeomorphism $f : M \to M$. By misuse of notation we will denote it by (f, df) again.
Following the lines of [Ch97], chapter 5.1, we consider the cotangent bundle T^*M and the projection maps $\pi_1 : TM \to M$ as well as $\pi_2 : TM^* \to M$. Then we define the canonical 1-form ω on TM^* by $\omega_{|\tau} = \pi_2^* \tau$, where $\omega_{|\tau}$ denotes the 1-form ω evaluated at $\tau \in TM^*$. Additionally we define the canonical 2-form Ω on TM^* by $\Omega = d\omega$, which is symplectic. In the next step, let M be a Riemannian manifold and $V : M \to \mathbb{R}$ be a function. Then we examine the Lagrangian $L : TM \to \mathbb{R}$ given by $L(\xi) = \frac{|\xi|}{2} - V \circ \pi_1(\xi)$, where $|\xi|$ is computed with respect to the Riemannian metric. To this Lagrangian we associate a bundle map $FL : TM \to TM^*$ defined by $FL(\xi)(\eta) = \frac{\mathrm{d}}{\mathrm{d}t} \left(L(\xi + t\eta)\right)_{|t=0}$ for $p \in M$, $\xi, \eta \in T_pM$. Hereby, we define $\Theta = FL^*\Omega$ and $\nu = FL^*\omega$.
In [Ch97], chapter 5.1, the differential form $\nu \wedge \Theta^{m-1}$ on the unit tangent bundle SM is considered. It is proven that it is the local product, up to a constant multiple, of the Riemannian volume on M with the Lebesgue $(m-1)$-form on the unit tangent spheres of M with respect to the Riemannian metric. In particular, for any $\nu \wedge \Theta^{m-1}$-integrable function g on SM we have "integrations over the fibers"

$$\int_{SM} g \, \nu \wedge \Theta^{m-1} = c(m) \cdot \int_M \mathrm{dVol}(p) \int_{S_pM} g|_{S_pM} \, \mathrm{d}\mu_p,$$

where Vol is the volume form induced by the Riemannian metric and μ_p is the Euclidean $(m-1)$-measure on the tangent sphere S_pM with respect to the Riemannian metric.
By the same approach we can deduce the same formula for the constructed invariant measurable Riemannian metric ω_∞ and for any integrable function on $\mathbb{P}TM$. The corresponding measure will be denoted by $\bar{\mu}$. Moreover, we point out that in our constructions the measure induced by the measurable Riemannian metric ω_∞ coincides with the measure μ on M. Since ω_∞ is f-invariant, we conclude that $\bar{\mu}$ is (f, df)-invariant.

1.2 First steps of the proof

By the same arguments as in [GKu15], section 2.2., constructions on $\mathbb{S}^1 \times [0,1]^{m-1}$ equipped with Lebesgue measure μ and standard circle action $\mathcal{R} = \{R_\alpha\}_{\alpha \in \mathbb{S}^1}$ comprising of diffeomorphisms $R_\alpha(\theta, r_1, ..., r_{m-1}) = (\theta + \alpha, r_1, ..., r_{m-1})$ can be transfered to a general compact connected smooth manifold M with a non-trivial circle action $\mathcal{S} = \{S_t\}_{t \in \mathbb{R}}$, $S_{t+1} = S_t$. Moreover, the density of the constructed diffeomorphisms follows if for every $\varepsilon > 0$ the parameters in the construction can be chosen in such a way that $d_{\infty}(f, R_{\alpha}) < \varepsilon$.

1.3 Outline of the proof

The constructions are based on the "approximation by conjugation"-method developed by D.V. Anosov and A. Katok in [AK70]. As indicated in the introduction, one constructs successively a sequence of measure-preserving diffeomorphisms $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$, where the conjugation maps $H_n = H_{n-1} \circ h_n$ and the rational numbers $\alpha_n = \frac{p_n}{q_n}$ are chosen in such a way that the functions f_n converge to a diffeomorphism f with the aimed properties.
Similar to the constructions in [GKu15] we will start by defining two sequences of partial partitions, which converge to the decomposition into points in each case. The first type of partial partition, called η_n, will satisfy the requirements in the proof of the weak mixing-property. On the partition elements of the even more detailed second type, called ζ_n, the conjugation map h_n will act as an isometry and this will enable us to construct an invariant measurable Riemannian metric. Afterwards, these conjugating diffeomorphisms $h_n = g_n \circ i_n \circ \phi_n$ will be constructed. In comparison to [GKu15], the construction of the map g_n is modified and an additional map i_n is introduced in order to prove the ergodicity of the projectivized derivative extension. On the one hand, the map g_n shall introduce shear in the θ-direction. On the other hand, the map $g_n \circ i_n$ has to be an isometry on the image under ϕ_n of any partition element $\check{I}_n \in \zeta_n$. Likewise the conjugation map ϕ_n will be built such that it acts on the elements of ζ_n as an isometry and on the elements of η_n in such a way that it satisfies the requirements of the aimed criterion for weak mixing. This criterion is established in section 4 and bases upon the notion of a $(\gamma, \delta, \epsilon)$-distribution of the map $\Phi_n = \phi_n \circ R^{m_n}_{\alpha_{n+1}} \circ \phi_n^{-1}$ with a specific sequence $(m_n)_{n\in\mathbb{N}}$ of natural numbers (see section 3). It is similar to the criterion in [GKu15], section 5, but modified in some places because of the new conjugation maps g_n and i_n.
In section 5 we will show convergence of the sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{A}_{\alpha}(M)$ for a given Liouville number α by the same approach as in [FS05]. For this purpose, we have to estimate the norms $|||H_n|||_k$ very carefully. Furthermore, we will see at the end of section 5 that the criterion for weak mixing applies to the obtained diffeomorphism $f = \lim_{n\to\infty} f_n$. By the same approach as in [GKu15] we will construct the aimed f-invariant measurable Riemannian metric in section 6. Finally, we will prove the ergodicity of the projectivized derivative extension. This proof bases upon the general method of approximation of measure-preserving transformation in Ergodic Theory which is outlined in subsection 7.1. In order to apply this method, we have to show that (f, df) admits a sufficiently fast approximation on $\mathbb{P}TM$ with respect to the measure $\bar{\mu}$. Therefore, we define a tower explicitly and examine the speed of approximation in subsection 7.2. For these examinations we use the same techniques as in [K2]. In particular, we require the map i_n to act as a rotation by a different angle on different parts of the tower element.

2 Explicit constructions

We present step n in our inductive process of construction. We assume that we have already defined the rational numbers $\alpha_1, ..., \alpha_n \in \mathbb{S}^1$ and the conjugation map $H_{n-1} = h_1 \circ ... \circ h_{n-1} \in \text{Diff}^{\infty}(M, \mu)$.
First of all, we choose $k_n \in \mathbb{Z}$ large enough such that for every subset $c \subset M$ of diameter $\text{diam}(c) < \frac{1}{2n}$ and every set $d = \{(r, \theta_1, ..., \theta_{m-1}) \,:\, r \in \mathbb{R},\ \theta_i \in [a_i, b_i]\}$ with $b_i - a_i \leq \frac{1}{k_n}$ we have

$$\{d_p H_{n-1}(d) \,:\, p \in c\} \subset \mathbb{R} \times [c_1, d_1] \times ... \times [c_{m-1}, d_{m-1}],$$

where $d_i - c_i \leq \frac{1}{2mn}$ for every $i \in \{1,...,m-1\}$.

2.1 Sequences of partial partitions

In this subsection we define the two announced sequences of partial partitions $(\eta_n)_{n\in\mathbb{N}}$ and $(\zeta_n)_{n\in\mathbb{N}}$ of $M = \mathbb{S}^1 \times [0,1]^{m-1}$.

2.1.1 Partial partition η_n

Remark 2.1. For convenience we will use the notation $\prod_{i=2}^{m}[a_i,b_i]$ for $[a_2,b_2]\times ... \times [a_m,b_m]$.

Initially, η_n will be constructed on the fundamental sector $\left[0, \frac{1}{q_n}\right] \times [0,1]^{m-1}$. For this purpose, we divide the fundamental sector in n sections:

- In case of $k \in \mathbb{N}$ and $2 \leq k \leq n-1$ on $\left[\frac{k-1}{n\cdot q_n}, \frac{k}{n\cdot q_n}\right] \times [0,1]^{m-1}$ the partial partition η_n consists of all multidimensional intervals of the following form:

$$\Bigg[\frac{k-1}{n\cdot q_n} + \frac{s}{n^2\cdot k_n^{m-1}\cdot q_n} + \frac{j_1^{(1)}}{n^2\cdot k_n^{m-1}\cdot q_n^2} + ... + \frac{j_1^{\left((m-1)\cdot\frac{(k+1)\cdot k}{2}\right)}}{n^2\cdot k_n^{m-1}\cdot q_n^{1+(m-1)\cdot\frac{(k+1)\cdot k}{2}}}$$
$$+ \frac{1}{10\cdot n^6\cdot q_{n-1}\cdot k_n^{m-1}\cdot q_n^{1+(m-1)\cdot\frac{(k+1)\cdot k}{2}}},$$
$$\frac{k-1}{n\cdot q_n} + \frac{s}{n^2\cdot k_n^{m-1}\cdot q_n} + \frac{j_1^{(1)}}{n^2\cdot k_n^{m-1}\cdot q_n^2} + ... + \frac{j_1^{\left((m-1)\cdot\frac{(k+1)\cdot k}{2}\right)}+1}{n^2\cdot k_n^{m-1}\cdot q_n^{1+(m-1)\cdot\frac{(k+1)\cdot k}{2}}}$$
$$- \frac{1}{10\cdot n^6\cdot q_{n-1}\cdot k_n^{m-1}\cdot q_n^{1+(m-1)\cdot\frac{(k+1)\cdot k}{2}}}\Bigg]$$
$$\times \prod_{i=2}^{m}\left[\frac{j_i^{(1)}}{q_n} + ... + \frac{j_i^{(k+1)}}{q_n^{k+1}} + \frac{1}{26n^4\cdot q_{n-1}\cdot q_n^{k+1}}, \frac{j_i^{(1)}}{q_n} + ... + \frac{j_i^{(k+1)}+1}{q_n^{k+1}} - \frac{1}{26n^4\cdot q_{n-1}\cdot q_n^{k+1}}\right],$$

  where $s \in \mathbb{Z}$ and $0 \leq s \leq nk_n^{m-1}-1$ as well as $j_1^{(l)} \in \mathbb{Z}$ and $\left\lceil \frac{q_n}{10n^4 q_{n-1}}\right\rceil \leq j_1^{(l)} \leq q_n - \left\lceil \frac{q_n}{10n^4 q_{n-1}}\right\rceil - 1$ for $l = 1, ..., (m-1)\cdot\frac{(k+1)\cdot k}{2}$ as well as $j_i^{(l)} \in \mathbb{Z}$ and $\left\lceil \frac{q_n}{10n^4 q_{n-1}}\right\rceil \leq j_i^{(l)} \leq q_n - \left\lceil \frac{q_n}{10n^4 q_{n-1}}\right\rceil - 1$ for $i = 2, ..., m$ and $l = 1, ..., k+1$.

- On $\left[0, \frac{1}{n\cdot q_n}\right] \times [0,1]^{m-1}$ as well as $\left[\frac{n-1}{n\cdot q_n}, \frac{1}{q_n}\right] \times [0,1]^{m-1}$ there are no elements of the partial partition η_n.

As the image under R_{l/q_n} with $l \in \mathbb{Z}$ this partial partition of $\left[0, \frac{1}{q_n}\right] \times [0,1]^{m-1}$ is extended to a partial partition of $\mathbb{S}^1 \times [0,1]^{m-1}$.

Remark 2.2. By construction this sequence of partial partitions converges to the decomposition into points.

2.1.2 Partial partition ζ_n

As in the previous case we will construct the partial partition ζ_n on the fundamental sector $\left[0, \frac{1}{q_n}\right] \times [0,1]^{m-1}$ initially and therefore divide this sector into n sections: In case of $k \in \mathbb{N}$ and $1 \leq k \leq n$ on $\left[\frac{k-1}{n \cdot q_n}, \frac{k}{n \cdot q_n}\right] \times [0,1]^{m-1}$ the partial partition ζ_n consists of all multidimensional intervals of the following form:

$$\Bigg[\frac{k-1}{n \cdot q_n} + \frac{s_1}{n^2 \cdot k_n^{m-1} \cdot q_n} + \frac{j_1^{(1)}}{n^2 \cdot k_n^{m-1} \cdot q_n^2} + ... + \frac{j_1^{\left((m-1)\cdot \frac{(n+1)\cdot n}{2}\right)}}{n^2 \cdot k_n^{m-1} \cdot q_n^{1+(m-1)\cdot \frac{(n+1)\cdot n}{2}}}$$
$$+ \frac{t_1}{n^2 \cdot q_{n-1} \cdot k_n^{m-1} \cdot q_n^{1+(m-1)\cdot \frac{n\cdot(n+1)}{2}}} + \frac{1}{5n^6 \cdot q_{n-1}^2 \cdot k_n^{m-1} \cdot q_n^{1+(m-1)\cdot \frac{n\cdot(n+1)}{2}}},$$
$$\frac{k-1}{n \cdot q_n} + \frac{s_1}{n^2 \cdot k_n^{m-1} \cdot q_n} + \frac{j_1^{(1)}}{n^2 \cdot k_n^{m-1} \cdot q_n^2} + ... + \frac{j_1^{\left((m-1)\cdot \frac{(n+1)\cdot n}{2}\right)}}{n^2 \cdot k_n^{m-1} \cdot q_n^{1+(m-1)\cdot \frac{(n+1)\cdot n}{2}}}$$
$$+ \frac{t_1+1}{n^2 \cdot q_{n-1} \cdot k_n^{m-1} \cdot q_n^{1+(m-1)\cdot \frac{n\cdot(n+1)}{2}}} - \frac{1}{5n^6 \cdot q_{n-1}^2 \cdot k_n^{m-1} \cdot q_n^{1+(m-1)\cdot \frac{n\cdot(n+1)}{2}}}\Bigg]$$
$$\times \prod_{i=2}^{m} \Bigg[\frac{j_i^{(1)}}{q_n} + ... + \frac{j_i^{\left(1+(m-1)\cdot \frac{n\cdot(n+1)}{2}\right)}}{q_n^{1+(m-1)\cdot \frac{n\cdot(n+1)}{2}}} + \frac{s_i}{n^2 \cdot k_n^{m-1} \cdot q_n^{1+(m-1)\cdot \frac{n\cdot(n+1)}{2}}}$$
$$+ \frac{t_i}{n^2 \cdot q_{n-1} \cdot k_n^{m-1} \cdot q_n^{1+(m-1)\cdot \frac{n\cdot(n+1)}{2}}} + \frac{1}{5n^6 \cdot q_{n-1}^2 \cdot k_n^{m-1} \cdot q_n^{1+(m-1)\cdot \frac{n\cdot(n+1)}{2}}},$$
$$\frac{j_i^{(1)}}{q_n} + ... + \frac{t_i+1}{n^2 \cdot q_{n-1} \cdot k_n^{m-1} \cdot q_n^{1+(m-1)\cdot \frac{n\cdot(n+1)}{2}}} - \frac{1}{5n^6 \cdot q_{n-1}^2 \cdot k_n^{m-1} \cdot q_n^{1+(m-1)\cdot \frac{n\cdot(n+1)}{2}}}\Bigg],$$

where

- $j_1^{(l)} \in \mathbb{Z}$, $\left\lceil \frac{q_n}{n^4 q_{n-1}} \right\rceil \leq j_1^{(l)} \leq q_n - \left\lceil \frac{q_n}{n^4 q_{n-1}} \right\rceil - 1$, for $l = 1, ..., (m-1) \cdot \frac{n\cdot(n+1)}{2}$
- $j_i^{(l)} \in \mathbb{Z}$, $\left\lceil \frac{q_n}{n^4 q_{n-1}} \right\rceil \leq j_i^{(l)} \leq q_n - \left\lceil \frac{q_n}{n^4 q_{n-1}} \right\rceil - 1$, for $l = 1, ..., (m-1) \cdot \frac{n\cdot(n+1)}{2} + 1$ and $i = 2, ..., m$
- $s_1 \in \mathbb{Z}$, $0 \leq s_1 \leq n k_n^{m-1} - 1$
- $s_i \in \mathbb{Z}$, $0 \leq s_i \leq n^2 k_n^{m-1} - 1$, for $i = 2, ..., m$
- $t_i \in \mathbb{Z}$, $1 \leq t_i \leq q_{n-1} - 2$, for $i = 1, ..., m$.

Remark 2.3. For every $n \geq m$ the partial partition ζ_n consists of disjoint sets, covers a set of measure at least $1 - \frac{3\cdot m}{q_{n-1}}$ and the sequence $(\zeta_n)_{n\in\mathbb{N}}$ converges to the decomposition into points.

2.2 The conjugation map g_n

Let $0.25 < \sigma < 0.5$. On the one hand, the map g_n shall introduce some kind of shear in the θ-direction as the map $\tilde{g}_{[nq_n^{\sigma}]}(\theta, r_1, ..., r_{m-1}) = (\theta + [nq_n^{\sigma}] \cdot r_1, r_1, ..., r_{m-1})$, which is helpful in the proof of the weak mixing-property. On the other hand, g_n must be an isometry on $i_n \circ \phi_n \left(\check{I}_n\right)$ for

all the partition elements $\check{I}_n \in \zeta_n$ in order to admit the construction of a f-invariant measurable Riemannian metric.
Inspired by the constructions in [Be13], section 4.1, let $a, b \in \mathbb{N}$, $\varepsilon > 0$ satisfying $\frac{1}{\varepsilon} \in \mathbb{Z}$ and $\rho : \mathbb{R} \to \mathbb{R}$ be a smooth increasing function that equals 0 for $x \leq -1$ and 1 for $x \geq 0$. Moreover, we consider $\delta > 0$ such that $\frac{1}{\delta} \in \mathbb{Z}$ and $a \cdot \delta = r \in \mathbb{N}$. Then we define the map $\tilde{\psi}_{a,b,\varepsilon,\delta} : [0,1] \to \mathbb{R}$ by

$$\tilde{\psi}_{a,b,\varepsilon,\delta}(x) = \frac{b \cdot r}{a} \cdot \rho\left(\frac{x}{\varepsilon} - \frac{r}{a \cdot \varepsilon}\right) + \frac{b}{a} \cdot \sum_{i=r+1}^{a-r-1} \rho\left(\frac{x}{\varepsilon} - \frac{i}{a \cdot \varepsilon}\right) + \frac{b \cdot (r+1)}{a} \cdot \rho\left(\frac{x}{\varepsilon} - \frac{a-r}{a \cdot \varepsilon}\right).$$

Note that $\tilde{\psi}_{a,b,\varepsilon,\delta}|_{[0,\frac{\delta}{2}] \cup [1-\frac{\delta}{2},1]} \equiv 0 \mod 1$ and for every $r \leq i \leq a-r-1$ we have $\tilde{\psi}_{a,b,\varepsilon,\delta}|_{[\frac{i}{a}, \frac{i+1}{a} - \varepsilon]} = b \cdot \frac{i}{a}$. Furthermore, we can estimate $\left\| D^l \tilde{\psi}_{a,b,\varepsilon,\delta} \right\|_0 \leq \frac{b}{\varepsilon^l} \cdot \left\| D^l \rho \right\|_0$.
Besides this map $\tilde{\psi}_{a,b,\varepsilon,\delta}$ we use a smooth map $\sigma_\delta : \mathbb{R} \to [0,1]$ satisfying $\sigma_\delta(x) = 0$ for $x \leq \frac{\delta}{2}$, $\sigma_\delta(x) = 1$ for $\delta \leq x \leq 1 - \delta$ and $\sigma_\delta(x) = 0$ for $x \geq 1 - \frac{\delta}{2}$. Then we define the measure-preserving diffeomorphism $g_{a,b,\varepsilon,\delta} : \mathbb{S}^1 \times [0,1]^{m-1} \to \mathbb{S}^1 \times [0,1]^{m-1}$ by

$$g_{a,b,\varepsilon,\delta}(\theta, r_1, ..., r_{m-1}) = \left(\theta + \tilde{\psi}_{a,b,\varepsilon,\delta}(r_1) \cdot \sigma_\delta(r_2) \cdot ... \sigma_\delta(r_{m-1}), r_1, ..., r_{m-1}\right).$$

We emphasize that the maps σ_δ are introduced to guarantee that $g_{a,b,\varepsilon,\delta}$ coincides with the identity in a neigbourhood of the boundary.
In our concrete constructions we will use

$$g_n = g_{n^2 \cdot k_n^{m-1} \cdot q_n^{1+(m-1)\cdot \frac{n\cdot(n+1)}{2}}, [nq_n^\sigma], \frac{1}{60n^4 q_{n-1}}, \frac{1}{30n^4 q_{n-1}}}.$$

Since $30n^4 q_{n-1}$ divides q_n due to Lemma 5.8, the condition $a\delta \in \mathbb{N}$ is satisfied. Moreover, we observe $g_n \circ R_{\frac{1}{q_n}} = R_{\frac{1}{q_n}} \circ g_n$ and $|||g_n|||_l \leq C_{l,n,q_{n-1},k_n} \cdot [nq_n^\sigma]$, where the constant C_{l,n,q_{n-1},k_n} depends on l, n and q_{n-1}.

2.3 The conjugation map i_n

In this subsection we define the so-called "inner rotations" i_n which will allow us to prove ergodicity of the projectivized derivative extension. For the construction we need the subsequent Lemma:

Lemma 2.4. *Let $c \in \mathbb{N}$, $c \geq 3$, $\varepsilon \in \left(0, \frac{1}{5c}\right]$ and $\beta_2, ..., \beta_m \in [0, \pi]$. Then there is a smooth measure-preserving diffeomorphism $\psi_{c,\varepsilon,\beta_2,...,\beta_m} : [0,1]^m \to [0,1]^m$ satisfying the following properties:*

- *$\psi_{c,\varepsilon,\beta_2,...,\beta_m}$ coincides with the identity on $[0,1]^m \setminus [\varepsilon, 1-\varepsilon]^m$.*
- *On every cube $\prod_{i=1}^m \left[\frac{j_i+\varepsilon}{c}, \frac{j_i+1-\varepsilon}{c}\right]$ with $1 \leq j_i \leq c-2$ the map $\psi_{c,\varepsilon,\beta_2,...,\beta_m}$ is equal to a composition of a translation and the rotations by arc β_i around a new center in the $x_1 - x_i$-coordinates.*

Proof. Similar to [GKu15], Lemma 3.4., such a measure-preserving diffemorphism is constructed with the aid of Moser's trick. $\square$

Using the dilation $D_a : \left[0, \frac{1}{a}\right]^m \to [0,1]^m$, $D_a(x_1, ..., x_m) = (a \cdot x_1, ..., a \cdot x_m)$ for $a \in \mathbb{Z}$ we define the map $\psi_{a,c,\varepsilon,\beta_2,...,\beta_m} : \left[0, \frac{1}{a}\right]^m \to \left[0, \frac{1}{a}\right]^m$, $\psi_{a,c,\varepsilon,\beta_2,...,\beta_m} = D_a^{-1} \circ \psi_{c,\varepsilon,\beta_2,...,\beta_m} \circ D_a$. Since it coincides with the identity in a neighbourhood of the boundary, we can extend it to a smooth diffeomorphism on $\mathbb{S}^1 \times [0,1]^{m-1}$ equivariantly by the description

$$\psi_{a,c,\varepsilon,\beta_2,...,\beta_m}\left(x_1+\frac{a_1}{a},...,x_m+\frac{a_m}{a}\right)=\left(\frac{a_1}{a},...,\frac{a_m}{a}\right)+\psi_{a,c,\varepsilon,\beta_2,...,\beta_m}\left(x_1,...,x_m\right)$$

for $a_1,...,a_m \in \mathbb{Z}$.
For the sake of convenience, we introduce the notation

$$\tilde{\psi}_{n,\beta_2,...,\beta_m}=\psi_{n^2\cdot k_n^{m-1}\cdot q_n^{1+(m-1)\cdot\frac{n\cdot(n+1)}{2}},q_{n-1},\frac{1}{5n^4 q_{n-1}},\beta_2,...,\beta_m}.$$

On $\left[\frac{i}{n^2\cdot k_n^{m-1}\cdot q_n},\frac{i+1}{n^2\cdot k_n^{m-1}\cdot q_n}\right]\times[0,1]^{m-1}$ we define for $j\in\{2,...,m\}$:

$$\beta_i^{(j)}=\frac{s\cdot\pi}{k_n}\text{ in case of } s\equiv\left\lceil\frac{i}{k_n^{j-2}}\right\rceil \mod k_n$$

as well as

$$i_n=\tilde{\psi}_{n,\beta_i^{(2)},...,\beta_i^{(m)}}.$$

Since each map coincides with the identity in a neighbourhood of the boundary, we can piece them together in order to get a smooth diffeomorphism on $\mathbb{S}^1\times[0,1]^{m-1}$.
On the elements of the partial partition η_n introduced in subsubsection 2.1.1 the diffeomorphism i_n satisfies the subsequent property which will be useful in the proof of Lemma 4.2.

Lemma 2.5. *For every element $\hat{I}_n\in\eta_n$ we have $i_n\left(\hat{I}_n\right)=\hat{I}_n$.*

Proof. Since $260n^4q_{n-1}$ divides q_n by Lemma 5.8, there is $u_1\in\mathbb{Z}$ such that

$$\frac{1}{10\cdot n^6\cdot q_{n-1}\cdot k_n^{m-1}\cdot q_n^{1+(m-1)\cdot\frac{(k+1)\cdot k}{2}}}=u_1\cdot\frac{1}{n^2\cdot k_n^{m-1}\cdot q_n^{1+(m-1)\cdot\frac{n\cdot(n+1)}{2}}}$$

and $u_2\in\mathbb{Z}$ such that

$$\frac{1}{26n^4q_{n-1}q_n^{k+1}}=u_2\cdot\frac{1}{n^2\cdot k_n^{m-1}\cdot q_n^{1+(m-1)\cdot\frac{n\cdot(n+1)}{2}}}.$$

Hence, $\hat{I}_n$ is a union of complete definition blocks of the map i_n. These blocks are mapped onto itself under the map i_n because i_n coincides with the identity in the neighbourhood of the boundary of each definition block. □

2.4 The conjugation map ϕ_n

In [GKu15], section 3.3, we constructed the smooth measure-preserving diffeomorphism $\tilde{\phi}_{\lambda,\varepsilon,i,j,\mu,\delta,\varepsilon_2}$ on $\mathbb{S}^1\times[0,1]^{m-1}$.
For the sake of convenience, we will use the notation $\phi^{(j)}_{\lambda,\mu}=\tilde{\phi}_{\lambda,\frac{1}{60n^4q_{n-1}},1,j,\mu,\frac{1}{10n^4q_{n-1}},\frac{1}{22n^4q_{n-1}}}$.
With this we define the diffeomorphism ϕ_n on the fundamental sector: On $\left[\frac{k-1}{n\cdot q_n},\frac{k}{n\cdot q_n}\right]\times[0,1]^{m-1}$ in case of $k\in\mathbb{N}$ and $1\le k\le n$:

$$\phi_n=\tilde{\phi}^{(m)}_{n^2\cdot k_n^{m-1}\cdot q_n^{1+(m-1)\cdot\frac{k\cdot(k-1)}{2}+(m-2)\cdot k},q_n^k}\circ\tilde{\phi}^{(m-1)}_{n^2\cdot k_n^{m-1}\cdot q_n^{1+(m-1)\cdot\frac{k\cdot(k-1)}{2}+(m-3)\cdot k},q_n^k}\circ...\circ\tilde{\phi}^{(2)}_{n^2\cdot k_n^{m-1}\cdot q_n^{1+(m-1)\cdot\frac{k\cdot(k-1)}{2}},q_n^k}$$

This is a smooth map because ϕ_n coincides with the identity in a neighbourhood of the different sections.
Now we extend ϕ_n to a diffeomorphism on $\mathbb{S}^1\times[0,1]^{m-1}$ using the description $\phi_n\circ R_{\frac{1}{q_n}}=R_{\frac{1}{q_n}}\circ\phi_n$.

2.5 The conjugation map h_n

With the aid of the previos constructions we define the conjugation map $h_n = g_n \circ i_n \circ \phi_n$. By the observations in the previous subsections we have $h_n \circ R_{\frac{1}{q_n}} = R_{\frac{1}{q_n}} \circ h_n$.

3 $(\gamma, \delta, \epsilon)$-distribution

We recall the notion of a $(\gamma, \delta, \epsilon)$-distribution, which was the central notion of the criterion for weak mixing deduced in [GKu15] and will be important in our proof of the weak mixing-property as well:

Definition 3.1. Let $\Phi : M \to M$ be a diffeomorphism. We say Φ $(\gamma, \delta, \epsilon)$-distributes an element $\hat{I}$ of a partial partition if the following properties are satisfied:

- $\pi_{\vec{r}}\left(\Phi\left(\hat{I}\right)\right)$ is a $(m-1)$-dimensional interval J, i.e. $J = I_1 \times ... \times I_{m-1}$ with intervals $I_k \subseteq [0,1]$, and $1 - \delta \leq \lambda\left(I_k\right) \leq 1$ for $k = 1, ..., m-1$. Here, $\pi_{\vec{r}}$ denotes the projection on the $(r_1, ..., r_{m-1})$-coordinates.
- $\Phi\left(\hat{I}\right)$ is contained in a set of the form $[c, c+\gamma] \times J$ for some $c \in \mathbb{S}^1$.
- For every $(m-1)$-dimensional interval $\tilde{J} \subseteq J$ it holds:

$$\left| \frac{\mu\left(\hat{I} \cap \Phi^{-1}\left(\mathbb{S}^1 \times \tilde{J}\right)\right)}{\mu\left(\hat{I}\right)} - \frac{\mu^{(m-1)}\left(\tilde{J}\right)}{\mu^{(m-1)}\left(J\right)} \right| \leq \epsilon \cdot \frac{\mu^{(m-1)}\left(\tilde{J}\right)}{\mu^{(m-1)}\left(J\right)},$$

  where $\mu^{(m-1)}$ is the Lebesgue measure on $[0,1]^{m-1}$.

Let $A \coloneqq 780 n^6 \cdot (n+1)^6 \cdot q_{n-1}^2 \cdot k_n^{m-1} \cdot q_n^{1+(m-1)\cdot \frac{n \cdot (n+1)}{2}}$. Analogous to [GKu15] we define the sequence of natural numbers $(m_n)_{n \in \mathbb{N}}$:

$$\begin{aligned} m_n &= \min\left\{ m \leq q_{n+1} \quad : \quad m \in \mathbb{N}, \ \inf_{k \in \mathbb{Z}} \left| m \cdot \frac{p_{n+1}}{q_{n+1}} - \frac{1}{n \cdot q_n} + \frac{k}{q_n} \right| \leq \frac{A}{q_{n+1}} \right\} \\ &= \min\left\{ m \leq q_{n+1} \quad : \quad m \in \mathbb{N}, \ \inf_{k \in \mathbb{Z}} \left| m \cdot \frac{q_n \cdot p_{n+1}}{q_{n+1}} - \frac{1}{n} + k \right| \leq \frac{A \cdot q_n}{q_{n+1}} \right\}. \end{aligned}$$

Lemma 3.2. *The set* $\left\{ m \leq q_{n+1} \quad : \quad m \in \mathbb{N}, \ \inf_{k \in \mathbb{Z}} \left| m \cdot \frac{q_n \cdot p_{n+1}}{q_{n+1}} - \frac{1}{n} + k \right| \leq \frac{A \cdot q_n}{q_{n+1}} \right\}$ *is nonempty for every* $n \in \mathbb{N}$, *i.e.* m_n *exists.*

Proof. In Lemma 5.8 we will construct the sequence $\alpha_n = \frac{p_n}{q_n}$ in such a way, that

$$\begin{aligned} q_n &\coloneqq 780 n^6 \cdot (n-1)^6 \cdot q_{n-2}^2 \cdot k_{n-1}^{m-1} \cdot q_{n-1}^{1+(m-1)\cdot \frac{n \cdot (n-1)}{2}} \cdot \tilde{q}_n \\ \text{and } p_n &\coloneqq 780 n^6 \cdot (n-1)^6 \cdot q_{n-2}^2 \cdot k_{n-1}^{m-1} \cdot q_{n-1}^{1+(m-1)\cdot \frac{n \cdot (n-1)}{2}} \cdot \tilde{p}_n \end{aligned}$$

with $\tilde{p}_n, \tilde{q}_n$ relatively prime. Then the proof follows along the lines of [GKu15], Lemma 4.3.. $\square$

Remark 3.3. We define

$$a_n = \left(m_n \cdot \frac{p_{n+1}}{q_{n+1}} - \frac{1}{n \cdot q_n} \right) \mod \frac{1}{q_n}$$

By the above construction of m_n it holds that $|a_n| \leq \frac{780 n^6 \cdot (n+1)^6 \cdot q_{n-1}^2 \cdot k_n^{m-1} \cdot q_n^{1+(m-1) \cdot \frac{n \cdot (n+1)}{2}}}{q_{n+1}}$. In Lemma 5.8 we will see that it is possible to choose $q_{n+1} \geq 30 \cdot 780 \cdot n^{14} \cdot (n+1)^6 \cdot q_{n-1}^3 \cdot k_n^{3m-3} \cdot q_n^{3+2 \cdot (m-1) \cdot n \cdot (n+1)}$. Thus, we get:

$$|a_n| \leq \frac{1}{30 \cdot n^8 \cdot q_{n-1} \cdot k_n^{2m-2} \cdot q_n^{2+(m-1) \cdot n \cdot (n+1)}}.$$

Our constructions are done in such a way that the following property is satisfied:

Lemma 3.4. *The map $\Phi_n := \phi_n \circ R_{\alpha_{n+1}}^{m_n} \circ \phi_n^{-1}$ with the conjugating maps ϕ_n defined in section 2.4 $\left(\frac{1}{n \cdot q_n^m}, \frac{1}{n^4}, \frac{1}{n} \right)$-distributes the elements of the partition η_n.*

Proof. The proof follows by the same calculations as in the proof of [GKu15], Lemma 4.5.. In this connection, we require the bound on a_n and that $260 n^4 q_{n-1}$ divides q_n. Then we obtain for a partition element $\hat{I}_{n,k} \in \eta_n$ on $\left[\frac{k-1}{n q_n}, \frac{k}{n q_n} \right] \times [0,1]^{m-1}$ that $\Phi_n \left(\hat{I}_{n,k} \right)$ is equal to:

$$\left[\frac{k}{n\cdot q_n}+\frac{s}{n^2\cdot k_n^{m-1}\cdot q_n}+\frac{j_1^{(1)}}{n^2\cdot k_n^{m-1}\cdot q_n^2}+...+\frac{j_1^{\left((m-1)\cdot\frac{(k-1)\cdot k}{2}\right)}}{n^2\cdot k_n^{m-1}\cdot q_n^{(m-1)\cdot\frac{(k-1)\cdot k}{2}+1}}\right.$$
$$+\frac{j_2^{(1)}}{n^2\cdot k_n^{m-1}\cdot q_n^{(m-1)\cdot\frac{(k-1)\cdot k}{2}+2}}+...+\frac{j_2^{(k)}}{n^2\cdot k_n^{m-1}\cdot q_n^{(m-1)\cdot\frac{(k-1)\cdot k}{2}+k+1}}$$
$$+\frac{j_3^{(1)}}{n^2\cdot k_n^{m-1}\cdot q_n^{(m-1)\cdot\frac{(k-1)\cdot k}{2}+k+2}}+...+\frac{j_m^{(k)}}{n^2\cdot k_n^{m-1}\cdot q_n^{(m-1)\cdot\frac{(k+1)\cdot k}{2}+1}}+\frac{j_1^{\left((m-1)\cdot\frac{(k-1)\cdot k}{2}+1\right)}}{n^2\cdot k_n^{m-1}\cdot q_n^{(m-1)\cdot\frac{(k+1)\cdot k}{2}+2}}$$
$$+...+\frac{j_1^{\left((m-1)\cdot\frac{(k-1)\cdot k}{2}+k\right)}+1}{n^2\cdot k_n^{m-1}\cdot q_n^{(m-1)\cdot\frac{(k+1)\cdot k}{2}+k+1}}-\frac{j_2^{(k+1)}+1}{n^2\cdot k_n^{m-1}\cdot q_n^{(m-1)\cdot\frac{(k+1)\cdot k}{2}+k+2}}$$
$$+\frac{j_1^{\left((m-1)\cdot\frac{(k-1)\cdot k}{2}+k+1\right)}}{n^2\cdot k_n^{m-1}\cdot q_n^{(m-1)\cdot\frac{(k+1)\cdot k}{2}+k+3}}+...+\frac{j_1^{\left((m-1)\cdot\frac{(k-1)\cdot k}{2}+2k\right)}+1}{n^2\cdot k_n^{m-1}\cdot q_n^{(m-1)\cdot\frac{(k+1)\cdot k}{2}+2k+2}}$$
$$-\frac{j_3^{(k+1)}+1}{n^2\cdot k_n^{m-1}\cdot q_n^{(m-1)\cdot\frac{(k+1)\cdot k}{2}+2k+3}}+...+\frac{j_1^{\left((m-1)\cdot\frac{(k+1)\cdot k}{2}\right)}+1}{n^2\cdot k_n^{m-1}\cdot q_n^{(m-1)\cdot\frac{(k+1)\cdot(k+2)}{2}}}$$
$$-\frac{j_m^{(k+1)}+1}{n^2\cdot k_n^{m-1}\cdot q_n^{(m-1)\cdot\frac{(k+1)\cdot(k+2)}{2}+1}}+\frac{1}{26\cdot n^6\cdot q_{n-1}\cdot k_n^{m-1}\cdot q_n^{(m-1)\cdot\frac{(k+1)\cdot(k+2)}{2}+1}},$$
$$\frac{k}{n\cdot q_n}+\frac{s}{n^2\cdot k_n^{m-1}\cdot q_n}+\frac{j_1^{(1)}}{n^2\cdot k_n^{m-1}\cdot q_n^2}+...-\frac{j_m^{(k+1)}}{n^2\cdot k_n^{m-1}\cdot q_n^{(m-1)\cdot\frac{(k+1)\cdot(k+2)}{2}+1}}$$
$$\left.-\frac{1}{26\cdot n^6\cdot q_{n-1}\cdot k_n^{m-1}\cdot q_n^{(m-1)\cdot\frac{(k+1)\cdot(k+2)}{2}+1}}\right]$$
$$\times\left[\frac{1}{10n^4q_{n-1}}+n^2\cdot k_n^{m-1}\cdot q_n^{(m-1)\cdot\frac{(k+1)\cdot k}{2}+1}\cdot a_n, 1-\frac{1}{10n^4q_{n-1}}+n^2\cdot k_n^{m-1}\cdot q_n^{(m-1)\cdot\frac{(k+1)\cdot k}{2}+1}\cdot a_n\right]$$
$$\times\prod_{i=3}^{m}\left[\frac{1}{26n^4q_{n-1}}, 1-\frac{1}{26n^4q_{n-1}}\right].$$

Thus, such a set $\Phi_n\left(\hat{I}_n\right)$ with $\hat{I}_n\in\eta_n$ has a θ-width of at most $\frac{1}{n^2\cdot k_n^{m-1}\cdot q_n^{3m+1}}$.
Moreover, we see that we can choose $\epsilon=0$ in the definition of a (γ,δ,ϵ)-distribution: With the notation $A_\theta:=\pi_\theta\left(\Phi_n\left(\hat{I}_n\right)\right)$ we have $\Phi_n\left(\hat{I}_n\right)=A_\theta\times J$ and so for every $(m-1)$-dimensional interval $\tilde{J}\subseteq J$:

$$\frac{\mu\left(\hat{I}_n\cap\Phi_n^{-1}\left(\mathbb{S}^1\times\tilde{J}\right)\right)}{\mu\left(\hat{I}_n\right)}=\frac{\mu\left(\Phi_n\left(\hat{I}_n\right)\cap\mathbb{S}^1\times\tilde{J}\right)}{\mu\left(\Phi_n\left(\hat{I}_n\right)\right)}=\frac{\tilde{\lambda}\left(A_\theta\right)\cdot\mu^{(m-1)}\left(\tilde{J}\right)}{\tilde{\lambda}\left(A_\theta\right)\cdot\mu^{(m-1)}\left(J\right)}=\frac{\mu^{(m-1)}\left(\tilde{J}\right)}{\mu^{(m-1)}\left(J\right)}$$

because Φ_n is measure-preserving. □

With the aid of the precedent calculations we prove the next property concerning the conjugation map i_n introduced in subsection 2.3:

Lemma 3.5. *For every $\hat{I}_n \in \eta_n$ we have: $i_n\left(\Phi_n\left(\hat{I}_n\right)\right) = \Phi_n\left(\hat{I}_n\right)$.*

Proof. In the proof of the precedent Lemma 3.4 we computed $\Phi_n\left(\hat{I}_{n,k}\right)$ for a partition element $\hat{I}_{n,k}$. Now we have to examine the effect of i_n on it.
Since $260n^4 q_{n-1}$ divides q_n by Lemma 5.8, there is $u_1 \in \mathbb{Z}$ such that

$$\frac{1}{10n^4 q_{n-1}} = u_1 \cdot \frac{1}{n^2 \cdot k_n^{m-1} \cdot q_n^{1+(m-1)\cdot\frac{n\cdot(n+1)}{2}}}$$

and $u_2 \in \mathbb{Z}$ such that

$$\frac{1}{26n^4 q_{n-1}} = u_2 \cdot \frac{1}{n^2 \cdot k_n^{m-1} \cdot q_n^{1+(m-1)\cdot\frac{n\cdot(n+1)}{2}}}.$$

Considering the θ-coordinate we observe that in case of $2 \leq k \leq n-2$ there exists $u_3 \in \mathbb{Z}$ such that

$$\frac{1}{26 \cdot n^6 \cdot q_{n-1} \cdot k_n^{m-1} \cdot q_n^{(m-1)\cdot\frac{(k+1)\cdot(k+2)}{2}+1}} = u_3 \cdot \frac{1}{n^2 \cdot k_n^{m-1} \cdot q_n^{1+(m-1)\cdot\frac{n\cdot(n+1)}{2}}}$$

In case of $k = n-1$ we use $\frac{1}{26n^4 q_{n-1}} < \varepsilon = \frac{1}{5n^4 q_{n-1}}$. By the bound on a_n the boundary of $\Phi_n\left(\hat{I}_{n,k}\right)$ lies in the domain where i_n coincides with the identity. □

4 Criterion for weak mixing

We will prove a criterion for weak mixing on $M = \mathbb{S}^1 \times [0,1]^{m-1}$. In [GKu15], Lemma 5.2., we deduced the subsequent characterisation of the weak mixing-property in the setting of the beforehand constructions.

Lemma 4.1. *Let $f = \lim_{n\to\infty} f_n$ be a diffeomorphism obtained by the constructions in the preceding sections and $(m_n)_{n\in\mathbb{N}}$ be a sequence of natural numbers fulfilling $d_0\left(f^{m_n}, f_n^{m_n}\right) < \frac{1}{2^n}$. Furthermore, let $(\nu_n)_{n\in\mathbb{N}}$ be a sequence of partial partitions, where $\nu_n \to \varepsilon$ and for every $n \in \mathbb{N}$ ν_n is the image of a partial partition η_n under a measure-preserving diffeomorphism F_n, satisfying the following property: For every m-dimensional cube $A \subseteq \mathbb{S}^1 \times (0,1)^{m-1}$ and for every $\epsilon \in (0,1]$ there exists $N \in \mathbb{N}$ such that for every $n \geq N$ and for every $\Gamma_n \in \nu_n$ we have*

$$\left|\mu\left(\Gamma_n \cap f_n^{-m_n}(A)\right) - \mu(\Gamma_n)\cdot\mu(A)\right| \leq \epsilon \cdot \mu(\Gamma_n)\cdot\mu(A). \tag{1}$$

Then f is weakly mixing.

Concerning the partial partitions we concentrate on the setting of our explicit constructions:

Lemma 4.2. *Consider the sequence of partial partitions $(\eta_n)_{n\in\mathbb{N}}$ constructed in section 2.1.1 and the diffeomorphisms g_n from section 2.2 as well as i_n from section 2.3. Furthermore, let $(H_n)_{n\in\mathbb{N}}$ be a sequence of measure-preserving smooth diffeomorphisms satisfying $\|DH_{n-1}\| \leq \frac{q_n^{0.25}}{2n^2\cdot\sqrt{m}}$ for every $n \in \mathbb{N}$ and we define the partial partitions $\nu_n = \left\{\Gamma_n = H_{n-1}\circ g_n \circ i_n\left(\hat{I}_n\right) \ : \ \hat{I}_n \in \eta_n\right\}$. Then we get $\nu_n \to \varepsilon$.*

Proof. By construction $\eta_n = \left\{ \hat{I}_n^i : i \in \Lambda_n \right\}$, where Λ_n is a countable set of indices. Because of $\eta_n \to \varepsilon$ it holds $\lim_{n\to\infty} \mu \left(\bigcup_{i\in\Lambda_n} \hat{I}_n^i \right) = 1$. Since $H_{n-1} \circ g_n \circ i_n$ is measure-preserving, we conclude:

$$\lim_{n\to\infty} \mu \left(\bigcup_{i\in\Lambda_n} \Gamma_n^i \right) = \lim_{n\to\infty} \mu \left(\bigcup_{i\in\Lambda_n} H_{n-1} \circ g_n \circ i_n \left(\hat{I}_n^i \right) \right) = \lim_{n\to\infty} \mu \left(H_{n-1} \circ g_n \circ i_n \left(\bigcup_{i\in\Lambda_n} \hat{I}_n^i \right) \right) = 1.$$

In Lemma 2.5 we observed $i_n \left(\hat{I}_n \right) = \hat{I}_n$ for every $\hat{I}_n \in \eta_n$. Additionally, by the definitions of an element $\hat{I}_n \in \eta_n$ and the map g_n we observe that $g_n \left(\hat{I}_n \right)$ is contained in a cuboid of θ-width $\frac{1}{n^2 \cdot k_n^{m-1} \cdot q_n^{3m-2}} + [nq_n^\sigma] \cdot \frac{1}{q_n^3}$ and edge length $\frac{1}{q_n^3}$ in the $r_1, ..., r_{m-1}$-coordinates. Hence, the diameter of $g_n \left(\hat{I}_n \right)$ is bounded by $\frac{2\sqrt{m} \cdot [nq_n^\sigma]}{q_n^3}$. Then we conclude for every $\Gamma_n = H_{n-1} \circ g_n \circ i_n \left(\hat{I}_n \right)$:

$$\operatorname{diam}(\Gamma_n) \leq \|DH_{n-1}\|_0 \cdot \operatorname{diam}\left(g_n \left(\hat{I}_n \right) \right) \leq \frac{q_n^{0.25}}{2n^2 \cdot \sqrt{m}} \cdot \frac{2\sqrt{m} \cdot [nq_n^\sigma]}{q_n^3} \leq \frac{1}{n \cdot q_n}$$

using that $\sigma < 1$. Hence, we have $\lim_{n\to\infty} \operatorname{diam}(\Gamma_n) \to 0$ and consequently $\nu_n \to \varepsilon$. □

In the following the Lebesgue measures on $\mathbb{S}^1$, $[0,1]^{m-2}$, $[0,1]^{m-1}$ are denoted by $\tilde{\lambda}$, $\mu^{(m-2)}$ and $\tilde{\mu}$ respectively. The next technical result is needed in the proof of Lemma 4.4. For the sake of convenience, we introduce the notation $a = n^2 \cdot k_n^{m-1} \cdot q_n^{1+(m-1)\cdot \frac{n\cdot(n+1)}{2}}$.

Lemma 4.3. *Given an interval K on the r_1-axis and a $(m-2)$-dimensional interval Z in the $(r_2, ..., r_{m-1})$-coordinates $K_{c,\gamma}$ denotes the cuboid $[c, c+\gamma] \times K \times Z$ for some $\gamma > 0$. We consider the diffeomorphism g_n constructed in subsection 2.2 and an interval $L = [l_1, l_2]$ of $\mathbb{S}^1$ satisfying $\tilde{\lambda}(L) \geq \frac{3\cdot[nq_n^\sigma]}{a}$.*
If $[nq_n^\sigma] \cdot \lambda(K) > 2$, then for the set $Q := \pi_{\vec{r}} \left(K_{c,\gamma} \cap g_n^{-1} (L \times K \times Z) \right)$ we have:

$$\begin{aligned} &\left| \tilde{\mu}(Q) - \lambda(K) \cdot \tilde{\lambda}(L) \cdot \mu^{(m-2)}(Z) \right| \\ &\leq \left(\frac{2}{[nq_n^\sigma]} \cdot \tilde{\lambda}(L) + \frac{2\cdot\gamma}{[nq_n^\sigma]} + \gamma \cdot \lambda(K) + \frac{[nq_n^\sigma] \cdot \lambda(K)}{a} + \frac{2}{a} \right) \cdot \mu^{(m-2)}(Z). \end{aligned}$$

Proof. We consider the diffeomorphism $\tilde{g}_b : M \to M$, $(\theta, r_1, ..., r_{m-1}) \mapsto (\theta + b \cdot r_1, r_1, ..., r_{m-1})$ and the set:

$$\begin{aligned} Q_b &:= \pi_{\vec{r}} \left(K_{c,\gamma} \cap \tilde{g}_b^{-1} (L \times K \times Z) \right) \\ &= \{ (r_1, r_2, ..., r_{m-1}) \in K \times Z \ : \ (\theta + b \cdot r_1, \vec{r}) \in L \times K \times Z, \theta \in [c, c+\gamma] \} \\ &= \{ (r_1, r_2, ..., r_{m-1}) \in K \times Z \ : \ b \cdot r_1 \in [l_1 - c - \gamma, l_2 - c] \mod 1 \}. \end{aligned}$$

The interval $b \cdot K$ seen as an interval in $\mathbb{R}$ does not intersect more than $b \cdot \lambda(K) + 2$ and not less than $b \cdot \lambda(K) - 2$ intervals of the form $[i, i+1]$ with $i \in \mathbb{Z}$.
Recall that g_n is constructed as a stepwise approximation of $\tilde{g}_{[nq_n^\sigma]}$. Obviously, $\tilde{g}_{[nq_n^\sigma]}(K_{c,\gamma})$ may hit (respectively leave) $L \times K \times Z$ at most one $\frac{1}{a}$-domain on the r_1-axis later than $\tilde{g}_{[nq_n^\sigma]}(K_{c,\gamma})$ (see figure 1). Thus, a resulting interval on the r_1-axis of $K_{c,\gamma} \cap \tilde{g}_{[nq_n^\sigma]}^{-1} (L \times K \times Z)$ and the corresponding r_1-projection of $K_{c,\gamma} \cap g_n^{-1} (L \times K \times Z)$ can differ by a length of at most $\frac{1}{a}$.

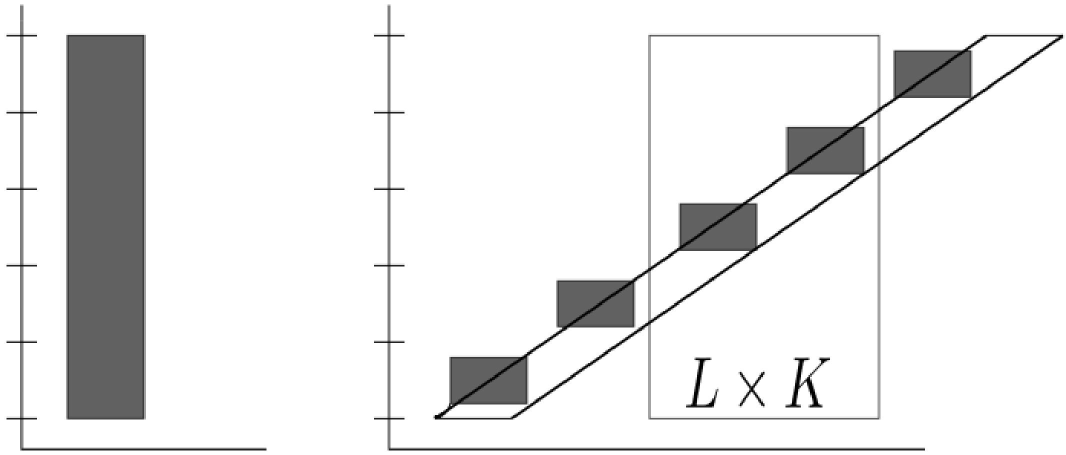

Figure 1: Qualitative shape of the action of g_n as well as $\tilde{g}_{[nq_n^\sigma]}$ on $K_{c,\gamma}$.

Therefore, we compute on the one side:

$$\begin{aligned}\tilde{\mu}(Q) &\leq \left([nq_n^\sigma]\cdot\lambda(K)+2\right)\cdot\left(\frac{l_2-(l_1-\gamma)}{[nq_n^\sigma]}+\frac{1}{a}\right)\cdot\mu^{(m-2)}(Z)\\ &=\left(\lambda(K)\cdot\tilde{\lambda}(L)+2\cdot\frac{\tilde{\lambda}(L)}{[nq_n^\sigma]}+\lambda(K)\cdot\gamma+\frac{2\cdot\gamma}{[nq_n^\sigma]}+\frac{[nq_n^\sigma]\cdot\lambda(K)}{a}+\frac{2}{a}\right)\cdot\mu^{(m-2)}(Z)\end{aligned}$$

and on the other side

$$\begin{aligned}\tilde{\mu}(Q) &\geq \left([nq_n^\sigma]\cdot\lambda(K)-2\right)\cdot\left(\frac{l_2-(l_1-\gamma)}{[nq_n^\sigma]}-\frac{1}{a}\right)\cdot\mu^{(m-2)}(Z)\\ &=\left(\lambda(K)\cdot\tilde{\lambda}(L)-2\cdot\frac{\tilde{\lambda}(L)}{[nq_n^\sigma]}+\lambda(K)\cdot\gamma-\frac{2\cdot\gamma}{[nq_n^\sigma]}-\frac{[nq_n^\sigma]\cdot\lambda(K)}{a}+\frac{2}{a}\right)\cdot\mu^{(m-2)}(Z).\end{aligned}$$

Both equations together yield:

$$\begin{aligned}&\left|\tilde{\mu}(Q)-\lambda(K)\cdot\tilde{\lambda}(L)\cdot\mu^{(m-2)}(Z)-\gamma\cdot\lambda(K)\cdot\mu^{(m-2)}(Z)-\frac{2}{a}\cdot\mu^{(m-2)}(Z)\right|\\ &\leq\left(\frac{2}{[nq_n^\sigma]}\cdot\tilde{\lambda}(L)+\frac{2\cdot\gamma}{[nq_n^\sigma]}+\frac{[nq_n^\sigma]\cdot\lambda(K)}{a}\right)\cdot\mu^{(m-2)}(Z).\end{aligned}$$

The claim follows because

$$\begin{aligned}&\left|\tilde{\mu}(Q)-\lambda(K)\cdot\tilde{\lambda}(L)\cdot\mu^{(m-2)}(Z)\right|-\gamma\cdot\lambda(K)\cdot\mu^{(m-2)}(Z)-\frac{2}{a}\cdot\mu^{(m-2)}(Z)\\ &\leq\left|\tilde{\mu}(Q)-\lambda(K)\cdot\tilde{\lambda}(L)\cdot\mu^{(m-2)}(Z)-\gamma\cdot\lambda(K)\cdot\mu^{(m-2)}(Z)-\frac{2}{a}\cdot\mu^{(m-2)}(Z)\right|.\end{aligned}$$

□

Lemma 4.4. *Let $n \geq 5$, g_n as in section 2.2, i_n as in section 2.3 and $\hat{I}_n \in \eta_n$, where η_n is the partial partition constructed in section 2.1.1. For the diffeomorphism ϕ_n constructed in section 2.4 and m_n as in chapter 3 we consider $\Phi_n = \phi_n \circ R^{m_n}_{\alpha_{n+1}} \circ \phi_n^{-1}$ and denote $\pi_{\vec{r}}\left(\Phi_n\left(\hat{I}_n\right)\right)$ by J. Then for every m-dimensional cube S of side length $q_n^{-\sigma}$ lying in $\mathbb{S}^1 \times J$ we get*

$$\left|\mu\left(\hat{I} \cap \Phi_n^{-1} \circ i_n^{-1} \circ g_n^{-1}(S)\right) \cdot \tilde{\mu}(J) - \mu\left(\hat{I}\right) \cdot \mu(S)\right| \leq \frac{20}{n} \cdot \mu\left(\hat{I}\right) \cdot \mu(S). \tag{2}$$

In other words this Lemma tells us that a partition element is "almost uniformly distributed" under $g_n \circ i_n \circ \Phi_n$ on the whole manifold $M = \mathbb{S}^1 \times [0,1]^{m-1}$.

Proof. Let S be a m-dimensional cube with sidelength $q_n^{-\sigma}$ lying in $\mathbb{S}^1 \times J$. Furthermore, we denote:

$$S_\theta = \pi_\theta(S) \qquad S_{r_1} = \pi_{r_1}(S) \qquad S_{\tilde{\vec{r}}} = \pi_{(r_2,\dots,r_{m-1})}(S) \qquad S_r = S_{r_1} \times S_{\tilde{\vec{r}}} = \pi_{\vec{r}}(S)$$

Obviously: $\tilde{\lambda}(S_\theta) = \lambda(S_{r_1}) = q_n^{-\sigma}$ and $\tilde{\lambda}(S_\theta) \cdot \lambda(S_{r_1}) \cdot \mu^{(m-2)}\left(S_{\tilde{\vec{r}}}\right) = \mu(S) = q_n^{-m\sigma}$.
According to Lemma 3.4 $\Phi_n\left(\frac{1}{n \cdot q_n^m}, \frac{1}{n^4}, \frac{1}{n}\right)$-distributes the partition element $\hat{I}_n \in \eta_n$, in particular $\Phi_n\left(\hat{I}_n\right) \subseteq [c, c+\gamma] \times J$ for some $c \in \mathbb{S}^1$ and some $\gamma \leq \frac{1}{n \cdot q_n^m}$. In particular, $2\gamma \leq \frac{2}{n \cdot q_n^m} < q_n^{-\sigma}$ for $n > 2$. So we can define a cuboid $S_1 \subseteq S$, where $S_1 := [s_1 + \gamma, s_2 - \gamma] \times S_r$ using the notation $S_\theta = [s_1, s_2]$.
Since g_n preserves the $\vec{r}$-coordinates, it holds: $\Phi_n\left(\hat{I}\right) \cap g_n^{-1}(S) \subseteq [c, c+\gamma] \times S_r =: K_{c,\gamma}$. We examine the two sets

$$Q := \pi_{\vec{r}}\left(K_{c,\gamma} \cap g_n^{-1}(S_\theta \times S_r)\right) \qquad Q_1 := \pi_{\vec{r}}\left(K_{c,\gamma} \cap g_n^{-1}([s_1+\gamma, s_2-\gamma] \times S_r)\right)$$

As seen above $\Phi_n\left(\hat{I}\right) \cap g_n^{-1}(S) \subseteq K_{c,\gamma}$. Hence, $\Phi_n\left(\hat{I}\right) \cap g_n^{-1}(S) \subseteq \Phi_n\left(\hat{I}\right) \cap g_n^{-1}(S) \cap K_{c,\gamma}$, which implies $\Phi_n\left(\hat{I}\right) \cap g_n^{-1}(S) \subseteq \Phi_n\left(\hat{I}\right) \cap \left(\mathbb{S}^1 \times Q\right)$.

Claim: On the other hand: $\Phi_n\left(\hat{I}\right) \cap \left(\mathbb{S}^1 \times Q_1\right) \subseteq \Phi_n\left(\hat{I}\right) \cap g_n^{-1}(S)$.

Proof of the claim: For $(\theta, \vec{r}) \in \Phi_n\left(\hat{I}\right) \cap \left(\mathbb{S}^1 \times Q_1\right)$ arbitrary it holds $(\theta, \vec{r}) \in \Phi_n\left(\hat{I}\right)$, i. e. $\theta \in [c, c+\gamma]$, and $\vec{r} \in \pi_{\vec{r}}\left(K_{c,\gamma} \cap g_n^{-1}([s_1+\gamma, s_2-\gamma] \times S_r)\right)$, i. e. in particular $\vec{r} \in S_r$. This implies the existence of $\bar{\theta} \in [c, c+\gamma]$ satisfying $\left(\bar{\theta}, \vec{r}\right) \in K_{c,\gamma} \cap g_n^{-1}(S_1)$. Hence, there is $\beta \in [s_1+\gamma, s_2-\gamma]$ such that $g_n\left(\bar{\theta}, \vec{r}\right) = (\beta, \vec{r})$. Moreover, we observe that the map g_n maps sets of the form $I \times \{\vec{r}\}$ with I an interval in $\mathbb{S}^1$ onto sets of the form $\tilde{I} \times \{\vec{r}\}$ with $\tilde{I}$ an interval in $\mathbb{S}^1$ and the length of the interval is preserved. Since $\left|\theta - \bar{\theta}\right| \leq \gamma$ there is $\bar{\beta} \in [s_1, s_2]$ such that $g_n(\theta, \vec{r}) = \left(\bar{\beta}, \vec{r}\right)$. So $(\theta, \vec{r}) \in \Phi_n\left(\hat{I}\right) \cap g_n^{-1}(S)$. □

Altogether, the following inclusions are true:

$$\Phi_n\left(\hat{I}\right) \cap \left(\mathbb{S}^1 \times Q_1\right) \subseteq \Phi_n\left(\hat{I}\right) \cap g_n^{-1}(S) \subseteq \Phi_n\left(\hat{I}\right) \cap \left(\mathbb{S}^1 \times Q\right)$$

Thus, we obtain:

$$\begin{aligned}
&\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(g_n^{-1}(S)\right)\right) \cdot \tilde{\mu}(J) - \mu\left(\hat{I}\right) \cdot \mu(S)\right| \\
&\leq \max\Bigg(\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(\mathbb{S}^1 \times Q\right)\right) \cdot \tilde{\mu}(J) - \mu\left(\hat{I}\right) \cdot \mu(S)\right|, \\
&\qquad \left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(\mathbb{S}^1 \times Q_1\right)\right) \cdot \tilde{\mu}(J) - \mu\left(\hat{I}\right) \cdot \mu(S)\right| \Bigg)
\end{aligned} \tag{3}$$

We want to apply Lemma 4.3 for $K = S_{r_1}$, $L = S_\theta$, $Z = S_{\tilde{\vec{r}}}$ and $b = [n \cdot q_n^\sigma]$ (note that $\frac{3[n\cdot q_n^\sigma]}{n^2\cdot k_n^{m-1}\cdot q_n^{1+(m-1)\cdot\frac{n\cdot(n+1)}{2}}} \leq \frac{3}{n\cdot q_n^m} < \frac{1}{q_n^\sigma} = \tilde{\lambda}(L)$ and for $n > 4$: $b \cdot \lambda(K) \geq \frac{1}{2} n q_n^\sigma \cdot q_n^{-\sigma} > 2$):

$$
\begin{aligned}
&|\tilde{\mu}(Q) - \mu(S)| \\
&\leq \left(\frac{2}{[n\cdot q_n^\sigma]} \cdot \tilde{\lambda}(S_\theta) + \frac{2\gamma}{[n\cdot q_n^\sigma]} + \gamma \cdot \lambda(S_{r_1}) + \frac{[n q_n^\sigma]\cdot \lambda(S_{r_1})}{a} + \frac{2}{a}\right) \cdot \mu^{(m-2)}\left(S_{\tilde{\vec{r}}}\right) \\
&\leq \left(\frac{4}{n\cdot q_n^\sigma}\tilde{\lambda}(S_\theta) + \frac{4}{n\cdot q_n^\sigma\cdot q_n^\sigma} + \frac{1}{n\cdot q_n^\sigma}\lambda(S_{r_1}) + \frac{[n q_n^\sigma]\cdot\lambda(S_{r_1})}{n^2\cdot k_n^{m-1}\cdot q_n^{1+(m-1)\cdot\frac{n\cdot(n+1)}{2}}} + \frac{2}{n\cdot q_n^m}\right)\cdot \mu^{(m-2)}\left(S_{\tilde{\vec{r}}}\right) \\
&\leq \frac{14}{n}\cdot \mu(S).
\end{aligned}
$$

In particular, we receive from this estimate: $\frac{14}{n} \cdot \mu(S) \geq \tilde{\mu}(Q) - \mu(S) \geq \tilde{\mu}(Q) - \mu(S)$, hence: $\tilde{\mu}(Q) \leq \left(1 + \frac{14}{n}\right) \cdot \mu(S) \leq 4 \cdot \mu(S)$.
Analogously we obtain: $\tilde{\mu}(Q_1) \leq 4 \cdot \mu(S)$ as well as $|\tilde{\mu}(Q_1) - \mu(S_1)| \leq \frac{14}{n} \cdot \mu(S)$.
Since Q as well as Q_1 are a finite union of disjoint $(m-1)$-dimensional intervals contained in J and Φ_n $\left(\frac{1}{n\cdot q_n^m}, \frac{1}{n^4}, \frac{1}{n}\right)$-distributes the interval $\hat{I}$, we get:

$$\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(\mathbb{S}^1 \times Q\right)\right) \cdot \tilde{\mu}(J) - \mu\left(\hat{I}\right) \cdot \tilde{\mu}(Q)\right| \leq \frac{1}{n} \cdot \mu\left(\hat{I}\right) \cdot \tilde{\mu}(Q) \leq \frac{4}{n} \cdot \mu\left(\hat{I}\right) \cdot \mu(S)$$

as well as

$$\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(\mathbb{S}^1 \times Q_1\right)\right) \cdot \tilde{\mu}(J) - \mu\left(\hat{I}\right) \cdot \tilde{\mu}(Q_1)\right| \leq \frac{1}{n} \cdot \mu\left(\hat{I}\right) \cdot \tilde{\mu}(Q_1) \leq \frac{4}{n} \cdot \mu\left(\hat{I}\right) \cdot \mu(S).$$

Now we can proceed

$$
\begin{aligned}
&\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(\mathbb{S}^1 \times Q\right)\right) \cdot \tilde{\mu}(J) - \mu\left(\hat{I}\right) \cdot \mu(S)\right| \\
&\leq \left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(\mathbb{S}^1 \times Q\right)\right) \cdot \tilde{\mu}(J) - \mu\left(\hat{I}\right) \cdot \tilde{\mu}(Q)\right| + \mu\left(\hat{I}\right) \cdot |\tilde{\mu}(Q) - \mu(S)| \\
&\leq \frac{4}{n} \cdot \mu\left(\hat{I}\right) \cdot \mu(S) + \mu\left(\hat{I}\right) \cdot \frac{14}{n} \cdot \mu(S) = \frac{18}{n} \cdot \mu\left(\hat{I}\right) \cdot \mu(S).
\end{aligned}
$$

Noting that $\mu(S_1) = \mu(S) - 2\gamma \cdot \tilde{\mu}(S_r)$ and so $\mu(S) - \mu(S_1) \leq 2 \cdot \frac{1}{n\cdot q_n^\sigma} \cdot \tilde{\mu}(S_r) \leq \frac{2}{n} \cdot \mu(S)$ we obtain in the same way as above:

$$\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(\mathbb{S}^1 \times Q_1\right)\right) \cdot \tilde{\mu}(J) - \mu\left(\hat{I}\right) \cdot \mu(S)\right| \leq \frac{20}{n} \cdot \mu\left(\hat{I}\right) \cdot \mu(S).$$

Using equation 3 this yields:

$$\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(g_n^{-1}(S)\right)\right) \cdot \tilde{\mu}(J) - \mu\left(\hat{I}\right) \cdot \mu(S)\right| \leq \frac{20}{n} \cdot \mu\left(\hat{I}\right) \cdot \mu(S).$$

Since i_n and Φ_n are measure-preserving and $i_n\left(\Phi_n\left(\hat{I}\right)\right) = \Phi_n\left(\hat{I}\right)$ by Lemma 3.5, we have

$$
\begin{aligned}
\mu\left(\hat{I} \cap \Phi_n^{-1}\left(g_n^{-1}(S)\right)\right) &= \mu\left(\Phi_n\left(\hat{I}\right) \cap g_n^{-1}(S)\right) = \mu\left(i_n \circ \Phi_n\left(\hat{I}\right) \cap g_n^{-1}(S)\right) \\
&= \mu\left(\hat{I} \cap \Phi_n^{-1} \circ i_n^{-1}\left(g_n^{-1}(S)\right)\right)
\end{aligned}
$$

and we conclude the statement of the Lemma. $\square$

Now we are able to prove the aimed criterion for weak mixing.

Proposition 4.5 (Criterion for weak mixing). *Let $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ and the sequence $(m_n)_{n\in\mathbb{N}}$ be constructed as in the previous sections. Suppose additionally that $d_0\left(f^{m_n}, f_n^{m_n}\right) < \frac{1}{2^n}$ for every $n \in \mathbb{N}$, $\|DH_{n-1}\|_0 \leq \frac{q_n^{0.25}}{2n^2\cdot\sqrt{m}}$ and that the limit $f = \lim_{n\to\infty} f_n$ exists.*
Then f is weakly mixing.

Proof. To apply Lemma 4.1 we consider the partial partitions $\nu_n := H_{n-1}\circ g_n \circ i_n(\eta_n)$. As proven in Lemma 4.2 these partial partitions satisfy $\nu_n \to \varepsilon$. We have to establish equation 1. For it let $\varepsilon > 0$ and a m-dimensional cube $A \subseteq \mathbb{S}^1 \times (0,1)^{m-1}$ be given. There exists $N \in \mathbb{N}$ such that $A \subseteq \mathbb{S}^1 \times \left[\frac{1}{n^4}, 1-\frac{1}{n^4}\right]^{m-1}$ for every $n \geq N$. Because of Lemma 3.4 and the properties of a $\left(\frac{1}{n\cdot q_n^m}, \frac{1}{n^4}, \frac{1}{n}\right)$-distribution we obtain for every $\hat{I}_n \in \eta_n$: $\pi_{\vec{r}}\left(\Phi_n\left(\hat{I}_n\right)\right) \supseteq \left[\frac{1}{n^4}, 1-\frac{1}{n^4}\right]^{m-1}$. Furthermore, we note $f_n^{m_n} = H_n \circ R_{\alpha_{n+1}}^{m_n} \circ H_n^{-1} = H_{n-1}\circ g_n \circ i_n \circ \Phi_n \circ i_n^{-1}\circ g_n^{-1}\circ H_{n-1}^{-1}$.
Let S_n be a m-dimensional cube of side length $q_n^{-\sigma}$ contained in $\mathbb{S}^1 \times \left[\frac{1}{n^4}, 1-\frac{1}{n^4}\right]^{m-1}$. We look at $C_n := H_{n-1}(S_n)$, $\Gamma_n \in \nu_n$, and compute (since i_n, g_n and H_{n-1} are measure-preserving):

$$\begin{aligned}&\left|\mu\left(\Gamma_n \cap f_n^{-m_n}(C_n)\right) - \mu(\Gamma_n)\cdot\mu(C_n)\right| = \left|\mu\left(\hat{I}_n \cap \Phi_n^{-1}\circ i_n^{-1}\circ g_n^{-1}(S_n)\right) - \mu\left(\hat{I}_n\right)\cdot\mu(S_n)\right| \\ &\leq \frac{1}{\tilde{\mu}(J)}\cdot\left|\mu\left(\hat{I}_n \cap \Phi_n^{-1}\circ i_n^{-1}\circ g_n^{-1}(S_n)\right)\cdot\tilde{\mu}(J) - \mu\left(\hat{I}_n\right)\cdot\mu(S_n)\right| + \frac{1-\tilde{\mu}(J)}{\tilde{\mu}(J)}\cdot\mu\left(\hat{I}_n\right)\cdot\mu(S_n)\end{aligned}$$

Bernoulli's inequality yields: $\tilde{\mu}(J) \geq \left(1-\frac{1}{n}\right)^{m-1} \geq 1 + (m-1)\cdot\left(-\frac{1}{n}\right) = 1 - \frac{m-1}{n}$. Hence, we obtain for $n > 2\cdot(m-1)$: $\tilde{\mu}(J) \geq \frac{1}{2}$ and so: $\frac{1-\tilde{\mu}(J)}{\tilde{\mu}(J)} \leq 2\cdot(1-\tilde{\mu}(J)) \leq \frac{2\cdot(m-1)}{n}$. We continue by applying Lemma 4.4:

$$\begin{aligned}\left|\mu\left(\Gamma_n \cap f_n^{-m_n}(C_n)\right) - \mu(\Gamma_n)\cdot\mu(C_n)\right| &\leq 2\cdot\frac{20}{n}\cdot\mu\left(\hat{I}_n\right)\cdot\mu(S_n) + \frac{2\cdot(m-1)}{n}\cdot\mu\left(\hat{I}_n\right)\cdot\mu(S_n) \\ &= \frac{38+2\cdot m}{n}\cdot\mu\left(\hat{I}_n\right)\cdot\mu(S_n)\end{aligned}$$

Moreover, it holds $\operatorname{diam}(C_n) \leq \|DH_{n-1}\|_0 \cdot \operatorname{diam}(S_n) \leq \sqrt{m}\cdot\frac{q_n^{0.25}}{2n^2\cdot\sqrt{m}\cdot q_n^{\sigma}}$. Since $0.25 < \sigma < 0.5$ we conclude $\operatorname{diam}(C_n) \to 0$ as $n \to \infty$. Thus, we can approximate A by a countable disjoint union of sets $C_n = H_{n-1}(S_n)$ with $S_n \subseteq \mathbb{S}^1 \times \left[\frac{1}{n^4}, 1-\frac{1}{n^4}\right]^{m-1}$ a m-dimensional cube of sidelength $q_n^{-\sigma}$ in given precision, when n is chosen big enough. Consequently for n sufficiently large there are sets $A_1 = \dot{\bigcup}_{i\in\Sigma_n^1} C_n^i$ and $A_2 = \dot{\bigcup}_{i\in\Sigma_n^2} C_n^i$ with countable sets Σ_n^1 and Σ_n^2 of indices satisfying $A_1 \subseteq A \subseteq A_2$ as well as $|\mu(A) - \mu(A_i)| \leq \frac{\epsilon}{3}\cdot\mu(A)$ for $i = 1, 2$.
Additionally we choose n such that $\frac{38+2\cdot m}{n} < \frac{\epsilon}{3}$ holds. It follows:

$$\begin{aligned}&\mu\left(\Gamma_n \cap f_n^{-m_n}(A)\right) - \mu(\Gamma_n)\cdot\mu(A) \\ &\leq \mu\left(\Gamma_n \cap f_n^{-m_n}(A_2)\right) - \mu(\Gamma_n)\cdot\mu(A_2) + \mu(\Gamma_n)\cdot(\mu(A_2) - \mu(A)) \\ &\leq \sum_{i\in\Sigma_n^2}\left(\mu\left(\Gamma_n \cap f_n^{-m_n}\left(C_n^i\right)\right) - \mu(\Gamma_n)\cdot\mu\left(C_n^i\right)\right) + \frac{\epsilon}{3}\cdot\mu(\Gamma_n)\cdot\mu(A) \\ &\leq \sum_{i\in\Sigma_n^2}\left(\frac{38+2\cdot m}{n}\cdot\mu\left(\hat{I}_n\right)\cdot\mu\left(S_n^i\right)\right) + \frac{\epsilon}{3}\cdot\mu(\Gamma_n)\cdot\mu(A) \\ &= \frac{38+2\cdot m}{n}\cdot\mu(\Gamma_n)\cdot\mu\left(\bigcup_{i\in\Sigma_n^2} C_n^i\right) + \frac{\epsilon}{3}\cdot\mu(\Gamma_n)\cdot\mu(A) \leq \frac{\epsilon}{3}\cdot\mu(\Gamma_n)\cdot\mu(A_2) + \frac{\epsilon}{3}\cdot\mu(\Gamma_n)\cdot\mu(A)\end{aligned}$$

$$= \frac{\epsilon}{3} \cdot \mu(\Gamma_n) \cdot \mu(A) + \frac{\epsilon}{3} \cdot \mu(\Gamma_n) \cdot (\mu(A_2) - \mu(A)) + \frac{\epsilon}{3} \cdot \mu(\Gamma_n) \cdot \mu(A) \leq \epsilon \cdot \mu(\Gamma_n) \cdot \mu(A).$$

Analogously we estimate: $\mu\left(\Gamma_n \cap f_n^{-m_n}(A)\right) - \mu(\Gamma_n) \cdot \mu(A) \geq -\epsilon \cdot \mu(\Gamma_n) \cdot \mu(A)$. Both estimates enable us to conclude: $\left|\mu\left(\Gamma_n \cap f_n^{-m_n}(A)\right) - \mu(\Gamma_n) \cdot \mu(A)\right| \leq \epsilon \cdot \mu(\Gamma_n) \cdot \mu(A)$. □

5 Convergence

In the following we show that the sequence of constructed measure-preserving smooth diffeomorphisms $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ converges. For this purpose, we need precise norm estimates on the conjugation maps.

5.1 Properties of the conjugation maps

First of all, we examine the conjugation map i_n introduced in subsection 2.3:

Lemma 5.1. *For every $l \in \mathbb{N}$ it holds*

$$|||i_n|||_l \leq C_{l,n,q_{n-1},k_n} \cdot q_n^{(l-1)\cdot\left(1+(m-1)\cdot\frac{n\cdot(n+1)}{2}\right)}$$

with a constant C_{l,n,q_{n-1},k_n} depending on l, n, q_{n-1} and k_n but independent of q_n.

Proof. The map i_n was defined by $i_n = D_a^{-1} \circ \psi_{q_{n-1},\frac{1}{5n^4 q_{n-1}},\beta_2,\dots,\beta_m} \circ D_a$. Hence, we have

$$|||i_n|||_l \leq a^{l-1} \cdot |||\psi_{q_{n-1},\frac{1}{5n^4 q_{n-1}},\beta_2,\dots,\beta_m}|||_l.$$

Since $a = n^2 \cdot k_n^{m-1} \cdot q_n^{1+(m-1)\cdot\frac{n\cdot(n+1)}{2}}$ and the rotation arcs depend on the number k_n, we conclude:

$$|||i_n|||_l \leq C_{l,n,q_{n-1},k_n} \cdot q_n^{(l-1)\cdot\left(1+(m-1)\cdot\frac{n\cdot(n+1)}{2}\right)},$$

where the constant C_{l,n,q_{n-1},k_n} depends on l, n, q_{n-1} and k_n but is independent of q_n. □

In the next step, we consider the composition $g_n \circ i_n$:

Lemma 5.2. *For every $l \in \mathbb{N}$ we have*

$$|||g_n \circ i_n|||_l \leq C_{l,n,q_{n-1},k_n} \cdot q_n^{l\cdot\left(2+(m-1)\cdot\frac{n\cdot(n+1)}{2}\right)}$$

with a constant C_{l,n,q_{n-1},k_n} depending on l, n, q_{n-1} and k_n but independent of q_n.

Proof. At the end of section 2.2 we saw $|||g_n|||_l \leq C_{l,n,q_{n-1},k_n} \cdot [nq_n^{\sigma}]$. Using Lemma 5.1 and the formula of Faà di Bruno as in [GKu15], Remark 6.3., we can estimate

$$|||g_n \circ i_n|||_l \leq \check{C}_{l,n,q_{n-1},k_n} \cdot [nq_n^{\sigma}]^l \cdot q_n^{(l-1)\cdot\left(1+(m-1)\cdot\frac{n\cdot(n+1)}{2}\right)}$$

□

By the same approach as in [GKu15], Lemma 6.4., we deduce the subsequent norm estimate of the map ϕ_n:

Lemma 5.3. *For every $l \in \mathbb{N}$ it holds*

$$|||\phi_n|||_l \leq C_{l,n,q_{n-1},k_n} \cdot q_n^{(m-1)^2 \cdot l \cdot n \cdot (n+1)},$$

where the constant C_{l,n,q_{n-1},k_n} is depending on l, n, q_{n-1} and k_n but is independent of q_n.

Proof. Compared to the proof of [GKu15], Lemma 6.4., we have $\varepsilon_1 = \frac{1}{60n^4 \cdot q_{n-1}}$, $\varepsilon_2 = \frac{1}{22n^4 \cdot q_{n-1}}$, $\lambda_{\max} = n^2 \cdot k_n^{m-1} \cdot q_n^{1+(m-1)\cdot\frac{n\cdot(n-1)}{2}+(m-2)\cdot n}$ and $\mu_{\max} = q_n^n$. Thus:

$$\begin{aligned} |||\phi_n|||_l &\leq \tilde{C} \cdot \left(n^2 \cdot k_n^{m-1} \cdot q_n^{1+(m-1)\cdot\frac{n\cdot(n-1)}{2}+(m-2)\cdot n}\right)^{(m-1)\cdot l} \cdot \left(q_n^n\right)^{(m-1)\cdot l} \\ &\leq C_{l,n,q_{n-1},k_n} \cdot q_n^{(m-1)^2 \cdot l \cdot n \cdot (n+1)}, \end{aligned}$$

where C_{l,n,q_{n-1},k_n} is a constant independent of q_n. □

Using the formula of Faà di Bruno again we prove for the conjugation map $h_n = g_n \circ i_n \circ \phi_n$:

Lemma 5.4. *For every $l \in \mathbb{N}$ it holds*

$$|||h_n|||_l \leq C_{l,n,q_{n-1},k_n} \cdot q_n^{2\cdot m^2 \cdot l \cdot n \cdot (n+1)},$$

where the constant C_{l,n,q_{n-1},k_n} is depending on l, n, q_{n-1} and k_n but is independent of q_n.

Finally, we are able to prove an estimate on the norms of the map H_n as in [GKu15], Lemma 6.6.:

Lemma 5.5. *For every $l \in \mathbb{N}$ we get:*

$$|||H_n|||_l \leq \breve{C} \cdot q_n^{2\cdot m^2 \cdot l \cdot n \cdot (n+1)},$$

where $\breve{C}$ is a constant depending solely on l, n, q_{n-1}, k_n and H_{n-1}. Since H_{n-1} and k_n are independent of q_n in particular, the same is true for $\breve{C}$.

5.2 Proof of convergence

For the proof of the convergence of the sequence $(f_n)_{n\in\mathbb{N}}$ in the $\mathrm{Diff}^\infty(M)$-topology the next result, that can be found in [FSW07], Lemma 4, is very useful.

Lemma 5.6. *Let $k \in \mathbb{N}_0$ and h be a C^∞-diffeomorphism on M. Then we get for every $\alpha, \beta \in \mathbb{R}$:*

$$d_k\left(h \circ R_\alpha \circ h^{-1}, h \circ R_\beta \circ h^{-1}\right) \leq C_k \cdot |||h|||_{k+1}^{k+1} \cdot |\alpha - \beta|,$$

where the constant C_k depends solely on k and m. In particular $C_0 = 1$.

The subsequent Lemma ([GKu15], Lemma 6.8.) shows that under some assumptions on the sequence $(\alpha_n)_{n\in\mathbb{N}}$ the sequence $(f_n)_{n\in\mathbb{N}}$ converges to $f \in \mathcal{A}_\alpha(M)$ in the $\mathrm{Diff}^\infty(M)$-topology.

Lemma 5.7. *Let $\varepsilon > 0$ be arbitrary and $(l_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers satisfying $\sum_{n=1}^\infty \frac{1}{l_n} < \varepsilon$. Furthermore, we assume that in our constructions the following conditions are fulfilled:*

$$|\alpha - \alpha_1| < \varepsilon \qquad \text{and} \qquad |\alpha - \alpha_n| \leq \frac{1}{2 \cdot l_n \cdot C_{l_n} \cdot |||H_n|||_{l_n+1}^{l_n+1}} \text{ for every } n \in \mathbb{N},$$

where C_{l_n} are the constants from Lemma 5.6.

1. *Then the sequence of diffeomorphisms $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ converges in the Diff$^\infty(M)$-topology to a measure-preserving smooth diffeomorphism f, for which $d_\infty(f, R_\alpha) < 3 \cdot \varepsilon$ holds.*
2. *Also the sequence of diffeomorphisms $\hat{f}_n = H_n \circ R_\alpha \circ H_n^{-1} \in \mathcal{A}_\alpha(M)$ converges to f in the Diff$^\infty(M)$-topology. Hence, $f \in \mathcal{A}_\alpha(M)$.*

We show that we can satisfy the conditions from this Lemma in our constructions:

Lemma 5.8. *Let $(l_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers with $\sum_{n=1}^{\infty} \frac{1}{l_n} < \infty$ and C_{l_n} be the constants from Lemma 5.6. For any Liouvillean number α there exists a sequence $\alpha_n = \frac{p_n}{q_n}$ of rational numbers with*

$$780 n^6 \cdot (n-1)^6 \cdot q_{n-2}^2 \cdot k_{n-1}^{m-1} \cdot q_{n-1}^{1+(m-1)\cdot\frac{n\cdot(n-1)}{2}} \text{ divides } q_n \tag{A}$$

$$(\alpha_n)_{n\in\mathbb{N}} \text{ converges to } \alpha \text{ monotonically} \tag{B}$$

such that our conjugation maps H_n constructed in section 2 fulfil the following conditions:

1. *For every $n \in \mathbb{N}$:*
$$|\alpha - \alpha_n| < \frac{1}{2 \cdot l_n \cdot C_{l_n} \cdot |||H_n|||_{l_n+1}^{l_n+1}}.$$
2. *For every $n \in \mathbb{N}$:*
$$n^{2m} \cdot k_n^{m\cdot(m-1)} \cdot q_{n-1}^m < q_n$$
3. *For every $n \in \mathbb{N}$:*
$$30 \cdot 780 n^6 \cdot (n-1)^{14} \cdot q_{n-2}^3 \cdot k_{n-1}^{3m-3} \cdot q_{n-1}^{3+2\cdot(m-1)\cdot n\cdot(n-1)} < q_n$$
4. *For every $n \in \mathbb{N}$:*
$$\|DH_{n-1}\|_0 < \frac{q_n^{0.25}}{2\sqrt{m}\cdot n^2}.$$

Proof. The sequence of rational numbers $\alpha_n = \frac{p_n}{q_n}$ will be created out of $\tilde{\alpha}_n = \frac{\tilde{p}_n}{\tilde{q}_n}$, at which $\tilde{p}_n \le p_n$ and $\tilde{q}_n \le q_n$ are relatively prime.
In Lemma 5.5 we saw $|||H_n|||_{l_n+1} \le \breve{C}_n \cdot q_n^{2\cdot m^2\cdot(l_n+1)\cdot n\cdot(n+1)}$, where the constant $\breve{C}_n$ was independent of q_n. Thus, we can require $\tilde{q}_n \ge \breve{C}_n$ for every $n \in \mathbb{N}$. Hereby, we get the estimate $|||H_n|||_{l_n+1} \le q_n^{3\cdot m^2\cdot(l_n+1)\cdot n\cdot(n+1)}$. Furthermore, we can demand

$$\tilde{q}_n > 30 \cdot (n-1)^8 \cdot q_{n-2} \cdot k_{n-1}^{2m-2} \cdot q_{n-1}^{2+(m-1)\cdot n\cdot(n-1)},$$
$$\tilde{q}_n > n^{2m} \cdot k_n^{m\cdot(m-1)} \cdot q_{n-1}^m$$

and $\|DH_{n-1}\|_0 < \frac{q_n^{0.25}}{2\sqrt{m}\cdot n^2}$ because H_{n-1} is independent of q_n.
Since α is a Liouvillean number, we find a sequence of rational numbers $\tilde{\alpha}_n = \frac{\tilde{p}_n}{\tilde{q}_n}$, $\tilde{p}_n, \tilde{q}_n$ relatively prime, under the above restrictions satisfying:

$$|\alpha - \tilde{\alpha}_n| = \left|\alpha - \frac{\tilde{p}_n}{\tilde{q}_n}\right|$$
$$< \frac{|\alpha - \alpha_{n-1}|}{2 \cdot l_n \cdot C_{l_n} \cdot \left(780 n^6 \cdot (n-1)^6 \cdot q_{n-2}^2 \cdot k_{n-1}^{m-1} \cdot q_{n-1}^{1+(m-1)\cdot\frac{n\cdot(n-1)}{2}}\right)^{3\cdot m^2\cdot(l_n+1)^2\cdot n\cdot(n+1)} \cdot \tilde{q}_n^{3\cdot m^2\cdot(l_n+1)^2\cdot n\cdot(n+1)}}.$$

Put

$$\begin{aligned} q_n &:= 780n^6 \cdot (n-1)^6 \cdot q_{n-2}^2 \cdot k_{n-1}^{m-1} \cdot q_{n-1}^{1+(m-1)\cdot \frac{n\cdot(n-1)}{2}} \cdot \tilde{q}_n \\ \text{and } p_n &:= 780n^6 \cdot (n-1)^6 \cdot q_{n-2}^2 \cdot k_{n-1}^{m-1} \cdot q_{n-1}^{1+(m-1)\cdot \frac{n\cdot(n-1)}{2}} \cdot \tilde{p}_n. \end{aligned}$$

Then we obtain:

$$|\alpha - \alpha_n| < \frac{|\alpha - \alpha_{n-1}|}{2 \cdot l_n \cdot C_{l_n} \cdot q_n^{3\cdot m^2 \cdot (l_n+1)^2 \cdot n \cdot (n+1)}}.$$

Thus, we have $|\alpha - \alpha_n| \to 0$ monotonically as $n \to \infty$.
Because of $|||H_n|||_{l_n+1}^{l_n+1} \leq q_n^{3\cdot m^2 \cdot (l_n+1)^2 \cdot n \cdot (n+1)}$ this yields: $|\alpha - \alpha_n| < \frac{1}{2 \cdot l_n \cdot C_{l_n} \cdot |||H_n|||_{l_n+1}^{l_n+1}}$. Thus, the first property of this Lemma is fulfilled. □

Remark 5.9. Lemma 5.8 shows that the conditions of Lemma 5.7 are satisfied. Therefore, our sequence of constructed diffeomorphisms f_n converges in the $\text{Diff}^\infty(M)$-topology to a diffeomorphism $f \in \mathcal{A}_\alpha$.

In particular, we have

$$|\alpha_{n+1} - \alpha_n| \leq 2 \cdot |\alpha - \alpha_n| \leq \frac{1}{l_n \cdot C_{l_n} \cdot q_n^{3\cdot m^2 \cdot (l_n+1)^2 \cdot n \cdot (n+1)}}. \tag{4}$$

Remark 5.10. Analogous to [GKu15], Lemma 6.11., we prove $d_0\left(f^{m_n}, f_n^{m_n}\right) < \frac{1}{2^n}$ for every $n \in \mathbb{N}$, where $(m_n)_{n\in\mathbb{N}}$ is the sequence of natural numbers defined in chapter 3.
Concluding we have checked that all the assumptions of Proposition 4.5 are satisfied. Thus, this criterion guarantees that the constructed diffeomorphism $f \in \mathcal{A}_\alpha(M)$ is weakly mixing. In addition, for every $\varepsilon > 0$ we can choose the parameters by Lemma 5.7 in such a way, that $d_\infty(f, R_\alpha) < \varepsilon$ holds.

6 Construction of the f-invariant measurable Riemannian metric

Let ω_0 denote the standard Riemannian metric on $M = \mathbb{S}^1 \times [0,1]^{m-1}$. The following Lemma shows that the conjugation map $h_n = g_n \circ i_n \circ \phi_n$ constructed in section 2 is an isometry with respect to ω_0 on the elements of the partial partition ζ_n.

Lemma 6.1. *Let $\check{I}_n \in \zeta_n$. Then $h_n|_{\check{I}_n}$ is an isometry with respect to ω_0.*

Proof. The proof is similar to the proof of [GKu15], Lemma 7.1.
Let $\check{I}_{n,k} \in \zeta_n$ be a partition element on $\left[\frac{k-1}{n\cdot q_n}, \frac{k}{n\cdot q_n}\right] \times [0,1]^{m-1}$. This element $\check{I}_{n,k}$ is positioned in such a way that all the occurring maps $\varphi_{\varepsilon,1,j}$ and $\varphi^{-1}_{\varepsilon_2,1,j}$ act as rotations on it. Thus, $\phi_n|_{\check{I}_{n,k}}$ is an isometry and $\phi_n\left(\check{I}_{n,k}\right)$ is equal to

$$\Bigg[\frac{k-1}{nq_n}+\frac{s_1}{n^2\cdot k_n^{m-1}\cdot q_n}+\frac{j_1^{(1)}}{n^2\cdot k_n^{m-1}\cdot q_n^2}+\ldots+\frac{j_1^{\left((m-1)\cdot\frac{k\cdot(k-1)}{2}\right)}+1}{n^2\cdot k_n^{m-1}\cdot q_n^{(m-1)\cdot\frac{k\cdot(k-1)}{2}+1}}$$
$$-\frac{j_2^{(1)}}{n^2\cdot k_n^{m-1}\cdot q_n^{(m-1)\cdot\frac{k\cdot(k-1)}{2}+2}}-\ldots-\frac{j_2^{(k)}}{n^2\cdot k_n^{m-1}\cdot q_n^{(m-1)\cdot\frac{k\cdot(k-1)}{2}+k+1}}-\frac{j_3^{(1)}}{n^2\cdot k_n^{m-1}\cdot q_n^{(m-1)\cdot\frac{k\cdot(k-1)}{2}+k+2}}$$
$$-\ldots-\frac{j_m^{(k)}+1}{n^2\cdot k_n^{m-1}\cdot q_n^{(m-1)\cdot\frac{k\cdot(k+1)}{2}+1}}+\frac{j_1^{\left((m-1)\cdot\frac{k\cdot(k+1)}{2}+1\right)}}{n^2\cdot k_n^{m-1}\cdot q_n^{(m-1)\cdot\frac{k\cdot(k+1)}{2}+2}}+\ldots+\frac{j_1^{\left((m-1)\cdot\frac{(n+1)\cdot n}{2}\right)}}{n^2\cdot k_n^{m-1}\cdot q_n^{1+(m-1)\cdot\frac{(n+1)\cdot n}{2}}}$$
$$+\frac{t_1}{n^2\cdot q_{n-1}\cdot k_n^{m-1}\cdot q_n^{1+(m-1)\cdot\frac{n\cdot(n+1)}{2}}}+\frac{1}{5n^6\cdot q_{n-1}^2\cdot k_n^{m-1}\cdot q_n^{1+(m-1)\cdot\frac{n\cdot(n+1)}{2}}},$$
$$\frac{k-1}{n\cdot q_n}+\frac{s_1}{n^2\cdot k_n^{m-1}q_n}+\frac{j_1^{(1)}}{n^2\cdot k_n^{m-1}\cdot q_n^2}+\ldots-\frac{j_m^{(k)}+1}{n^2\cdot k_n^{m-1}\cdot q_n^{(m-1)\cdot\frac{k\cdot(k+1)}{2}+1}}$$
$$+\ldots+\frac{t_1+1}{n^2\cdot q_{n-1}\cdot k_n^{m-1}\cdot q_n^{1+(m-1)\cdot\frac{n\cdot(n+1)}{2}}}-\frac{1}{5n^6\cdot q_{n-1}^2\cdot k_n^{m-1}\cdot q_n^{1+(m-1)\cdot\frac{n\cdot(n+1)}{2}}}\Bigg]$$
$$\times\prod_{i=2}^{m}\Bigg[\frac{j_1^{\left((m-1)\cdot\frac{k\cdot(k-1)}{2}+(i-2)\cdot k+1\right)}}{q_n}+\ldots+\frac{j_1^{\left((m-1)\cdot\frac{k\cdot(k-1)}{2}+(i-1)\cdot k\right)}}{q_n^k}+\frac{j_i^{(k+1)}}{q_n^{k+1}}+\ldots+\frac{j_i^{\left(1+(m-1)\cdot\frac{n\cdot(n+1)}{2}\right)}}{q_n^{1+(m-1)\cdot\frac{n\cdot(n+1)}{2}}}$$
$$+\frac{s_i}{n^2k_n^{m-1}q_n^{1+(m-1)\cdot\frac{n\cdot(n+1)}{2}}}+\frac{t_i}{n^2q_{n-1}k_n^{m-1}\cdot q_n^{1+(m-1)\cdot\frac{n(n+1)}{2}}}+\frac{1}{5n^6q_{n-1}^2\cdot k_n^{m-1}\cdot q_n^{1+(m-1)\cdot\frac{n(n+1)}{2}}},$$
$$\frac{j_1^{\left((m-1)\cdot\frac{k\cdot(k-1)}{2}+(i-2)\cdot k+1\right)}}{q_n}+\ldots+\frac{j_1^{\left((m-1)\cdot\frac{k\cdot(k-1)}{2}+(i-1)\cdot k\right)}}{q_n^k}+\frac{j_i^{(k+1)}}{q_n^{k+1}}+\ldots+\frac{j_i^{\left(1+(m-1)\cdot\frac{n\cdot(n+1)}{2}\right)}}{q_n^{1+(m-1)\cdot\frac{n\cdot(n+1)}{2}}}$$
$$+\frac{s_i}{n^2k_n^{m-1}\cdot q_n^{1+(m-1)\cdot\frac{n(n+1)}{2}}}+\frac{t_i+1}{n^2q_{n-1}k_n^{m-1}\cdot q_n^{1+(m-1)\cdot\frac{n(n+1)}{2}}}-\frac{1}{5n^6q_{n-1}^2k_n^{m-1}\cdot q_n^{1+(m-1)\cdot\frac{n\cdot(n+1)}{2}}}\Bigg].$$

On this set $i_n=\psi_{n^2\cdot k_n^{m-1}\cdot q_n^{1+(m-1)\cdot\frac{n\cdot(n+1)}{2}},q_{n-1},\frac{1}{5n^4q_{n-1}},\beta_k^{(2)},\ldots,\beta_k^{(m)}}$ is equal to the composition of a translation and the respective rotations. Additionally, $i_n\circ\phi_n\left(\check{I}_{n,k}\right)$ is contained in the domain where $g_n=g_{n^2\cdot k_n^{m-1}\cdot q_n^{1+(m-1)\cdot\frac{n\cdot(n+1)}{2}},[nq_n^\sigma],\frac{1}{60n^4q_{n-1}},\frac{1}{30n^4q_{n-1}}}$ acts as a translation. □

Remark 6.2. As observed in Lemma 6.1 the map $h_n=g_n\circ i_n\circ\phi_n$ acts as the composition of the respective rotations and translations on every $\check{I}_n\in\zeta_n$. Hence, h_n^{-1} is a composition of rotations and translations on $h_n\left(\check{I}_n\right)$. In the following $G_n:=\bigcup_{\check{I}_n\in\zeta_n}h_n\left(\check{I}_n\right)$ will be called the "good domain" of h_n^{-1}. Similarly, $\bigcup_{\check{I}_n\in\zeta_n}\check{I}_n$ is the "good domain" of h_n and its corresponding parts on the θ-axis are called the "good length" of h_n. By the same arguments as in Remark 2.3 observe that for an interval $\left[\frac{l}{q_n},\frac{l+1}{q_n}\right]$ on the θ-axis the length $\left(1-\frac{3m}{q_{n-1}}\right)\cdot\frac{1}{q_n}$ is part of the "good length".

Since the elements of the partial partition ζ_n cover a set of M of measure at least $1-\frac{3m}{q_{n-1}}$ (see Remark 2.3), we are able to apply the same approach as in [GKu15], section 7, and construct the aimed measurable f-invariant Riemannian metric as the limit of the smooth metrics $\omega_n=\left(H_n^{-1}\right)^*\omega_0$.

7 Ergodicity of the derivative extension

7.1 General informations on Approximation in Ergodic Theory

This section provides a short introduction to the method of approximation of measure-preserving transformations in Ergodic Theory. A more comprehensive presentation can be found in [Ka03]. In [KS67] Katok and Stepin introduced the concept of periodic approximation: Let (X, μ) be a Lebesgue space. A tower t of height $h(t) = h$ is an ordered sequence of disjoint measurable sets $t = \{c_0, ..., c_{h-1}\}$ of X having equal measure, which is denoted by $m(t)$. The sets c_i are called the levels of the tower, especially c_0 is the base. Associated with a tower there is a cyclic permutation σ sending c_0 to c_1, c_1 to c_2,... and c_{h-1} to c_0.

Definition 7.1. A periodic process is a collection of disjoint towers covering the space X together with an equivalence relation among these towers identifying their bases.

There are two partial partitions associated with a periodic process: The partition ξ into all sets of all towers and the partition η consisting of the union of bases of towers in each equivalence class and their images under the iterates of σ, where when we go beyond the height of a certain tower in the class we drop this tower and continue until the highest tower in the equivalence class has been exhausted. Obviously, we have $\eta \leq \xi$. A sequence $(\xi_n, \eta_n, \sigma_n)$ of periodic processes is called exhaustive if $\eta_n \rightarrow \varepsilon$.

Definition 7.2. Let $T : (X, \mu) \rightarrow (X, \mu)$ be a measure-preserving transformation. An exhaustive sequence of periodic processes $(\xi_n, \eta_n, \sigma_n)$ forms a periodic approximation of T if

$$d(\xi_n, T, \sigma_n) = \sum_{c \in \xi_n} \mu(T(c) \triangle \sigma_n(c)) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Given a sequence $g(n)$ of positive numbers we will say that the transformation T admits a periodic approximation with speed $g(n)$ if for a certain subsequence $(n_k)_{k\in\mathbb{N}}$ there exists an exhaustive sequence of periodic processes $(\xi_k, \eta_k, \sigma_k)$ such that $d(\xi_k, T, \sigma_k) < g(n_k)$.

This notion was generalised by Schwartzbauer in [S70], Definition 3.1 and the adjacent remarks:

Definition 7.3. Let $\varphi(n)$ be a monotonic sequence of positive numbers such that $\lim_{n\rightarrow\infty} \varphi(n) = 0$. We say that the automorphism $T : (X, \mu) \rightarrow (X, \mu)$ admits an approximation with speed $\varphi(n)$ if for each $n \in \mathbb{N}$ there exists a partial partition $\xi_n = \left\{ c_i^{(n)} \ : \ i = 0, ..., q_n - 1 \right\}$ such that

1. $\xi_n \rightarrow \varepsilon$ as $n \rightarrow \infty$,
2. $\lim_{n\rightarrow\infty} \sum_{i=0}^{q_n-1} \left| \mu\left(c_i^{(n)}\right) - \frac{1}{q_n} \right| = 0$,
3. $\sum_{i=0}^{q_n-1} \mu\left(Tc_i^{(n)} \Delta c_{i+1}^{(n)}\right) < \varphi(q_n)$, where $c_{q_n}^{(n)}$ is understood to be $c_0^{(n)}$.

In particular, the tower levels are not required to have equal measure anymore. Since in our constructions the maps (f_n, df_n) are not necessarily measure-preserving with respect to $\bar{\mu}$ and the tower sets will be defined with the aid of these maps, we require this more general concept. From the different types of approximations various ergodic properties can be derived. For example in [S70], Corollary 4.1., the subsequent Lemma is proven.

Lemma 7.4. *Let $T : (X, \mu) \rightarrow (X, \mu)$ be a measure-preserving transformation. If T admits an approximation with speed $\varphi(n) = \frac{\theta}{n}$ with $\theta < 4$, then T is ergodic.*

We will use this Lemma as a criterion for the ergodicity of the projectivized derivative extension.

7.2 Application of the criterion

We prove the ergodicity of the projectivized derivative extension with the aid of Lemma 7.4. In order to apply it, we have to prove that (f, df) admits a sufficiently fast approximation on $\mathbb{P}TM$ with respect to the measure $\bar{\mu}$ introduced in section 1.1. For this purpose, we define a tower explicitly and examine the speed of approximation.

7.2.1 Tower for good cyclic approximation

Using the "good domains" G_n introduced in Remark 6.2 we define

$$\bar{G}_n := G_{n+1} \cap \bigcap_{j=1}^{\infty} h_{n+1} \circ ... \circ h_{n+j} \left(G_{n+j+1}\right).$$

In particular, for every $s \in \mathbb{N}$ the map $h_{n+s}^{-1} \circ ... \circ h_{n+1}^{-1}$ is a composition of rotations and translations on $\bar{G}_n$.

Furthermore, let $\check{c}_0^{(n)} \subset \mathbb{S}^1 \times [0,1]^{m-1} = M$ be the set

$$\begin{aligned}
\bigcup \Bigg[& \frac{s_1^{(1)}}{n^2 \cdot k_n^{m-1} \cdot q_n} + \frac{j_1^{(1)}}{n^2 \cdot k_n^{m-1} \cdot q_n^2} + ... + \frac{j_1^{(m-1)}}{n^2 \cdot k_n^{m-1} \cdot q_n^m} + \frac{1}{n^6 \cdot q_{n-1} \cdot k_n^{m-1} \cdot q_n^m} + ... \\
& + \frac{1}{n^6 \cdot q_{n-1} \cdot k_n^{m-1} \cdot q_n^{(m-1) \cdot \frac{(n+1) \cdot n}{2}}} + \frac{1}{n^2 \cdot q_{n-1} \cdot k_n^{m-1} \cdot q_n^{1+(m-1) \cdot \frac{n \cdot (n+1)}{2}}} \\
& + \frac{1}{5n^6 \cdot q_{n-1}^2 \cdot k_n^{m-1} \cdot q_n^{1+(m-1) \cdot \frac{n \cdot (n+1)}{2}}} + \frac{s_1^{(2)}}{q_{n+1}}, \\
& \frac{s_1^{(1)}}{n^2 \cdot k_n^{m-1} \cdot q_n} + \frac{j_1^{(1)}}{n^2 \cdot k_n^{m-1} \cdot q_n^2} + ... + \frac{j_1^{(m-1)}}{n^2 \cdot k_n^{m-1} \cdot q_n^m} + \frac{1}{n^6 \cdot q_{n-1} \cdot k_n^{m-1} \cdot q_n^{m+1}} + ... \\
& + \frac{1}{n^6 \cdot q_{n-1} \cdot k_n^{m-1} \cdot q_n^{(m-1) \cdot \frac{(n+1) \cdot n}{2}}} + \frac{1}{n^2 \cdot q_{n-1} \cdot k_n^{m-1} \cdot q_n^{1+(m-1) \cdot \frac{n \cdot (n+1)}{2}}} \\
& + \frac{1}{5n^6 \cdot q_{n-1}^2 \cdot k_n^{m-1} \cdot q_n^{1+(m-1) \cdot \frac{n \cdot (n+1)}{2}}} + \frac{s_1^{(2)} + 1}{q_{n+1}} \Bigg] \\
\times \prod_{i=2}^{m} \Bigg[& \frac{1}{n^4 \cdot q_{n-1}} + \frac{j_i^{(2)}}{q_n^2} + ... + \frac{j_i^{\left(1+(m-1) \cdot \frac{n \cdot (n+1)}{2}\right)}}{q_n^{1+(m-1) \cdot \frac{n \cdot (n+1)}{2}}} + \frac{s_i}{n^2 \cdot k_n^{m-1} \cdot q_n^{1+(m-1) \cdot \frac{n \cdot (n+1)}{2}}} \\
& + \frac{u_i}{n^2 \cdot q_{n-1} \cdot k_n^{m-1} \cdot q_n^{1+(m-1) \cdot \frac{n \cdot (n+1)}{2}}} + \frac{1}{5n^6 \cdot q_{n-1}^2 \cdot k_n^{m-1} \cdot q_n^{1+(m-1) \cdot \frac{n \cdot (n+1)}{2}}} + \frac{t_i}{q_{n+1}}, \\
& \frac{1}{n^4 \cdot q_{n-1}} + \frac{j_i^{(2)}}{q_n^2} + ... + \frac{j_i^{\left(1+(m-1) \cdot \frac{n \cdot (n+1)}{2}\right)}}{q_n^{1+(m-1) \cdot \frac{n \cdot (n+1)}{2}}} + \frac{s_i}{n^2 \cdot k_n^{m-1} \cdot q_n^{1+(m-1) \cdot \frac{n \cdot (n+1)}{2}}} \\
& + \frac{u_i}{n^2 \cdot q_{n-1} \cdot k_n^{m-1} \cdot q_n^{1+(m-1) \cdot \frac{n \cdot (n+1)}{2}}} + \frac{1}{5n^6 \cdot q_{n-1}^2 \cdot k_n^{m-1} \cdot q_n^{1+(m-1) \cdot \frac{n \cdot (n+1)}{2}}} + \frac{t_i + 1}{q_{n+1}} \Bigg],
\end{aligned}$$

where the union is taken over

- $s_1^{(1)} \in \mathbb{Z}$, $0 \leq s_1^{(1)} \leq k_n^{m-1} - 1$
- $s_1^{(2)} \in \mathbb{Z}$, $0 \leq s_1^{(2)} \leq A-1$ using the notation $A := 780n^6 \cdot (n+1)^6 \cdot q_{n-1}^2 \cdot k_n^{m-1} \cdot q_n^{1+(m-1)\cdot \frac{n\cdot(n+1)}{2}}$
- $s_i \in \mathbb{Z}$, $0 \leq s_i \leq n^2 k_n^{m-1} - 1$, for $i = 2, ..., m$
- $j_1^{(t)} \in \mathbb{Z}$, $\left\lceil \frac{q_n}{n^4 q_{n-1}} \right\rceil \leq j_1^{(t)} \leq q_n - \left\lceil \frac{q_n}{n^4 q_{n-1}} \right\rceil - 1$, for $t = 1, ..., m-1$
- $j_i^{(s)} \in \mathbb{Z}$, $\left\lceil \frac{q_n}{n^4 q_{n-1}} \right\rceil \leq j_i^{(s)} \leq q_n - \left\lceil \frac{q_n}{n^4 q_{n-1}} \right\rceil - 1$, for $s = 2, ..., 1 + (m-1) \cdot \frac{n\cdot(n+1)}{2}$ and $i = 2, ..., m$
- $u_i \in \mathbb{Z}$, $1 \leq u_i \leq q_{n-1} - 2$, for $i = 2, ..., m$
- $t_i \in \mathbb{Z}$, $0 \leq t_i \leq \frac{q_{n+1}}{n^2 \cdot q_{n-1} \cdot k_n^{m-1} \cdot q_n^{1+(m-1)\cdot \frac{n\cdot(n+1)}{2}}} - 2 \cdot \left\lceil \frac{q_{n+1}}{5n^6 \cdot q_{n-1}^2 \cdot k_n^{m-1} \cdot q_n^{1+(m-1)\cdot \frac{n\cdot(n+1)}{2}}} \right\rceil - 1$ for $i = 2, ..., m$.

Remark 7.5. Note that all the parts of $\check{c}_0^{(n)}$ are positioned in the domain, where i_n acts as a translation and rotation as well as g_n is a translation on $i_n\left(\check{c}_0^{(n)}\right)$. At this juncture, the requirement that $5n^6 \cdot q_{n-1}^2 \cdot k_n^{m-1} \cdot q_n^{1+(m-1)\cdot \frac{n\cdot(n+1)}{2}}$ divides q_{n+1} (see Lemma 5.8) is important. In particular, the rotation arcs of i_n are different for all the occurring $s_1^{(1)}$.

Remark 7.6. We compute that $\phi_n^{-1}\left(\check{c}_0^{(n)}\right)$ contains at least $A \cdot q_{n+1}^{m-1} \cdot k_n^{m-1} \cdot \left(1 - \frac{3m}{q_{n-1}}\right)$ many $\prod_{i=1}^m \left[\frac{j_i}{q_{n+1}}, \frac{j_i+1}{q_{n+1}}\right]$-domains, where $\left\lceil \frac{q_{n+1}}{(n+1)^4 q_n} \right\rceil \leq j_i \leq q_{n+1} - \left\lceil \frac{q_{n+1}}{(n+1)^4 q_n} \right\rceil - 1$ for $j = 2, ..., m$. On each of these cubes there are at most $(n+1)^{2m} \cdot k_{n+1}^{m\cdot(m-1)} \cdot q_n^m \cdot q_{n+1}^{m\cdot(m-1)\cdot \frac{(n+1)\cdot(n+2)}{2}}$ elements $h_{n+1}\left(\check{I}_{n+1}\right)$ with $\check{I}_{n+1} \in \zeta_{n+1}$ and a measure of at least $\frac{1-\frac{3m}{q_n}}{q_{n+1}^m}$ is covered by sets of G_{n+1} (in case of $n \geq m$). Similarly, we observe that for any $h_{n+1}\left(\check{I}_{n+1}\right) \subset G_{n+1}$ we have

$$\begin{aligned} \mu\left(h_{n+1}\left(\check{I}_{n+1}\right) \cap h_{n+1}\left(G_{n+2}\right)\right) &= \mu\left(\check{I}_{n+1} \cap \bigcup_{\check{I}_{n+2} \in \zeta_{n+2}} h_{n+2}\left(\check{I}_{n+2}\right)\right) \\ &\geq \left(1 - \frac{3m}{q_{n+1}}\right) \cdot \mu\left(h_{n+1}\left(\check{I}_{n+1}\right)\right). \end{aligned}$$

In the next step, we define $\tilde{c}_0^{(n)} := \check{c}_0^{(n)} \cap \phi_n\left(G_{n+1}\right)$ and $\check{c}_0^{(n)} := \check{c}_0^{(n)} \cap \phi_n\left(\bar{G}_n\right)$. With the aid of Remark 7.6 we estimate

$$\frac{k_n^{m-1}}{\tilde{q}_{n+1}} \geq \mu\left(\tilde{c}_0^{(n)}\right) \geq A \cdot q_{n+1}^{m-1} \cdot k_n^{m-1} \cdot \left(1 - \frac{3m}{q_{n-1}}\right) \cdot \frac{1 - \frac{3m}{q_n}}{q_{n+1}^m} \geq \frac{k_n^{m-1}}{\tilde{q}_{n+1}} \cdot \left(1 - \frac{4m}{q_{n-1}}\right).$$

Then we define $\bar{c}_0^{(n)} := g_n \circ i_n\left(\tilde{c}_0^{(n)}\right)$ and we consider $\bar{c}_0^{(n)} \times \left[0, \frac{1}{k_n}\right]^{m-1} \subset \mathbb{P}TM$ with respect to ω_0. The base element of the tower in $\mathbb{P}TM$ is $c_0^{(n)} = (H_{n-1}, dH_{n-1})\left(\bar{c}_0^{(n)} \times \left[0, \frac{1}{k_n}\right]^{m-1}\right) \subset \mathbb{P}TM$ with respect to ω_∞. Finally, the tower elements are

$$c_i^{(n)} = \left(f_n^i, df_n^i\right)\left(c_0^{(n)}\right) \text{ for } i = 0, ..., \tilde{q}_{n+1} - 1.$$

Lemma 7.7. *We have*

$$\sum_{i=0}^{\tilde{q}_{n+1}-1} \left| \bar{\mu}\left(c_i^{(n)}\right) - \frac{1}{\tilde{q}_{n+1}} \right| \leq \frac{4m \cdot \left(k_n^{m-1}-1\right)}{\tilde{q}_{n+1}},$$

which converges to 0 *as* $n \to \infty$ *by Lemma 5.8. Thus, the second requirement in the definition 7.3 of an approximation is fulfilled.*

Proof. For $(y, v) = \left(f_n^i(x), d_x f_n^i(\tilde{v})\right) = \left(f_n^i\left(H_{n-1} \circ g_n \circ i_n(z)\right), d_x f_n^i\left(d_{g_n \circ i_n(z)} H_{n-1}(\bar{v})\right)\right)$ and $(y, w) = \left(f_n^i(x), d_x f_n^i(\tilde{w})\right) = \left(f_n^i\left(H_{n-1} \circ g_n \circ i_n(z)\right), d_x f_n^i\left(d_{g_n \circ i_n(z)} H_{n-1}(\bar{w})\right)\right)$ with $z \in \check{c}_0^{(n)}$ as well as $\bar{v}, \bar{w} \in \left[0, \frac{1}{k_n}\right]^{m-1}$ we calculate with the aid of the construction of the f-invariant Riemannian metric ω_∞

$$\begin{aligned}
&\omega_\infty|_y(v,w) = \lim_{k\to\infty} \left(H_k^{-1}\right)^* \omega_0|_y(v,w) \\
&= \lim_{k\to\infty} \omega_0|_{H_k^{-1}(y)} \left(d_y H_k^{-1}(v), d_y H_k^{-1}(w)\right) \\
&= \lim_{k\to\infty} \omega_0|_{h_k^{-1}\circ \ldots \circ h_{n+1}^{-1} \circ R^i_{\alpha_{n+1}} \circ \phi_n^{-1}(z)} \Big(d_{R^i_{\alpha_{n+1}} \circ H_n^{-1}(x)} \left(h_k^{-1} \circ \ldots \circ h_{n+1}^{-1}\right) \cdot d_{H_{n-1}^{-1}(x)} h_n^{-1}(\bar{v}), \\
&\qquad\qquad d_{R^i_{\alpha_{n+1}} \circ H_n^{-1}(x)} \left(h_k^{-1} \circ \ldots \circ h_{n+1}^{-1}\right) \cdot d_{H_{n-1}^{-1}(x)} h_n^{-1}(\bar{w}) \Big) \\
&= \omega_0|_{\phi_n^{-1}(z)} \left(d_{H_{n-1}^{-1}(x)} h_n^{-1}(\bar{v}), d_{H_{n-1}^{-1}(x)} h_n^{-1}(\bar{w}) \right).
\end{aligned}$$

In the last step we exploited that $h_k^{-1} \circ \ldots \circ h_{n+1}^{-1}$ is an isometry with respect to ω_0 on $\bar{G}_n$. Additionally, h_n^{-1} is an isometry on $g_n \circ \phi_n\left(\check{c}_0^{(n)}\right)$ and ω_0 is independent from the base point. Hence, we conclude $\omega_\infty|_y(v,w) = \omega_0|_{g_n \circ i_n(z)}(\bar{v}, \bar{w})$ and then $\omega_\infty|_y(v,w) = \omega_\infty|_x(\tilde{v}, \tilde{w})$.
Thus, (f_n, df_n) ist $\bar{\mu}$-preserving on sets with base points in $H_{n-1} \circ g_n \circ \phi_n\left(\check{c}_0^{(n)}\right)$. Since $\mu\left(\tilde{c}_0^{(n)}\right) \geq \mu\left(\check{c}_0^{(n)}\right) \geq \left(1 - \frac{4m}{q_{n+1}}\right) \cdot \mu\left(\tilde{c}_0^{(n)}\right)$ we have

$$\left(1 - \frac{4m}{q_{n+1}}\right) \cdot \frac{1}{\tilde{q}_{n+1}} \cdot \left(1 - \frac{4m}{q_{n-1}}\right) \leq \bar{\mu}\left(c_i^{(n)}\right) \leq \left(1 + \frac{4m \cdot \left(k_n^{m-1} - 1\right)}{q_{n+1}}\right) \cdot \frac{1}{\tilde{q}_{n+1}} \cdot \left(1 - \frac{4m}{q_{n-1}}\right). \tag{5}$$

In particular, this yields

$$\sum_{i=0}^{\tilde{q}_{n+1}-1} \left| \bar{\mu}\left(c_i^{(n)}\right) - \frac{1}{\tilde{q}_{n+1}} \right| \leq \frac{4m \cdot \left(k_n^{m-1}-1\right)}{\tilde{q}_{n+1}}.$$

□

Furthermore, we observe that these tower elements are disjoint sets in $\mathbb{P}TM$ by construction. Hence, we are able to define a partial partition

$$\xi_n := \left\{ c_i^{(n)} \ : \ i = 0, 1, \ldots, \tilde{q}_{n+1} - 1 \right\}$$

(using the notation from section 7.1) and have to show $\xi_n \to \varepsilon$ as $n \to \infty$.

Lemma 7.8. *We have*

$$\xi_n \to \varepsilon \text{ as } n \to \infty.$$

Proof. This property is fulfilled if we show that the partial partitions $\tilde{\xi}_n := \left\{c \in \xi_n : \operatorname{diam}(c) < \frac{1}{n}\right\}$ satisfy $\bar{\mu}\left(\bigcup_{c\in\tilde{\xi}_n} c\right) \to 1$ as $n \to \infty$. For this purpose, we examine which tower elements satisfy the condition on their diameter. Due to the requirement on the number k_n (see the beginning of section 2) it is satisfied if $\operatorname{diam}\left(h_n \circ R^i_{\alpha_{n+1}} \circ H_n^{-1}\left(c_0^{(n)}\right)\right) < \frac{1}{2n}$. Since the map h_n is $\frac{1}{q_n}$-equivariant and

$$\begin{aligned} h_n \circ R^i_{\alpha_{n+1}} \circ H_n^{-1}\left(H_{n-1}\left(\bar{c}_0^{(n)}\right)\right) &= h_n \circ R^i_{\alpha_{n+1}} \circ \phi_n^{-1} \circ i_n^{-1} \circ g_n^{-1} \circ H_{n-1}^{-1}\left(H_{n-1} \circ g_n \circ i_n\left(\tilde{c}_0^{(n)}\right)\right) \\ &= h_n \circ R^i_{\alpha_{n+1}} \circ \phi_n^{-1}\left(\tilde{c}_0^{(n)}\right), \end{aligned}$$

we have to check for how many iterates i the set $R^i_{\alpha_{n+1}} \circ \phi_n^{-1}\left(\tilde{c}_0^{(n)}\right)$ is contained in the "good domain" of h_n and the deviation $i \cdot |\alpha_{n+1} - \alpha_n|$ is not in $\left[\frac{k}{nq_n} + \frac{n-1}{n^2 q_n}, \frac{k+1}{nq_n}\right)$ for any $k \in \mathbb{Z}$, $0 \le k \le n-1$ (otherwise the different definitions of ϕ_n on the abutting domains may cause some problems). Under these assumptions we have

$$\operatorname{diam}\left(h_n \circ R^i_{\alpha_{n+1}} \circ \phi_n^{-1}\left(\tilde{c}_0^{(n)}\right)\right) \le [nq_n^\sigma] \cdot \frac{\sqrt{m}}{q_n}.$$

Because of $0.25 < \sigma < 0.5$ and Lemma 5.8, 4., we deduce the aimed estimate

$$\operatorname{diam}\left(H_{n-1} \circ h_n \circ R^i_{\alpha_{n+1}} \circ \phi_n^{-1}\left(\tilde{c}_0^{(n)}\right)\right) \le \|DH_{n-1}\|_0 \cdot [nq_n^\sigma] \cdot \frac{\sqrt{m}}{q_n} \le \frac{q_n^{0.25}}{2n^2 \cdot \sqrt{m}} \cdot [nq_n^\sigma] \cdot \frac{\sqrt{m}}{q_n} < \frac{1}{2n}$$

Note that the base of the tower is positioned in this "good domain". Since $R^i_{\alpha_{n+1}} = R^i_{\frac{\tilde{p}_{n+1}}{\tilde{q}_{n+1}}}$ is equidistributed on $\mathbb{S}^1$ and a length of at least $\left(1 - \frac{4m}{q_{n-1}}\right) \cdot \left(1 - \frac{1}{n}\right)$ corresponds to the "good domain" by Remark 6.2, we can estimate the number of allowed iterates $i \in \{0, 1, ..., \tilde{q}_{n+1} - 1\}$ by $\left(1 - \frac{4m}{q_{n-1}}\right) \cdot \left(1 - \frac{1}{n}\right) \cdot \tilde{q}_{n+1}$. This corresponds to a measure

$$\begin{aligned} \bar{\mu}\left(\bigcup_{c\in\tilde{\xi}_n} c\right) &\ge \left(1 - \frac{4m}{q_{n-1}}\right) \cdot \left(1 - \frac{1}{n}\right) \cdot \tilde{q}_{n+1} \cdot \bar{\mu}\left(c_i^{(n)}\right) \\ &\ge \left(1 - \frac{4m}{q_{n-1}}\right)^2 \cdot \left(1 - \frac{1}{n}\right) \cdot \left(1 - \frac{4m}{q_{n+1}}\right), \end{aligned}$$

which converges to 1 as $n \to \infty$. □

7.2.2 Speed of approximation

For the speed of approximation it holds:

$$\begin{aligned} &\sum_{c\in\xi_n} \bar{\mu}\left(\left(f, df\right)(c) \triangle \left(f_n, df_n\right)(c)\right) \\ \le &\sum_{c\in\xi_n} \left(\bar{\mu}\left(\left(f, df\right)(c) \triangle \left(f_{n+1}, df_{n+1}\right)(c)\right) + \bar{\mu}\left(\left(f_{n+1}, df_{n+1}\right)(c) \triangle \left(f_n, df_n\right)(c)\right)\right). \end{aligned}$$

Lemma 7.9. *We have*

$$\sum_{c\in\xi_n}\bar{\mu}\left(\left(f_n,df_n\right)(c)\triangle\left(f_{n+1},df_{n+1}\right)(c)\right)\leq\frac{q_{n+1}}{A}\cdot\frac{q_{n+1}^{m\cdot(m-1)\cdot(n+2)\cdot(n+1)}}{l_{n+1}\cdot C_{l_{n+1}}\cdot q_{n+1}^{3\cdot m^2\cdot(l_{n+1}+1)^2\cdot(n+1)\cdot(n+2)}}. \tag{6}$$

Proof. First of all, we aim for estimating $\bar{\mu}\left(\left(f_{n+1},df_{n+1}\right)\left(f_n^i,df_n^i\right)\left(c_0^{(n)}\right)\triangle\left(f_n^{i+1},df_n^{i+1}\right)\left(c_0^{(n)}\right)\right)$. For this purpose, we consider

$$\begin{aligned}
&\mu\left(f_{n+1}\circ f_n^i\left(H_{n-1}\left(\bar{c}_0^{(n)}\right)\right)\triangle f_n^{i+1}\left(H_{n-1}\left(\bar{c}_0^{(n)}\right)\right)\right)\\
=&\mu\left(H_{n+1}\circ R_{\alpha_{n+2}}\circ R^i_{\alpha_{n+1}}\circ h^{-1}_{n+1}\circ\phi_n^{-1}\left(\tilde{c}_0^{(n)}\right)\triangle H_{n+1}\circ R^{i+1}_{\alpha_{n+1}}\circ h^{-1}_{n+1}\circ\phi_n^{-1}\left(\tilde{c}_0^{(n)}\right)\right)\\
=&\mu\left(R_{\alpha_{n+2}}\circ R^i_{\alpha_{n+1}}\circ h^{-1}_{n+1}\circ\phi_n^{-1}\left(\tilde{c}_0^{(n)}\right)\triangle R^{i+1}_{\alpha_{n+1}}\circ h^{-1}_{n+1}\circ\phi_n^{-1}\left(\tilde{c}_0^{(n)}\right)\right).
\end{aligned}$$

Since $h^{-1}_{n+1}\circ\phi_n^{-1}\left(\tilde{c}_0^{(n)}\right)$ consists of at most

$$A\cdot q_{n+1}^{m-1}\cdot k_n^{m-1}\cdot\left(1-\frac{3m}{q_{n-1}}\right)\cdot(n+1)^{2m}\cdot k_{n+1}^{m\cdot(m-1)}\cdot q_n^m\cdot q_{n+1}^{m\cdot(m-1)\cdot\frac{(n+2)\cdot(n+1)}{2}}$$

elements $\check{I}_{n+1}\in\zeta_{n+1}$ by Remark 7.6 and the measure difference is at most $|\alpha_{n+2}-\alpha_{n+1}|$ for any such element, we estimate with the aid of Lemma 5.8 and equation 4

$$\begin{aligned}
&\mu\left(R_{\alpha_{n+2}}\circ R^i_{\alpha_{n+1}}\circ h^{-1}_{n+1}\circ\phi_n^{-1}\left(\tilde{c}_0^{(n)}\right)\triangle R^{i+1}_{\alpha_{n+1}}\circ h^{-1}_{n+1}\circ\phi_n^{-1}\left(\tilde{c}_0^{(n)}\right)\right)\\
\leq&A\cdot q_{n+1}^{m-1}\cdot k_n^{m-1}\cdot\left(1-\frac{3m}{q_{n-1}}\right)\cdot(n+1)^{2m}\cdot k_{n+1}^{m\cdot(m-1)}\cdot q_n^m\cdot q_{n+1}^{m\cdot(m-1)\cdot\frac{(n+2)\cdot(n+1)}{2}}\cdot|\alpha_{n+2}-\alpha_{n+1}|\\
\leq&q_{n+1}^{m+2}\cdot q_{n+1}^{m\cdot(m-1)\cdot\frac{(n+2)\cdot(n+1)}{2}}\cdot\frac{1}{l_{n+1}\cdot C_{l_{n+1}}\cdot q_{n+1}^{3\cdot m^2\cdot(l_{n+1}+1)^2\cdot(n+1)\cdot(n+2)}}\\
\leq&q_{n+1}^{m\cdot(m-1)\cdot(n+2)\cdot(n+1)}\cdot\frac{1}{l_{n+1}\cdot C_{l_{n+1}}\cdot q_{n+1}^{3\cdot m^2\cdot(l_{n+1}+1)^2\cdot(n+1)\cdot(n+2)}}.
\end{aligned}$$

For $y\in R_{\alpha_{n+2}}\circ R^i_{\alpha_{n+1}}\circ h^{-1}_{n+1}\circ\phi_n^{-1}\left(\tilde{c}_0^{(n)}\right)\cap R^{i+1}_{\alpha_{n+1}}\circ h^{-1}_{n+1}\circ\phi_n^{-1}\left(\tilde{c}_0^{(n)}\right)$ there are $x_1,x_2\in\tilde{c}_0^{(n)}$ such that $y=R_{\alpha_{n+2}}\circ R^i_{\alpha_{n+1}}\circ h^{-1}_{n+1}\circ\phi_n^{-1}(x_1)$, $y=R^{i+1}_{\alpha_{n+1}}\circ h^{-1}_{n+1}\circ\phi_n^{-1}(x_2)$ and

$$\begin{aligned}
&d_{g_n\circ i_n(x_1)}\left(h^{-1}_{n+1}\circ\phi_n^{-1}\circ i_n^{-1}\circ g_n^{-1}\right)\left(\left[0,\frac{1}{k_n}\right]^{m-1}\right)\\
=&d_{g_n\circ i_n(x_2)}\left(h^{-1}_{n+1}\circ\phi_n^{-1}\circ i_n^{-1}\circ g_n^{-1}\right)\left(\left[0,\frac{1}{k_n}\right]^{m-1}\right)
\end{aligned}$$

(because they are close to each other and are positioned in the domain where the maps act as the respective rotations and translations). Hence, we conclude

$$\begin{aligned}
&\bar{\mu}\left(\left(f_{n+1},df_{n+1}\right)\left(f_n^i,df_n^i\right)\left(c_0^{(n)}\right)\triangle\left(f_n^{i+1},df_n^{i+1}\right)\left(c_0^{(n)}\right)\right)\\
=&\mu\left(f_{n+1}\circ f_n^i\left(H_{n-1}\left(\bar{c}_0^{(n)}\right)\right)\triangle f_n^{i+1}\left(H_{n-1}\left(\bar{c}_0^{(n)}\right)\right)\right)\\
\leq&q_{n+1}^{m\cdot(m-1)\cdot(n+2)\cdot(n+1)}\cdot\frac{1}{l_{n+1}\cdot C_{l_{n+1}}\cdot q_{n+1}^{3\cdot m^2\cdot(l_{n+1}+1)^2\cdot(n+1)\cdot(n+2)}}.
\end{aligned}$$

This difference occours for every $i \in \{0, ..., \tilde{q}_{n+1}-1\}$ and thus we can estimate

$$\sum_{c \in \xi_n} \bar{\mu}\left(\left(f_n, df_n\right)(c) \triangle \left(f_{n+1}, df_{n+1}\right)(c)\right) \leq \frac{q_{n+1}}{A} \cdot \frac{q_{n+1}^{m\cdot(m-1)\cdot(n+2)\cdot(n+1)}}{l_{n+1} \cdot C_{l_{n+1}} \cdot q_{n+1}^{3\cdot m^2 \cdot (l_{n+1}+1)^2 \cdot (n+1)\cdot(n+2)}}.$$

□

In the next step we consider $\sum_{c \in \xi_n} \bar{\mu}\left(\left(f, df\right)(c) \triangle \left(f_{n+1}, df_{n+1}\right)(c)\right)$:

Lemma 7.10. *We have*

$$\sum_{c \in \xi_n} \bar{\mu}\left(\left(f, df\right)(c) \triangle \left(f_{n+1}, df_{n+1}\right)(c)\right) \leq \frac{5m \cdot k_n^{m-1}}{q_{n+1}}.$$

Proof. We compute for every $c = \left(f_n^i, df_n^i\right)\left(c_0^{(n)}\right) \in \xi_n$:

$$\begin{aligned}
&\mu\left(f_{n+2}\left(f_n^i\left(H_{n-1}\bar{c}_0^{(n)}\right)\right) \triangle f_{n+1}\left(f_n^i\left(H_{n-1}\bar{c}_0^{(n)}\right)\right)\right) \\
=&\mu\left(H_{n+2} \circ R_{\alpha_{n+3}} \circ h_{n+2}^{-1} \circ h_{n+1}^{-1}\left(R_{\alpha_{n+1}}^i \circ \phi_n^{-1}\left(\tilde{c}_0^{(n)}\right)\right) \triangle H_{n+1} \circ R_{\alpha_{n+2}} \circ h_{n+1}^{-1}\left(R_{\alpha_{n+1}}^i \circ \phi_n^{-1}\left(\tilde{c}_0^{(n)}\right)\right)\right) \\
=&\mu\left(R_{\alpha_{n+3}} \circ h_{n+2}^{-1}\left(R_{\alpha_{n+1}}^i \circ h_{n+1}^{-1} \circ \phi_n^{-1}\left(\tilde{c}_0^{(n)}\right)\right) \triangle R_{\alpha_{n+2}} \circ h_{n+2}^{-1}\left(R_{\alpha_{n+1}}^i \circ h_{n+1}^{-1} \circ \phi_n^{-1}\left(\tilde{c}_0^{(n)}\right)\right)\right) \\
=&\mu\left(R_{\alpha_{n+3}} \circ R_{\alpha_{n+1}}^i\left(h_{n+2}^{-1} \circ h_{n+1}^{-1} \circ \phi_n^{-1}\left(\tilde{c}_0^{(n)}\right)\right) \triangle R_{\alpha_{n+2}} \circ R_{\alpha_{n+1}}^i\left(h_{n+2}^{-1} \circ h_{n+1}^{-1} \circ \phi_n^{-1}\left(\tilde{c}_0^{(n)}\right)\right)\right),
\end{aligned}$$

where we exploited that h_{n+2} comutes with $R_{\frac{1}{q_{n+2}}}$ and q_{n+2} is a multiple of q_{n+1}.
Since we have no controll on $h_{n+2}^{-1} \circ h_{n+1}^{-1} \circ \phi_n^{-1}\left(\tilde{c}_0^{(n)}\right)$ for these areas of $d := h_{n+1}^{-1} \circ \phi_n^{-1}\left(\tilde{c}_0^{(n)}\right)$, that do not belong to the "good domain" of the map h_{n+2}^{-1}, they will be part of the measure difference in our estimates. Using Remark 7.6 the "good domain" of the map h_{n+2}^{-1} on an element $\check{I}_{n+1} \in \zeta_{n+1}$ has measure at least $\left(1 - \frac{3m}{q_{n+1}}\right) \cdot \mu\left(h_{n+1}\left(\check{I}_{n+1}\right)\right)$. On the other hand, for every $h_{n+2}\left(\check{I}_{n+2}\right)$ belonging to d the difference is caused by the deviation $|\alpha_{n+3} - \alpha_{n+2}|$. We observe that there are at most

$$\left(\frac{1 - \frac{2}{5(n+1)^4 q_n}}{(n+1)^2 \cdot q_n \cdot k_{n+1}^{m-1} \cdot q_{n+1}^{(m-1)\cdot\frac{(n+1)\cdot(n+2)}{2}}}\right)^m \cdot (n+2)^{2m} \cdot k_{n+2}^{m\cdot(m-1)} \cdot q_{n+2}^{m\cdot\left(1+(m-1)\cdot\frac{(n+2)\cdot(n+3)}{2}\right)}$$

elements $h_{n+2}\left(\check{I}_{n+2}\right)$ contained in $\check{I}_{n+1} \in \zeta_{n+1}$. Altogether, the measure difference caused by $R_{\alpha_{n+3}} \circ R_{\alpha_{n+1}}^i \circ h_{n+2}^{-1}$ and $R_{\alpha_{n+2}} \circ R_{\alpha_{n+1}}^i \circ h_{n+2}^{-1}$ on an element $\check{I}_{n+1} \in \zeta_{n+1}$ contained in d is at most

$$\begin{aligned}
&\frac{3m}{q_{n+1}} \cdot \left(\frac{1}{(n+1)^2 \cdot q_n \cdot k_{n+1}^{m-1} \cdot q_{n+1}^{1+(m-1)\cdot\frac{(n+1)\cdot(n+2)}{2}}}\right)^m \\
+&\left(\frac{1 - \frac{2}{5(n+1)^4 q_n}}{(n+1)^2 q_n k_{n+1}^{m-1} q_{n+1}^{(m-1)\frac{(n+1)\cdot(n+2)}{2}}}\right)^m \cdot (n+2)^{2m} \cdot k_{n+2}^{m(m-1)} \cdot q_{n+2}^{m\left(1+(m-1)\frac{(n+2)\cdot(n+3)}{2}\right)} \cdot \left|\alpha_{n+3} - \alpha_{n+2}\right|.
\end{aligned}$$

Moreover, we recall that d consists of at most

$$A \cdot q_{n+1}^{m-1} \cdot k_n^{m-1} \cdot \left(1 - \frac{3m}{q_{n-1}}\right) \cdot (n+1)^{2m} \cdot k_{n+1}^{m\cdot(m-1)} \cdot q_n^m \cdot q_{n+1}^{m\cdot(m-1)\cdot\frac{(n+2)\cdot(n+1)}{2}}$$

elements $\check{I}_{n+1} \in \zeta_{n+1}$. Hereby, we obtain

$$\begin{aligned}
&\mu\left(f_{n+2}(c) \triangle f_{n+1}(c)\right) \\
\leq &\frac{3m \cdot A \cdot k_n^{m-1}}{q_{n+1}^2} + A \cdot q_{n+1}^{m-1} \cdot k_n^{m-1} \cdot (n+2)^{2m} \cdot k_{n+2}^{m\cdot(m-1)} \cdot q_{n+2}^{m\cdot\left(1+(m-1)\cdot\frac{(n+2)\cdot(n+3)}{2}\right)} \cdot \left|\alpha_{n+3}-\alpha_{n+2}\right|.
\end{aligned}$$

We note that for

$$y \in R_{\alpha_{n+3}} \circ R^i_{\alpha_{n+1}}\left(h^{-1}_{n+2} \circ h^{-1}_{n+1} \circ \phi_n^{-1}\left(\tilde{c}_0^{(n)}\right)\right) \cap R_{\alpha_{n+2}} \circ R^i_{\alpha_{n+1}}\left(h^{-1}_{n+2} \circ h^{-1}_{n+1} \circ \phi_n^{-1}\left(\tilde{c}_0^{(n)}\right)\right),$$

where $y = R_{\alpha_{n+3}} \circ R^i_{\alpha_{n+1}}\left(h^{-1}_{n+2} \circ h^{-1}_{n+1} \circ \phi_n^{-1}(x_1)\right)$ and $y = R_{\alpha_{n+2}} \circ R^i_{\alpha_{n+1}}\left(h^{-1}_{n+2} \circ h^{-1}_{n+1} \circ \phi_n^{-1}(x_2)\right)$ with $x_1, x_2 \in \tilde{c}_0^{(n)}$ close to each other contained in the "good domain" of $h^{-1}_{n+2} \circ h^{-1}_{n+1} \circ \phi_n^{-1}$ we have

$$\begin{aligned}
&d_{g_n \circ i_n(x_1)}\left(h^{-1}_{n+2} \circ h^{-1}_{n+1} \circ \phi_n^{-1} \circ i_n^{-1} \circ g_n^{-1}\right)\left(\left[0, \frac{1}{k_n}\right]^{m-1}\right) \\
=&d_{g_n \circ i_n(x_2)}\left(h^{-1}_{n+2} \circ h^{-1}_{n+1} \circ \phi_n^{-1} \circ i_n^{-1} \circ g_n^{-1}\right)\left(\left[0, \frac{1}{k_n}\right]^{m-1}\right)
\end{aligned}$$

Thus, we conclude

$$\begin{aligned}
&\bar{\mu}\left(\left(f_{n+2}, df_{n+2}\right)(c) \triangle \left(f_{n+1}, df_{n+1}\right)(c)\right) \\
\leq &\mu\left(f_{n+2}\left(f_n^i\left(H_{n-1}\bar{c}_0^{(n)}\right)\right) \triangle f_{n+1}\left(f_n^i\left(H_{n-1}\bar{c}_0^{(n)}\right)\right)\right) \\
\leq &\frac{3m \cdot A \cdot k_n^{m-1}}{q_{n+1}^2} + A \cdot q_{n+1}^{m-1} \cdot k_n^{m-1} \cdot (n+2)^{2m} \cdot k_{n+2}^{m\cdot(m-1)} \cdot q_{n+2}^{m\cdot\left(1+(m-1)\cdot\frac{(n+2)\cdot(n+3)}{2}\right)} \cdot \left|\alpha_{n+3}-\alpha_{n+2}\right|.
\end{aligned}$$

Every of the $\tilde{q}_{n+1} = \frac{q_{n+1}}{A}$ elements $c \in \xi_n$ contributes and so we obtain

$$\begin{aligned}
&\sum_{c \in \xi_n} \bar{\mu}\left(\left(f_{n+1}, df_{n+1}\right)(c) \triangle \left(f_{n+2}, df_{n+2}\right)(c)\right) \\
\leq &\frac{3m \cdot k_n^{m-1}}{q_{n+1}} + q_{n+1}^{m} \cdot k_n^{m-1} \cdot (n+2)^{2m} \cdot k_{n+2}^{m\cdot(m-1)} \cdot q_{n+2}^{m\cdot\left(1+(m-1)\cdot\frac{(n+2)\cdot(n+3)}{2}\right)} \cdot \left|\alpha_{n+3}-\alpha_{n+2}\right| \\
\leq &\frac{4m \cdot k_n^{m-1}}{q_{n+1}}.
\end{aligned}$$

using Lemma 5.8 in the last step.
Analogously estimating the other summands we get

$$\begin{aligned}
&\sum_{c \in \xi_n} \bar{\mu}\left(\left(f, df\right)(c) \triangle \left(f_{n+1}, df_{n+1}\right)(c)\right) \\
\leq &\sum_{k=1}^{\infty} \sum_{i=0}^{\tilde{q}_{n+1}-1} \mu\Bigg(f_{n+k+1}\left(f_n^i\left(H_{n-1}\left(\bar{c}_0^{(n)} \cap \bigcap_{j=1}^{k-1} h_n \circ \ldots \circ h_{n+j}\left(G_{n+j+1}\right)\right)\right)\right) \\
&\qquad \triangle f_{n+k}\left(f_n^i\left(H_{n-1}\left(\bar{c}_0^{(n)} \cap \bigcap_{j=1}^{k-1} h_n \circ \ldots \circ h_{n+j}\left(G_{n+j+1}\right)\right)\right)\right)\Bigg) \\
\leq &\sum_{j=n+1}^{\infty} \frac{4m \cdot k_{j-1}^{m-1}}{q_j} \leq \frac{5m \cdot k_n^{m-1}}{q_{n+1}}.
\end{aligned}$$

□

Using this estimate and equation 6 we conclude

$$\sum_{c\in\xi_n}\bar{\mu}\left(\left(f,df\right)(c)\triangle\left(f_n,df_n\right)(c)\right)\leq\frac{1}{l_{n+1}\cdot C_{l_{n+1}}\cdot q_{n+1}^{2\cdot m^2\cdot(l_{n+1}+1)^2\cdot(n+1)\cdot(n+2)}}+\frac{5m\cdot k_n^{m-1}}{q_{n+1}}\leq\frac{6m\cdot k_n^{m-1}}{q_{n+1}}.$$

In order to prove that this speed of approximation is of order $o\left(\frac{1}{\tilde{q}_{n+1}}\right)$ we compute

$$\frac{\frac{6m\cdot k_n^{m-1}}{q_{n+1}}}{\frac{1}{\tilde{q}_{n+1}}}=\frac{q_{n+1}}{A}\cdot\frac{6m\cdot k_n^{m-1}}{q_{n+1}}\leq\frac{m}{n^6\cdot(n+1)^6\cdot q_{n-1}^2\cdot q_n^{1+(m-1)\cdot\frac{n\cdot(n+1)}{2}}}.$$

Since this converges to 0 as $n\to\infty$, the third requirement of definition 7.3 is satisfied. Hence, we can apply Lemma 7.4 and obtain the ergodicity of (f,df) with respect to $\bar{\mu}$.

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