23 The Listing Theorem

We promised to explain the word *enumerable* in 'computably enumerable'.

23.1 Theorem. (i) A nonempty set is c.e. iff it is the range of a total computable function.
(ii) A set is c.e. iff it is either finite or the range of a 1-1 total computable function.

Proof. (i) ($\Rightarrow$) Suppose $W_e \neq \emptyset$. Then there is a least s such that $W_{e,s} \neq \emptyset$; let a be its least element. Now define

$$f(\langle s, x\rangle) = x \text{ if } x \in W_{e,s+1} - W_{e,s}; \quad a \text{ otherwise.}$$

Then $W_e = \mathrm{Ran} f$.
($\Leftarrow$) By (4) of §22, the range of a p.c. function is c.e.
(ii) Suppose W_e is infinite; construct f as in (i). From the infinite sequence

$$f(0), f(1), f(2), f(3), \ldots$$

remove the recurrences of a. The enumeration of the resulting sequence is a 1-1 total computable function. ☒

The gumball gauge principle.

23.2 Corollary. There are binary p.c. functions η and θ such that for all e,
(i) $W_e = \mathrm{Ran}(\lambda x.\eta(e, x))$, and $\lambda x.\eta(e, x)$ is total if $W_e \neq \emptyset$;
(ii) $W_e = \mathrm{Ran}(\lambda x.\theta(e, x))$, and $\lambda x.\theta(e, x)$ is 1-1, and total if W_e is infinite.

24 C.e. sets under inclusion

24.1 Sublattice Theorem. There exist computable functions f and g such that for all x, y, $W_{f(x,y)} = W_x \cup W_y$ and $W_{g(x,y)} = W_x \cap W_y$.

Proof. Define: $\varphi_{f(x,y)}(z) \simeq \varphi_x(z)$ if $\exists s(z \in W_{x,s} - W_{y,s})$,
$\varphi_y(z)$ otherwise;
and $\varphi_{g(x,y)}(z) \simeq \varphi_x(z) + \varphi_y(z)$. ☒

So the c.e. sets form a sublattice of the Boolean algebra of subsets of ω.

24.2 Reduction Principle for c.e. sets. For any c.e. sets A and B, there exist c.e. sets $A_1 \subseteq A$ and $B_1 \subseteq B$ such that $A_1 \cap B_1 = \emptyset$ and $A_1 \cup B_1 = A \cup B$.

Proof. Define $R := (\{0\} \times A) \cup (\{1\} \times B)$. By the previous, R is c.e. Let ψ be a selector function for R. Take $A_1 = \psi^{-1}\{0\}$ and $B_1 = \psi^{-1}\{1\}$. ☒

25 Δ_1 sets

25.1 Definition. (i) $\Pi_1 = \{\bar{A} \mid A \in \Sigma_1\}$;
(ii) $\Delta_1 = \Sigma_1 \cap \Pi_1$.

25.2 Complementation Theorem (Post). A set is computable iff it is Δ_1.

Proof. ($\Rightarrow$) If A is computable, then so is $\bar{A}$.
($\Leftarrow$) Suppose $A = W_e$ and $\bar{A} = W_i$. Define: $f(x) = \mu s(x \in W_{e,s}$ or $x \in W_{i,s})$. Then f is computable, and $x \in A$ iff $x \in W_{e,f(x)}$. ☒

It follows that the computable sets form a *subalgebra* of the Boolean algebra of subsets of ω.

25.3 Corollary. $\bar{K}$ is not c.e.

Proof. If it were, K would be computable. ☒

26 Exercises
:1 (a) Prove: if $A \leq_m B \in \Sigma_1$, then $A \in \Sigma_1$.
(b) Show that Fin and Tot are not c.e.
(c) Show that Cof is not c.e.
:2 Prove: if A is c.e. and ψ is computable, then $\psi[A]$ and $\psi^{-1}[A]$ are c.e.
:3 Let f be a total function. Prove: f is a computable function iff it is a computable relation.
(S. 2.1.23 is misstated; compare 23.2.)

27 Static and dynamic

A c.e. set W_e is the union of a chain

$$W_{e,0} \subseteq W_{e,1} \subseteq W_{e,2} \subseteq \ldots \subseteq W_{e,s} \subseteq \ldots$$

of decidable finite sets, a *computable enumeration* of W_e. We refer to properties of this chain as *dynamic* properties of W_e. The attributes of W_e proper, independent of the way it is enumerated, we call *static*.

28 Uniform sequences and simultaneous enumerations

An initial segment of ω is either a finite set $\{0,\ldots, n-1\}$ (which means Ø in case $n = 0$) or ω itself.

28.1 Definition. A sequence $\mathbb{V} = (V_e |\ e \in \omega)$ of c.e. sets is *uniformly c.e.* (*u.c.e.*) if there is a computable function f such that $V_e = W_{f(e)}$.

Examples. (i) The standard sequence $\mathbb{W} := (W_e |\ e \in \omega)$.
(ii) The sequence $(W_{e,i} |\ i \in \omega)$ of decidable finite sets that approximate the c.e. set W_e.
(iii) The sequence $(W_{e,n} |\ e \in \omega)$. Its elements are subsets of $\{0,\ldots, n-1\}$; if $e \geq n$, $W_{e,n} = $ Ø.

28.2 Definition. (i) Let $\mathbb{V}$ be a u.c.e. sequence. A 1-1 function h from an initial segment of ω into ω is a *simultaneous computable enumeration* (*s.c.e.*) of $\mathbb{V}$ if $\operatorname{Ran} h = \{\langle x, e\rangle |\ x \in V_e\}$. Assuming h, we define:

$$V_{e,s} = \{x|\ \exists t \le s\ h(t) = \langle x, e\rangle\}.$$

(ii) Given h, we may define $V_e := \{x|\ \exists s\ h(s) = \langle x, e\rangle\}$, and denote the resulting u.c.e. sequence by $\mathbb{V}_h$.

Examples. (i) An s.c.e. for $\mathbb{W} := (W_e|\ e \in \omega)$ is $h_1 = \pi_1 \circ g$, with g defined by

$$g(0) = \mu z.T(\pi_2(\pi_1(z)), \pi_1(\pi_1(z)), \pi_2(z)),$$

$$g(x+1) = \mu z[T(\pi_2(\pi_1(z)), \pi_1(\pi_1(z)), \pi_2(z))\ \&\ z > g(x)].$$

Unfortunately, this implies a new definition of $W_{e,s}$ that is *not* equivalent to the original one. The new definition has the virtue, over the old one, that $W_{e,s+1} - W_{e,s}$ always contains exactly one element. The old definition, on the other hand, has the property

$$x \in W_{e,s} \Rightarrow x, e < s \qquad (*).$$

We should like to have some variant h of h_1 that induces (*) to hold, but this is impossible: $h(0)$ would have to be a pair of numbers less than 0.

There are, however, a few little tricks we can apply. First relax (*) to

$$x \in W_{e,s} \Rightarrow x, e \le s \qquad (**).$$

Now observe that (**) implies something like

$$\Sigma_e |W_e \cap \{0,\dots, n-e\}| \ge n \qquad (\dagger).$$

To ensure (†) is possible, we move a few fat sets to the beginning of the enumeration; it suffices to stipulate $W_0 = W_1 = \omega$. Once (†) has been secured, there is always room to solve the finite puzzle of defining the next value of h. So when it suits us, we may assume we have an s.c.e. h_0 of $\mathbb{W}$ that satisfies (**); or even (*), allowing exceptions for $s \le 1$.

(ii) Define an s.c.e. h of $(W_{e,i}|\ i \in \omega)$ by

$$h(s) = \langle x, i\rangle \text{ iff } \sum_{j<i} |W_{e,j}| + |\{y \in W_{e,i}|\ y < x\}| = s.$$

The domain of this enumeration may be finite.

(iii) The domain of an s.c.e. of $(W_{e,n}|\ e \in \omega)$ will certainly be finite.

28.2 Definition. Let h be an s.c.e. of a u.c.e. sequence $\mathbb{V}$. For some i, j, put $X_s = V_{i,s}$ and $Y_s = V_{j,s}$. Then

(i) $X \setminus Y = \{z|\ \exists s\ z \in X_s - Y_s\}$;

(ii) $X \searrow Y = (X \setminus Y) \cap Y$.

Beware: much is suppressed in these dynamic notations.

28.3 Dynamic Flow Theorem. Fix an s.c.e. of $\mathbb{W}$, and for some b, put $B_s = W_{b,s}$ and $B = W_b$. If B is noncomputable, then for every e such that W_e includes $\overline{B}$, $W_e \searrow B$ is infinite.

Proof. If $W_e \searrow B$ is finite, then $W_e \setminus B =^* \overline{B}$, which makes B computable. ☒

By essentially the same argument, $W_e \searrow B$ is noncomputable.

29 Exercise

Let $(X_s|\ s \in \omega)$ and $(Y_s|\ s \in \omega)$ be computable enumerations of c.e. sets X and Y. Prove:

(a) $X \setminus Y$ and $X \searrow Y$ are c.e.;

(b) $X \setminus Y = (X - Y) \cup (X \searrow Y)$;

(c) if $X - Y$ is not c.e., then $X \searrow Y$ is noncomputable;

(d) the Reduction Principle, by putting, for $A = W_x$ and $B = W_y$, $A_1 = W_x \setminus W_y$ and $B_1 = W_y \setminus W_x$.

30 Friedberg's Splitting Theorem

In the proof of the next theorem, we build a set that is to satisfy an infinite list $R_0, R_1, R_2, \ldots, R_n, \ldots$ $(n \in \omega)$ of requirements. The earlier a requirement appears in the list, the higher its *priority*. At any stage in the construction, there may be requirements *demanding attention*; then we satisfy one of them, the one with highest priority.

In the present case, once a requirement has been acted on, it remains satisfied. In more advanced applications of the method, violations may occur.

Theorem. For any noncomputable c.e. set B, there exist disjoint noncomputable c.e. sets A_0 and A_1 such that $B = A_0 \cup A_1$.

Proof. Let f be a 1-1 computable listing of B; define: $B_s = \{f(0), \ldots, f(s)\}$. To ensure A_0 and A_1 are noncomputable, we meet the requirements

$$R_{\langle e,\, i\rangle} : W_e \neq \overline{A_i}\,,\ \ e \in \omega,\ \ i = 0, 1.$$

Stage 0: $A_0 = \{f(0)\}$, $A_1 = \emptyset$.

Suppose we have constructed $A_{i,s}$, for $i = 0, 1$.

Stage $s + 1$: If there are $\langle e, i\rangle$ such that

$$f(s+1) \in W_{e,s} \quad \& \quad W_{e,s} \cap A_{i,s} = \emptyset \qquad (\ddagger),$$

take the least one and add $f(s + 1)$ to A_i; so $A_{i,s+1} = A_{i,s} \cup \{f(s + 1)\}$, and $A_{\overline{\mathrm{sg}}(i),s+1} = A_{\overline{\mathrm{sg}}(i),s}$. If there are no such $\langle e, i\rangle$, add $f(s + 1)$ to A_0.

Observe that a requirement, once met, is satisfied forever after. Put $A_i = \bigcup_s A_{i,s}$. Clearly A_0 and A_1 are disjoint, and $A_0 \cup A_1 = B$.

If the requirements are not all met, take the least $\langle e, i\rangle$ for which $R_{\langle e,\, i\rangle}$ is failed. Then $W_e = \overline{A_i}$, so $W_e \supseteq \overline{B}$. So $W_e \searrow B$ is infinite, by the Dynamic Flow Theorem. Hence there will be s where (‡) holds after all $R_{\langle d,j\rangle}$ with $\langle d, j\rangle < \langle e, i\rangle$ have been met. But by construction, at the stage following the first such s, $R_{\langle e,\, i\rangle}$ is satisfied: a contradiction. ☒