15 Reduction

15.1 Definition. Let A and B be sets (of natural numbers).
(i) A is *many-one reducible* (*m-reducible*) to B, notation $A \leq_m B$, if there exists a computable function f such that $x \in A \Leftrightarrow f(x) \in B$.
(ii) A is *one-one reducible* (1-*reducible*) to B, notation $A \leq_1 B$, if there exists a 1-1 computable function f such that $x \in A \Leftrightarrow f(x) \in B$.

For example, $K \leq_1 K_0$. Observe that $A \leq_m B$ implies $\overline{A} \leq_m \overline{B}$, by the same function. These reducibilities are easily seen to be reflexive and transitive, so $\leq_m \cap \geq_m$ and $\leq_1 \cap \geq_1$ are equivalence relations. We denote them by $\equiv_m$ and $\equiv_1$, respectively. The *m-degree* $\deg_m(A)$ is $A/\equiv_m$; the 1-*degree* $\deg_1(A)$ is $A/\equiv_1$.

15.2 Proposition. If $A \leq_m B$ and B is computable, then A is computable.

15.3 Theorem. $K \leq_1 \mathrm{Tot} := \{x \mid \mathrm{Dom}\, \varphi_x = \omega\}$.

Proof. There exists a 1-1 computable function f such that $\varphi_{f(x)}(y) \simeq \varphi_x(x)$. ⊠

The proof shows that we cannot decide either whether a p.c. function is a constant function, or whether it is empty. Moreover, we can substitute any c.e. set for K.

16 Index sets

The method of Theorem 15.3 applies to almost all classes that correspond to properties of *functions*.

16.1 Definition. A is an *index set* if

$$x \in A \;\&\; \varphi_x = \varphi_y \Rightarrow y \in A.$$

For example, Tot is an index set.

16.2 Index Set Theorem. If A is a nontrivial (i.e. other than Ø and ω) index set, then $K \leq_1 A$ or $K \leq_1 \overline{A}$.

Proof. Let e_0 be an index of the empty function. If $e_0 \in \overline{A}$, then we show $K \leq_1 A$ as follows. Take $e_1 \in A$. Then $\varphi_{e_1} \neq \varphi_{e_0}$ since A is an index set. By the s-m-n Theorem, construct a 1-1 computable function f such that

$$\varphi_{f(x)}(y) \simeq \varphi_{e_1}(y) + 0 \cdot \varphi_x(x).$$ ⊠

16.3 Rice's Theorem. Let C be a class of (unary) p.c. functions. Then the set of indices of elements of C is computable only if C is empty or C contains all p.c. functions.

Here are some more index sets:
$K_1 = \{x \mid W_x \neq \emptyset\}$;

Fin = $\{x \mid W_x \text{ is finite}\}$;
Inf = ω – Fin;
Con = $\{x \mid \exists n\ \varphi_x = \lambda y.n\}$ (indices of *constant* functions);
Cof = $\{x \mid W_x \text{ is cofinite}\}$;
Cput = $\{x \mid W_x \text{ is computable}\}$;
Ext = $\{x \mid \exists y \in \text{Tot}\ \varphi_x \subseteq \varphi_y\}$ (*extendible* to total functions).

17 Complete sets, degrees and lattices

17.1 Definition. A c.e. set A is 1-*complete* if $B \leq_1 A$ for every c.e. set B.

For example, K_0 is 1-complete.

Classifying sets by degrees of unsolvability and comparing degrees are major concerns of recursion theory. As to comparing degrees: clearly the original quasi-ordering induces an ordering of the degrees. We have

$$\mathbf{a} \leq \mathbf{b} \text{ iff } \exists A \in \mathbf{a}\ \exists B \in \mathbf{b}\ A \leq B$$
$$\text{iff } \forall A \in \mathbf{a}\ \forall B \in \mathbf{b}\ A \leq B$$

An order (partially ordered set) $(X, \leq)$ is an *upper semilattice* if every two elements x, y have a *join* (*least upper bound*, *supremum*) $x \vee y$; that is,

(*) $$\forall u \in X\ (x \leq u\ \&\ y \leq u \Leftrightarrow x \vee y \leq u).$$

The join is unique, for if a and b are joins of x and y, then by (*), since $a \leq a$, $x \leq a$ and $y \leq a$. Hence by (*) again, $b \leq a$. Switching a and b in the argument, we get $b \leq a$. So $a = b$ by antisymmetry.

The order $(X, \leq)$ is an *lower semilattice* if every two elements x, y have a *meet* (*greatest lower bound*, *infimum*) $x \wedge y$; that is,

(*) $$\forall u \in X\ (x \geq u\ \&\ y \geq u \Leftrightarrow x \wedge y \geq u).$$

The meet is unique as well, being the join in the upper semilattice $(X, \geq)$.

A *lattice* is an order that is both an upper and a lower semilattice.

By Exercise 18:2, the m-degrees form an upper semilattice.

17.2 Definition. Let A and B be sets. Then

$$A \oplus B = \{2a \mid a \in A\} \cup \{2b + 1 \mid b \in B\}.$$

This *join* contains, in an obvious sense, precisely the information contained in A and B. Your proof of 18:2, however, will not carry over to 1-reducibility.

18 Exercises

:1 Suppose $B = A \oplus \overline{A}$ for some set $A \subset \omega$. Prove $B \leq_1 \overline{B}$.
:2 Prove that $\deg_m(A \oplus B) = \deg_m(A) \vee \deg_m(B)$.
:3 Prove that K_0, K_1 and K are 1-equivalent.
:4 Prove that $K \leq_1$ Fin *directly*, that is, without using Rice's Theorem.